Quantum computation based on *d*-level cluster state

D. L. Zhou,[1] B. Zeng,[2] Z. Xu,[2] and C. P. Sun[3]
[1]*Center for Advanced Study, Tsinghua University, Beijing 100084, China*
[2]*Department of Physics, Tsinghua University, Beijing, 100084, China*
[3]*Institute of Theoretical Physics, The Chinese Academy of Sciences, Beijing, 100080, China*
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The concept of a qudit (a d-level system) cluster state is proposed by generalizing the qubit cluster state [Phys. Rev. Lett. **86**, 910 (2001)] to higher-dimensional Hilbert space according to the finite-dimensional representations of quantum plane algebra. We demonstrate their quantum correlations and prove a theorem which guarantees the availability of the qudit cluster states in quantum computation. We explicitly construct the network to show the universality of the one-way computer based on the defined qudit cluster states and single-qudit measurement. A protocol of implementing one-way quantum computer is suggested using the high-dimensional ''Ising'' model which can be found in many magnetic systems.

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I. INTRODUCTION

Quantum computers can undertake computational tasks that are intractable for classical computers. The reason lies in the fact that quantum computing systems composed of qubits (two-level quantum systems) possess mysterious quantum coherence, such as entanglement (or quantum correlation), which has no counterpart in the classical realm [1]. Recently, an important kind of entangled states, namely, cluster states [2], was introduced. Cluster states enjoy the following remarkable property: each pair of qubits can be projected into maximally entangled state with certainty by single-qubit measurements on all the other qubits. This property might suitably be referred to as maximal connectedness. More surprisingly, it was shown that cluster states can be used to build a one-way universal quantum computer, in which all the operations can be implemented by single-qubit measurements only [3]. Raussendorf and Briegel provide a simple criterion for the functioning of gate simulations on such quantum computers [4]. This is a key theoretical step towards realizing such a scalable quantum computer. On the other hand, it was pointed out that the protocol of cluster state computers can be easily realized in practical physical systems since the creation of cluster states needs only Ising-type interactions [2]. In fact, such interactions appear naturally in the solid state lattice system with proper spin-spin interactions [2], and in the optical lattice cold atom system [5].

Theoretically, it is natural to ask whether the concept of cluster states can be generalized to the higher-dimensional case, or the so-called qudit case, since most available physical systems cannot be treated as two-level systems even in an approximate way. The answer to this question is affirmative. In our analysis, we make full use of the noncommutative operators X and Z, which provides a d-dimensional irreducible representation of Manin's quantum plane algebra (QPA) [6]. The crucial point is that the qudit cluster state $|\phi\rangle_{\mathcal{C}}$ can be defined as a common eigenstate of the tensor product operators

$$X_a^{\dagger} \bigotimes_{b\in\mathcal{N}(a)} Z_b, \tag{1}$$

where the lower indices a and b denote qudit a and qudit b in the cluster, and index b is taken in the neighborhood of index $a[\mathcal{N}(a)]$, depending on the cluster structure. Based on this definition of qudit cluster states, we manage to construct all single-qudit unitary gates and one imprimitive two-qudit gate, presenting an important ingredient for building a one way universal quantum computer.

This paper is organized as follows. We briefly review the finite-dimensional representations of quantum plane algebra in Sec. II. This prepares the main mathematical tools used in this paper. In Sec. III the qudit cluster state, as a nontrivial generalization of the qubit cluster state, is defined by means of the quantum plane algebra, and its essential properties of quantum correlations are analyzed. The main theorem is presented and proved in Sec. IV, expediting the introduction of a one-way quantum computer based on qudit cluster states. As in the qubit case [4], this theorem guarantees the functioning of gate simulations on the qudit cluster state quantum computers. In Sec. V the universality of the qudit cluster state computer is proved by constructing explicitly all the single- and one two-qudit logic gates. Finally, short conclusions are presented in Sec. VI.

II. FINITE-DIMENSIONAL REPRESENTATIONS OF QUANTUM PLANE ALGEBRA

In this section we will review some basic results about the finite-dimensional representations of Manin's quantum plane algebra. This will provide us with a useful mathematical tool to describe not only qudit cluster states but also unitary transformation on the Hilbert space. Manin's quantum plane is defined by

$$XZ = qZX, \tag{2}$$

where q is a complex number. It is well known that the associative algebra generated by Z, X possesses a d-dimensional irreducible representation only when $q^d=1$ [6]. In this article, we take $q \equiv q_d \equiv e^{i2\pi/d}$. This special case is first introduced and studied by Weyl [13] and Schwinger [14]. Obviously, when $d=1$, $q=1$, X and Z can be regarded as the ordinary coordinates of R^2 plane. When $d=2$, $q=-1$, X and Z can be identified with the Pauli matrices σ_x and σ_z. From this viewpoint Z and X can be regarded as the so-called "generalized Pauli operators" [7–12].

When $q=q_d$, Z^d and X^d commute with the algebra generators, so they belong to the center of QPA. From Schur's lemma it follows that Z^d and X^d are constant multiples of the d-dimensional identity matrix, i.e., $Z^d=zI$ and $X^d=xI$. In general we can normalize them to the identity. Since the complex field C is algebraically closed, there must exist an eigenstate $|0\rangle$, which satisfies

$$Z|0\rangle=|0\rangle. \tag{3}$$

Defining $|k\rangle=X^{\dagger k}|0\rangle$, then we have

$$Z|k\rangle=q_d^k|k\rangle, \quad (k\in Z_d), \tag{4}$$

according to Eq. (2), and

$$X|k\rangle=|k-1\rangle \tag{5}$$

by definition. Thus in the Z-diagonal representation, the matrices of X and Z are

$$Z=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & q_d & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_d^{d-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & q_d^{d-1} \end{bmatrix}, \tag{6}$$

$$X=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{7}$$

From Eq. (5), we have

$$X|x(0)\rangle=|x(0)\rangle, \tag{8}$$

where

$$|x(0)\rangle=\frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}|k\rangle. \tag{9}$$

Similar to the eigenvalue equation of Z, we have

$$X|x(j)\rangle=q_d^j|x(j)\rangle, \tag{10}$$

where

$$|x(j)\rangle=Z^j|x(0)\rangle=\frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}q_d^{jk}|k\rangle. \tag{11}$$

These equations define a representation of the algebra equivalent to the previous one.

Obviously,

$$B=\{Z^jX^k,(j,k\in Z_d)\} \tag{12}$$

is a basis of the algebra. The elements of this basis are called unitary operator bases in Ref. [14]. The following general commutation relations for any two basis elements can be checked immediately:

$$X^jZ^k=q_d^{jk}Z^kX^j. \tag{13}$$

We observe that many other elements in the algebra have the same commutation relation as the generators Z and X. Denote by (m,n) the greatest common factor of integers m and n. Let $m_1,\ n_1,\ m_2,\ n_2\in Z_d$ be integers such that $(m_1,n_1)=1$, $(m_2,n_2)=1$, and $m_1n_2-m_2n_1=1$. We take

$$\overline{Z}=q_d^{[-(d-1)/2]m_1n_1}Z^{m_1}X^{n_1}, \tag{14}$$

$$\overline{X}=q_d^{[-(d-1)/2]m_2n_2}Z^{m_2}X^{n_2}. \tag{15}$$

It then follows that $\overline{Z}$ and $\overline{X}$ respect the commutation relation Eq. (2). Note that the coefficients in the definitions of $\overline{Z}$ and $\overline{X}$ serve to guarantee that they possess the same eigenvalues as Z and X, respectively. It turns out that $\overline{Z}$ and $\overline{X}$ defines a unitary transformation U:

$$\overline{Z}=UZU^\dagger, \quad \overline{X}=UXU^\dagger. \tag{16}$$

It is easy to check that all this kind of unitary transformations form a group. This is the so-called Clifford group.

III. QUDIT CLUSTER STATES IN QUANTUM PLANE

Now we generalize the concept of qubit cluster states to the qudit case. For conceptual simplicity we first restrict ourselves to one-dimensional lattices. Let us recall the definition of one-dimensional cluster states for N qubits. Consider an N-site lattice, with a qubit attached to each site. As a novel multiqubit entangled state, the cluster state is written as

$$|\phi\rangle_{\mathcal{C}}=\frac{1}{2^{N/2}}\bigotimes_{a=1}^{N}[|0\rangle_a+|1\rangle_a(\sigma_z)_{a+1}], \tag{17}$$

where $(\sigma_i)_a$ $(i=x,y,z)$ are the Pauli matrices assigned for site a in the lattice, and

$$\sigma_z|s\rangle=(-1)^s|s\rangle, \quad (s\in\{0,1\}).$$

By analogy we naturally conjecture that the qudit cluster state in one dimension is

$$|\phi\rangle_{\mathcal{C}}=\frac{1}{d^{N/2}}\bigotimes_{a=1}^{N}\left(\sum_{k=0}^{d-1}|k\rangle_aZ_{a+1}^k\right), \tag{18}$$

where

$$Z_a|k\rangle_a=q_d^k|k\rangle_a, \quad \forall a. \tag{19}$$

Now we present one of our main results.

Theorem 1. The qudit cluster state in one dimension defined by Eq. (18) is a common eigenstate of eigenvalue 1 of the operators $X_a^\dagger \otimes_{b\in\mathcal{N}(a)} Z_b$, i.e.,

$$X_a^\dagger \bigotimes_{b\in\mathcal{N}(a)} Z_b |\phi\rangle_{\mathcal{C}} = |\phi\rangle_{\mathcal{C}}, \tag{20}$$

where

$$\mathcal{N}(a) = \begin{cases} \{2\}, & a=1 \\ \{N-1\}, & a=N \\ \{a-1,a+1\}, & a\notin\{1,N\}. \end{cases} \tag{21}$$

Proof. We notice that the qudit cluster state (18) can be constructed by the following procedure. We first prepare a product state

$$|+\rangle = \bigotimes_{a=1}^{N} |x(0)\rangle_a ,$$

then apply a unitary transformation

$$S = \prod_{b-a=1} S_{ab} \tag{22}$$

to the state $|+\rangle$. Here S_{ab} is defined as an intertwining operator

$$S_{ab}|j\rangle_a|k\rangle_b = q_d^{jk}|j\rangle_a|k\rangle_b . \tag{23}$$

It is easy to prove

$$|\phi\rangle_{\mathcal{C}} = S|+\rangle. \tag{24}$$

Since $X_a^\dagger|+\rangle = |+\rangle$, it is easy to check

$$SX_a^\dagger S^\dagger |\phi\rangle_{\mathcal{C}} = |\phi\rangle_{\mathcal{C}}. \tag{25}$$

Now it suffices to prove

$$SX_a^\dagger S^\dagger = X_a^\dagger \bigotimes_{b\in\mathcal{N}(a)} Z_b .$$

To this end, for $a,b,c\in\{1,...,N\}$, we observe that

$$S_{ab}X_a^\dagger S_{ab}^\dagger = X_a^\dagger \otimes Z_b , \tag{26}$$

$$S_{ab}X_b^\dagger S_{ab}^\dagger = Z_a \otimes X_b^\dagger , \tag{27}$$

$$S_{ab}X_c^\dagger S_{ab}^\dagger = X_c^\dagger , \quad \forall c\notin\{a,b\}, \tag{28}$$

and

$$S_{ab}Z_c S_{ab}^\dagger = Z_c , \quad \forall c. \tag{29}$$

The theorem then follows directly. ■

Although the above proof is proposed in the context of a one-dimensional cluster, it still works for the more complicated two- or three-dimensional cluster. The reason is as follows. For a general cluster C with one qudit on each site, the cluster state $|\phi\rangle_{\mathcal{C}}$ is defined by the following eigenequations:

$$X_a^\dagger \bigotimes_{b\in\mathcal{N}(a)} Z_b |\phi\rangle_{\mathcal{C}} = |\phi\rangle_{\mathcal{C}}. \tag{30}$$

Formally, this definition of a general cluster is the same as a one-dimensional cluster. Of course, different clusters have different definitions of neighbors.

Now we preceed to discuss the properties of quantum correlations in the above cluster states under single qudit measurements. For simplicity, we still restrict ourselves to the one-dimensional case.

First, let us consider the problem of describing a von Neumann measurement for a single qudit. Although the generator Z (or X) is not Hermitian, i.e., the eigenvalues of Z (or X) are not real, the nondegenerate eigenstates of Z (or X) can still be used to define a von Neumann measurement. For example, when we make a measurement marked by Z, we mean that we can obtain different results corresponding to different eigenstates of Z.

Next, we discuss the minimal number P_e of single qudit measurements needed to destroy all the quantum correlations in qudit cluster states. In the one-dimensional case, if we expand the cluster state $|\psi\rangle_{\mathcal{C}}$ with respect to a product basis, the minimum number of terms needed will grow exponentially with N (precisely, as $d^{[N/2]}$, where $[N/2]$ denotes the maximal integer not more than $N/2$). This can be shown by induction as in Ref. [3]. This observation gives us an upper bound of $P_e : P_e \leqslant [N/2]$. Further analysis shows that by measuring Z_{2a} ($a=1,2,...,[N/2]$) all the quantum entanglement will be destroyed. Let us illustrate the key steps to reach this conclusion with one-dimensional cluster states for four qudits. In this case the cluster state is defined as the common eigenstate of eigenvalue 1 of the set of operators $\{X_1^\dagger Z_2, Z_1X_2^\dagger Z_3, Z_2X_3^\dagger Z_4, Z_3X_4^\dagger\}$. If we measure Z_2 and Z_4, we obtain the results $q_d^{s_2}$ and $q_d^{s_4}$, and the qudits 2 and 4 collapse into states $|s_2\rangle_2$ and $|s_4\rangle_4$, respectively. Then from the definition of the cluster state, qudits 1 and 3 are in the eigenstates of eigenvalue 1 of $q_d^{s_2}X_1^\dagger$ and $q_d^{s_2+s_4}X_3$, namely, $|x(s_2)\rangle_2$ and $|x(s_2+s_4)\rangle_4$. Thus all the entanglement is destroyed by measuring Z_2 and Z_4. The general case can be dealt with similarly. In summary, we have $P_e = [N/2]$.

Finally, we probe the most remarkable property—maximal connectedness—of a qudit cluster state. It means that each pair of qudits in the cluster can be projected into a maximally entangled state with certainty by single-qudit measurement on all the other qudits. In fact, to project two arbitrary qudits in a one-dimensional cluster into a maximally entangled state, we only need to measure X for the qudits between them, and Z for all the other qudits. Let us use the same four qudits cluster state as above to prove this property case by case. To project qudits 1 and 2 into maximally entangled states, we need to measure Z_3 and Z_4. When we obtain the result $q_d^{s_3}$, qudits 1 and 2 collapse into the common eigenstate of eigenvalue 1 of the operators $X_1^\dagger Z_2$ and $q_d^{s_3}Z_1X_2^\dagger$, which is a maximally entangled state of qudits 1 and 2. To project qudits 1 and 3 into maximally entangled states, we need to measure $X_2^\dagger$ and Z_4. When we obtain the results $q_d^{s_2}$ and $q_d^{s_4}$, qudits 1 and 3 collapse into the common eigenstate of eigenvalue 1 of the operators $q_d^{s_2}Z_1Z_3$ and

$q_d^{s_4} X_1^\dagger X_3^\dagger$, which is a maximally entangled state of qudits 1 and 3. Other cases can be treated in the same way. For the general one-dimensional cluster state, this property can be proved similarly.

Moreover, we point out that two- or three-dimensional qudit cluster states are also maximally connected. As a matter of fact, the higher-dimensional problem can be reduced to the one-dimensional one. To this end, we only need to find a path connecting the two qudits under consideration and measure Z for all the other qudits which are not on the path.

Now it is in order to discuss the problem of physical implementation of our one-way quantum computer. Physically the qudit cluster state (18) can be created by the Hamiltonian

$$H = -\hbar g \sum_{(a,b)} N_a^{(z)} N_b^{(z)} \quad (g>0), \tag{31}$$

where (a,b) denotes sites a and b which are nearest neighbors in the cluster; and $N^{(z)}$ is defined as

$$N^{(z)} = \sum_{k=0}^{d-1} |k\rangle k \langle k|. \tag{32}$$

Then the intertwining operator S in Eq. (22) has the explicit form

$$S = \exp\left(-\frac{i}{\hbar} H t_{\mathcal{C}} \right), \tag{33}$$

with the evolution time $t_{\mathcal{C}} = 2\pi/dg$.

To associate with the more familiar Hamitonian in physics, let us define the spin-$(d-1)/2$ operator of z direction

$$s_z = N^{(z)} - \frac{d-1}{2}. \tag{34}$$

Then we can rewrite Eq. (31) as

$$H = \frac{d-1}{2} \hbar g \sum_a \nu_a (s_z)_a - \hbar g \sum_{(a,b)} (s_z)_a (s_z)_b, \tag{35}$$

where ν_a is the number of nearest neighbors of qudit a in the cluster. Obviously, the interaction Hamiltonian

$$H_I = -\hbar g \sum_{(a,b)} (s_z)_a (s_z)_b \tag{36}$$

is the ferromagnetic Ising-type interaction with spin-$(d-1)/2$.

By the way, we remark that higher spin Ising models have been actively studied systems in condensed matter and statistical physics due to their rich variety of critical and multicritical phenomena. For example, the spin-1 Ising model with nearest-neighbor interactions and a single-ion potential is known as the Blume-Emery-Griffiths (BEG) model [17], the spin-3/2 Ising model was introduced to explain phase transitions in $DyVO_4$, and its phase diagrams were obtained by the mean-field approximation [18]. In addition, the magnetic properties of some artificially fabricated superlattices can be explained in the framework of higher spin Ising models. Such superlattices consist of two or more ferromagnetic materials. They have been widely studied over the years because their physical properties differ dramatically from those of simple solids formed from one single material. Moreover, the development of film deposition techniques has aroused great interest in the study of superlattices of other materials. A number of experimental [19–22] and theoretical works [23–30] have been devoted to this direction.

IV. MEASUREMENT-BASED QUANTUM COMPUTATION WITH QUDIT CLUSTER STATES

We have seen that the qudit cluster states exhibit the same features in quantum entanglement as those of the qubit cluster states. Then a question arises naturally: how are these natures of quantum correlations related to constructing the universal quantum computation? We will answer this question in the following two sections.

In this section we further generalize the basic concept of "single-qudit quantum measurement" and study the measurement-based quantum computation (MBQC) on qudit clusters. Along the line to construct MBQC for the qubit case, we will establish the corresponding theorem which relates unitary transformation to quantum entanglement exhibited by the qudit cluster states. The quantum computations with qudit clusters inherit all basic concepts of those with qubit clusters. They include the basic procedure of simulation of any unitary gate, the concatenation of gate simulation, and the method of dealing with the random measurement results. Here we will give a d-dimensional version of the central theorem 1 in Ref. [4].

Before formulating our central theorem, let us prepare some basic elements of quantum computing with qudit clusters. The main problem concerning quantum computing with qudits is to simulate the arbitrary quantum gate g defined on n-qudit Hilbert space. This problem can be tackled in three steps. The first step is to find out a proper cluster $\mathcal{C}(g)$ and divide it into three subclusters: the input cluster $\mathcal{C}_I(g)$, the body cluster $\mathcal{C}_M(g)$, and the output cluster $\mathcal{C}_O(g)$. As usual we require that the input and output clusters have the same rank (i.e., the same number of qudits). Then we prepare the initial state as

$$|\Psi(\text{in})\rangle_{\mathcal{C}(g)} = |\psi(\text{in})\rangle_{\mathcal{C}_{I(g)}} |+\rangle_{\mathcal{C}_{M(g)} \cup \mathcal{C}_{O(g)}}, \tag{37}$$

and entangle the qudits on the qudit cluster by using the cluster state generator S,

$$|\Phi(\text{in})\rangle_{\mathcal{C}(g)} = S |\Psi(\text{in})\rangle_{\mathcal{C}(g)}. \tag{38}$$

This step brings the structure information of the qudit cluster into our computing process, and thus relates it with the corresponding qudit cluster state.

The second step is to measure all qudits on the cluster relative to a special space-time-dependent basis according to a given measurement pattern (MP). The definition of MP is given as follows.

Definition 1. A measurement pattern $\mathcal{M}_{\mathcal{C}}$ on a cluster $\mathcal{C}$ is a set of unitary matrices

$$\mathcal{M}_{\mathcal{C}}=\{u_a Z_a u_a^{\dagger}|a\in\mathcal{C},\quad u_a\in \mathrm{SU}(d)\},\tag{39}$$

which determines the one-qudit measured operators $N_a^{(u)}$ on $\mathcal{C}$, with the explicit form

$$N_a^{(u)}=\sum_{s=0}^{d-1} u_a|s\rangle_a s_a\langle s|u_a^{\dagger}.\tag{40}$$

If this measurement pattern $\mathcal{M}_{\mathcal{C}}$ operates on the initial state $|\Phi(\mathrm{in})\rangle_{\mathcal{C}(g)}$, the set of measurement outcomes

$$\{s\}_{\mathcal{C}}=\{s_a\in Z_d|a\in\mathcal{C}\}\tag{41}$$

is obtained. Then, modulo a norm factor, the resulting state $|\Psi_{\mathcal{M}}\rangle_{\mathcal{C}}$ is given by

$$|\Phi_{\mathcal{M}_{\mathcal{C}}}^{\{s\}}\rangle=P_{\mathcal{M}_{\mathcal{C}}}^{\{s\}}|\Phi(\mathrm{in})\rangle_{\mathcal{C}(g)},\tag{42}$$

where the pure state projection

$$P_{\mathcal{M}_{\mathcal{C}}}^{\{s\}}=\bigotimes_{k\in\mathcal{C}} u_k|s\rangle_{k\ k}\langle s|u_k^{\dagger}.\tag{43}$$

It is worthy to point out that we always measure X for the input qudits and Z for the output qudits, regardless of gate g, i.e.,

$$\mathcal{M}_{\mathcal{C}_I(g)}=\{X_i,\ \ i\in\mathcal{C}_I(g)\},\tag{44}$$

$$\mathcal{M}_{\mathcal{C}_O(g)}=\{Z_i,\ \ i\in\mathcal{C}_O(g)\}.\tag{45}$$

In this way we manage to associate the measurement values with the outcome of gate g acting on the initial state, completing the third step.

We notice that the essential point of the above standard procedure of qudit clusters quantum computation is to associate a given gate g with a measurement pattern. Although by now we have no general optimal operational procedure to do this for practical problems, the following theorem provides a useful tool in realizing specific gates on the qudit clusters.

Theorem 2. Suppose that the state $|\psi\rangle_{\mathcal{C}(g)}=P_{\mathcal{M}_{\mathcal{C}_M}(g)}^{\{s\}}|\phi\rangle_{\mathcal{C}(g)}$ obeys the 2n eigenvalue equations

$$X_{\mathcal{C}_I(g),i}(UX_iU^{\dagger})_{\mathcal{C}_O(g)}|\psi\rangle_{\mathcal{C}(g)}=q_d^{-\lambda_{x,i}}|\psi\rangle_{\mathcal{C}(g)}\tag{46}$$

$$Z_{\mathcal{C}_{I(g),i}}^{\dagger}(UZ_iU^{\dagger})_{\mathcal{C}_O(g)}|\psi\rangle_{\mathcal{C}(g)}=q_d^{-\lambda_{z,i}}|\psi\rangle_{\mathcal{C}(g)},\tag{47}$$

with $\lambda_{x,i},\ \lambda_{z,i}\in Z_d$ and $1\leqslant i\leqslant n$. Then, according to the above standard quantum computing procedure, we have

$$P_{\mathcal{M}_{\mathcal{C}_I(g)}}^{\{s\}}P_{\mathcal{M}_{\mathcal{C}_M(g)}}^{\{s\}}|\Phi(\mathrm{in})\rangle_{\mathcal{C}(g)}\propto\left(\prod|s_i\rangle\right)_{\mathcal{C}_I(g)\cup\mathcal{C}_M(g)}|\psi(\mathrm{out})\rangle_{\mathcal{C}_O(g)},\tag{48}$$

where the input and output states in the simulation of g are related via

$$|\psi(\mathrm{out})\rangle=UU_{\Sigma}|\psi(\mathrm{in})\rangle,\tag{49}$$

where U_{Σ} is a byproduct operator given by

$$U_{\Sigma}=\bigotimes_{[\mathcal{C}_O(g)\ni i]=1}^{n}(Z_i)^{-\lambda_{x,i}-s_i}(X_i)^{\lambda_{z,i}}.\tag{50}$$

Proof. Let us begin with the case that

$$|\psi(\mathrm{in})\rangle_{\mathcal{C}_I(g)}=|\{t\}\rangle_{\mathcal{C}_I(g)},\tag{51}$$

where

$$\{t\}=\{t_1t_2\cdots t_n\}.$$

To associate with $|\psi\rangle_{\mathcal{C}(g)}$, write the initial input state as

$$|\psi(\mathrm{in})\rangle_{\mathcal{C}_I(g)}=(\sqrt{d})^nP_{\mathcal{M}'_{\mathcal{C}_I(g)}}^{\{t\}}|+\rangle_{\mathcal{C}_I(g)},\tag{52}$$

where

$$\mathcal{M}'_{\mathcal{C}_I(g)}=\{Z_i,i\in\mathcal{C}_I(g)\}.\tag{53}$$

From Eqs. (37), (38), (48), (52), we have

$$\left(\prod|s_i\rangle\right)_{\mathcal{C}_I(g)\cup\mathcal{C}_M(g)}|\psi(\mathrm{out})\rangle_{\mathcal{C}_O(g)}\propto P_{\mathcal{M}_{\mathcal{C}_I(g)}}^{\{s\}}P_{\mathcal{M}'_{\mathcal{C}_I(g)}}^{\{t\}}|\psi\rangle_{\mathcal{C}(g)}.\tag{54}$$

To find out the equations for the final state $|\psi(\mathrm{out})\rangle_{\mathcal{C}_O(g)}$, let $P_{\mathcal{M}_{\mathcal{C}_I(g)}}^{\{s\}}P_{\mathcal{M}'_{\mathcal{C}_I(g)}}^{\{t\}}$ act on both sides of Eqs. (46) and (47),

$$(UX_iU^{\dagger})_{\mathcal{C}_O(g)}|\overline{\psi}(\mathrm{out})\rangle_{\mathcal{C}_O(g)}=q_d^{-s_i-\lambda_{x,i}}|\psi(\mathrm{out})\rangle_{\mathcal{C}_O(g)},\tag{55}$$

$$(UZ_iU^{\dagger})_{\mathcal{C}_O(g)}|\psi(\mathrm{out})\rangle_{\mathcal{C}_O(g)}=q_d^{t_i-\lambda_{z,i}}|\psi(\mathrm{out})\rangle_{\mathcal{C}_O(g)},\tag{56}$$

where the input state for $|\overline{\psi}(\mathrm{out})\rangle$

$$|\overline{\psi}(\mathrm{in})\rangle=X_i^{\dagger}|\{t\}\rangle.\tag{57}$$

Before drawing a conclusion, we need to check the final state is not a zero vector. In fact, from Eqs. (46) and (47), the state $U^{\dagger}|\psi\rangle_{\mathcal{C}(g)}$ is a simultaneous eigenvector of the operators $X_{\mathcal{C}_I(g),i}X_{\mathcal{C}_O(g),i}$ and $Z_{\mathcal{C}_I(g),i}^{\dagger}Z_{\mathcal{C}_O(g),i}$. We find that it has every component in the Z-diagonal representation of the input part. Consequently the final state is indeed a nonzero vector. So from Eqs. (55) and (56), we obtain

$$|\psi(\mathrm{out})\rangle_{\mathcal{C}_O(g)}=e^{i\eta(t)}UU_{\Sigma}|\{t\}\rangle_{\mathcal{C}_O(g)}.\tag{58}$$

To further determine the relation between the output state and the input state, let us consider another case such that

$$|\psi'(\mathrm{in})\rangle_{I(g)}=|+\rangle_{\mathcal{C}_I(g)}.\tag{59}$$

This time the final state

$$|\{s_i\}\rangle_{\mathcal{C}_I(g)\cup\mathcal{C}_M(g)}|\psi'(\text{out})\rangle_{O(g)}=P^{\{s\}}_{\mathcal{C}_I(g)}(X)|\psi\rangle_{\mathcal{C}(g)}. \quad (60)$$

Let $P^{\{s\}}_{\mathcal{M}_{\mathcal{C}_I(g)}}$ apply on both sides of Eqs. (46) and (47). Then we obtain

$$(UX_iU^\dagger)_{\mathcal{C}_O(g)}|\psi'(\text{out})\rangle_{O(g)}=q_d^{-s_i-\lambda_{x,i}}|\psi'(\text{out})\rangle_{O(g)}, \quad (61)$$

$$(UZ_iU^\dagger)_{\mathcal{C}_O(g)}|\overline{\psi'}(\text{out})\rangle_{O(g)}=q_d^{-\lambda_{z,i}}|\psi'(\text{out})\rangle_{O(g)}, \quad (62)$$

where for $|\overline{\psi'}(\text{out})\rangle$ the input state

$$|\overline{\psi'}(\text{in})\rangle=Z_i^\dagger|+\rangle. \quad (63)$$

From the above equations, it follows that

$$|\psi'(\text{out})\rangle_{O(g)}=e^{i\chi}UU_\Sigma|+\rangle_{\mathcal{C}_O(g)}. \quad (64)$$

Substituting Eq. (58) into Eq. (64), we have

$$|\psi'(\text{out})\rangle_{O(g)}=UU_\Sigma d^{-n/2}\sum_{\{t\}} e^{i\eta(t)}|\{t\}\rangle_{\mathcal{C}_O(g)}. \quad (65)$$

Comparing Eq. (64) with Eq. (65), we finally obtain

$$e^{i\eta(t)}=e^{i\chi}. \quad (66)$$

This completes the proof. ■

This theorem tells us that, since the cluster states have remarkable quantum correlations, they play an essential role in the realization of arbitrary unitary gates. More precisely, as long as one cluster can be used to process a unitary gate for the cluster state, it will work for arbitrary input states. Therefore, it is sufficient to check the conditions for the cluster states, i.e., Eq. (46) and Eq. (47). We would like to emphasize that what is characteristic of the above theorem is that it is expressed in terms of not only the unitary operators X and Z, but also their conjugates. For $d=2$, it exactly reduces to Theorem 1 of Ref. [4].

Before using the theorem to construct a specific unitary gate, we need to explain how to deal with the byproduct part U_Σ. The basic idea is to move U_Σ to the front of U according to the commutation relation between U_Σ and U. To complete this operation, the general strategy is to divide the measurements into several steps such that the subsequent measurements depend on the results of the previous measurements. In the next section, we will use specific examples to demonstrate how to construct all basic elementary gates with the help of this theorem.

V. UNIVERSALITY OF QUDIT CLUSTER QUANTUM COMPUTATION

It is well known that a finite collection of one qubit unitary operations and CNOT gate is enough to construct any unitary transformation in the qubit quantum computing network. This conclusion remains true in some sense for the qudit quantum computing [16]. To be precise, the collection of all one-qudit gates and any imprimitive two-qudit gate is exactly universal for arbitrary qudit quantum computing, where a primitive two-qudit gate means such a gate that maps separate states to separate states. Therefore, in order to prove the universality of quantum computation with qudit clusters, we only need to construct the basic elementary gates, namely, any one-qudit unitary operation and one imprimitive two-qudit unitary transformation, and then integrate them to realize an arbitrary unitary gate. We will do this in this section based on Theorem 2 in Sec. III.

A. Realizations of single-qudit unitary transformations

Let us start with single-qudit gates. First, we introduce a proposition for any unitary transformation in d-dimensional Hilbert space.

Proposition 1. Let $\{N_i,(i\in Z_{d^2-1})\}$ be a Hermitian basis of the operator space for d-dimensional Hilbert space, then any unitary transformation U has the form

$$U=q_d^{\sum_i\alpha_iN_i}=\prod_i q_d^{\beta_iN_i}, \quad (67)$$

where α_i and β_i are real numbers.

The first equality is obvious. For the proof of the second equality please refer to Ref. [15].

Thanks to this proposition, we can divide any qudit gate into a product of more basic ones—single parameter unitary transformations. Now, we need to find $d\times d$ independent N_i to simulate all unitary gates for one qudit. From the definition of qudit cluster states, we expect that the single parameter unitary transformations must have deep relations with the basis elements of QPA. This is indeed true. In fact, along this line we find a good way to introduce $d\times d$ one-parameter unitary transformations. The idea originates from the observation that some of the basis elements of QPA can be used to define a state basis. We find that all the unitary transformations that do not change the basis state up to a phase are defined by the property of multivalued complex functions. For example, for operator Z, we can define

$$Z^\beta(\{m\})=q_d^{\beta N(Z,\{m\})} \quad (\beta\in\mathbb{R}), \quad (68)$$

where

$$N(Z,\{m\})=\sum_{n=0}^{d-1}|n\rangle(n+m_nd)\langle n| \quad (\forall m_n\in\mathbb{Z}). \quad (69)$$

Although the above definitions concern infinite unitary transformations, there are only d independent ones, which can be used to describe the following type of unitary transformations:

$$U_Z(\{\alpha\})|n\rangle=q_d^{\alpha_n}|n\rangle \quad (\forall n\in Z_d,\alpha\in\mathbb{R}). \quad (70)$$

Obviously, these d independent unitary transformations, or the corresponding $N(Z,\{m\})$, can take the place of $\{Z^n,n\in Z_d\}$ in the unitary basis.

A similar argument can be applied to $\overline{Z}$ defined in Sec. II. To be precise, for operator $\overline{Z}$, we define

$$\overline{Z}^{\beta}(\{m\}) = q_d^{\beta N(\overline{Z},\{m\})} \quad (\beta \in \mathbb{R}), \tag{71}$$

where

$$N(\overline{Z},\{m\}) = \sum_{n=0}^{d-1} |n(\overline{Z})\rangle (n+m_n d) \langle n(\overline{Z})| \quad (\forall m_n \in \mathbb{Z}), \tag{72}$$

with

$$\overline{Z}|n(\overline{Z})\rangle = q_d^n |n(\overline{Z})\rangle. \tag{73}$$

In the following, we will show that we can select $d\times d$ independent Hermitian operators from $N(\overline{Z},\{m\})$.

When d is a prime number, a convenient choice is to take $\overline{Z}$ from the operator set

$$\{Z,X,ZX,ZX^2,\ldots,ZX^{d-1}\}. \tag{74}$$

Because each $\overline{Z}$ defines $d-1$ independent $N(\overline{Z},\{m\})$ besides the identity, we obtain d^2 $[=(d-1)(d+1)+1]$ independent Hermitian operators.

When d is not a prime number, we can choose the independent $N(\overline{Z},\{m\})$ by the following procedure. First, we take $\overline{Z}$ from $\{Z, X, ZX\}$, and thus obtain $3(d-1)$ independent $N(\overline{Z},\{m\})$ besides the identity, which can take the place of the set of basis elements

$$S = \{Z^n, X^n, Z^n, X^n, (n \in Z_d)\}. \tag{75}$$

Then we take an element $\overline{Z} \notin S$, find out the elements in $\{\overline{Z}^n, n \in Z_d\}$ that is not in the set S, add these elements to S, and take the new independent $N(\overline{Z},\{m\})$, whose number is the number of new elements in set S. We repeat the above step until $S=B(12)$, then we obtain d^2 independent Hermitian operators.

Now it is clear that if we can do all the above basic unitary transformations $\overline{Z}^{\beta}(\{m\})$, we can claim that we can do all single-qudit unitary transformations. Let us focus on the basic unitary transformations. Our strategy is as follows. We first realize $Z^{\alpha}(\{m\})$ on a five-qudit cluster, as a basic single-qudit transformation, and then associate it with the other single-qudit unitary transformation $\overline{Z}^{\alpha}(\{m\})$. We observe that

$$\overline{Z}^{\alpha}(\{m\}) = U Z^{\alpha}(\{m\}) U^{\dagger}, \tag{76}$$

where U satisfies

$$\overline{Z} = U Z U^{\dagger} \tag{77}$$

and belongs to the Cliford group defined in Sec. II. Equation (77) is a consequence of Eq. (76). In fact, $Z^{\alpha}(\{m\})$ and $\overline{Z}^{\alpha}(\{m\})$ are diagonal in the Z and the $\overline{Z}$ representations, respectively, and they can be expanded as

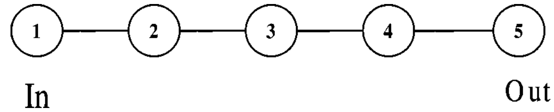

FIG. 1. Five-qudit cluster used in realization of $X\alpha(\{m\})$ and $Z\alpha(\{m\})$. A circle represents one qudit, number n in the circle means the nth qudit, *in* or *out* denote the input or output part of the cluster, and two qudits which are connected by a line are neighbors.

$$Z^{\alpha}(\{m\}) = \sum_{n=0}^{d-1} f(n,m,\alpha) Z^n, \tag{78}$$

$$\overline{Z}^{\alpha}(\{m\}) = \sum_{n=0}^{d-1} f(n,m,\alpha) \overline{Z}^n, \tag{79}$$

where

$$f(n,m,\alpha) = \frac{1}{\sqrt{d}} \mathrm{Tr}[Z^{\dagger d} Z^{\alpha}(\{m\})]. \tag{80}$$

Then the claim follows directly. Consequently, to realize the unitary gate $\overline{Z}^{\alpha}(\{m\})$ we only need to connect the concerned clusters in proper order. Thus, in principle, we can make any single qudit unitary gate in $SU(d)$.

1. Five-qudit cluster realization of $X\alpha(\{m\})$ and $Z\alpha(\{m\})$

In this section we realize the basic single-qudit unitary transformation $Z^{\alpha}(\{m\})$ on a five-qudit cluster designed as in Fig. 1, which is a linear array of five qudits. We also use the same cluster to implement $X^{\alpha}(\{m\})$. As a byproduct, it is shown that the same cluster with different measurement patterns can realize different unitary transformations. The corresponding cluster state is defined by the following system of equations:

$$X_1^{\dagger} Z_2 |\phi\rangle_{\mathcal{C}} = |\phi\rangle_{\mathcal{C}}, \tag{81}$$

$$Z_1 X_2^{\dagger} Z_3 |\phi\rangle_{\mathcal{C}} = |\phi\rangle_{\mathcal{C}}, \tag{82}$$

$$Z_2 X_3^{\dagger} Z_4 |\phi\rangle_{\mathcal{C}} = |\phi\rangle_{\mathcal{C}}, \tag{83}$$

$$Z_3 X_4^{\dagger} Z_5 |\phi\rangle_{\mathcal{C}} = |\phi\rangle_{\mathcal{C}}, \tag{84}$$

$$Z_4 X_5^{\dagger} |\phi\rangle_{\mathcal{C}} = |\phi\rangle_{\mathcal{C}}. \tag{85}$$

It follows from Eqs. (81)–(85) that

$$X_1 X_3^{\dagger} X_5 |\phi\rangle_{\mathcal{C}} = |\phi\rangle_{\mathcal{C}}, \tag{86}$$

$$Z_1^{\dagger} X_2 X_4^{\dagger} Z_5 |\phi\rangle_{\mathcal{C}} = |\phi\rangle_{\mathcal{C}}. \tag{87}$$

From Eq. (85), we obtain

$$Z_4^{\dagger\alpha}(\{m\}) X_5^{\alpha}(\{m\}) |\phi\rangle_{\mathcal{C}} = |\phi\rangle_{\mathcal{C}}. \tag{88}$$

Notice that here we have used the following condition on $\{m\}$. If $n_4+n_5=0\,[\mathrm{Mod}(d)]$, then

$$n_4+m_{n_4}d+n_5+m_{n_5}d=0 \quad (n_4,n_5\in Z_d). \tag{89}$$

From the above four equations, we have

$$X_1X_3^\dagger[X_5^\alpha(\{m\})X_5X_5^{\dagger\alpha}(\{m\})]|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{90}$$

$$Z_1^\dagger X_2[Z_4^{\dagger\alpha}(\{m\})X_4^\dagger Z_4^\alpha(\{m\})] \times[X_5^\alpha(\{m\})Z_5X_5^{\dagger\alpha}(\{m\})]|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}. \tag{91}$$

For the measurement pattern

$$\{X_2,X_3^\dagger,Z_4^{\dagger\alpha}(\{m\})X_4^\dagger Z_4^\alpha(\{m\})\},$$

Theorem 2 says that the simulated unitary transformation is $X_5^\alpha(\{m\})U_\Sigma$, where

$$U_\Sigma=Z_5^{-s_1-s_3}X_5^{s_2+s_4}.$$

Because U_Σ depends on the measurement results and cannot be moved to the front of $X_5^\alpha(\{m\})$ trivially, different measurement results lead to different unitary transformations. In order to realize the gate $X_5^\alpha(\{m\})$, we complete the measurement in two steps. We first measure $\{X_1,X_2,X_3^\dagger\}$. When the outcomes are s_1, λ_x $(=s_3)$, and s_2, the byproduct operator reads $U_\Sigma=Z^{-s_3-s_1}X^{\lambda_z}$. At this time, λ_z is still unknown since it depends on the measurement result s_4. However, as $Z_5^{-s_3-s_1}X_5^\alpha(\{m\})Z_5^{s_3+s_3}$ is diagonal in the X representation, we have

$$Z_5^{-s_3-s_1}X_5^\alpha(\{m\})Z_5^{s_3+s_1}=\prod_{\{m\}}X_5^{\alpha_{\{m\}}^{s_1s_3}}(\{m\}). \tag{92}$$

From Eq. (92), we can obtain d equations of $\{\alpha_{\{m\}}^{s_1s_3}\}$, which determine the values of $\{\alpha_{\{m\}}^{s_1s_3}\}$.

Now we make a new choice depending on the known measurement results. Also from Eqs. (81)–(85), we obtain the following equation instead of Eq. (91):

$$Z_1^\dagger X_2\left(\prod_{\{m\}}Z_4^{\dagger\alpha_{\{m\}}^{s_1s_3}}(\{m\})X_4^\dagger\prod_{\{m\}}Z_4^{\alpha_{\{m\}}^{s_1s_3}}(\{m\})\right) \times\left(\prod_{\{m\}}X_5^{\alpha_{\{m\}}^{s_1s_3}}(\{m\})Z_5\prod_{\{m\}}X_5^{\dagger\alpha_{\{m\}}^{s_1s_3}}(\{m\})\right)|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}. \tag{93}$$

Measuring the fourth qudit relative to the basis

$$\prod_{\{m\}}Z_4^{\dagger\alpha_{\{m\}}^{s_1s_2}}(\{m\})X_4^\dagger\prod_{\{m\}}Z_4^{\alpha_{\{m\}}^{s_1s_3}}(\{m\}),$$

we obtain the value s_4 and $\lambda_z=s_2+s_4$. According to Theorem 2, we obtain the final operation

$$UU_\Sigma=Z_5^{-s_3-s_1}X_5^{s_2+s_4}X_5^\alpha(\{m\}). \tag{94}$$

Finally, by measuring Z_5 we obtain the correct result

$$s=s_5+s_2+s_4. \tag{95}$$

The above equation does not mean that the final result depends only on the measurements on the second, fourth, and fifth qudits. The reason is that different values of s_1 and s_3 correspond to different measurements on the fourth qudit.

Based on the same cluster, we can also implement the single-qudit rotation $Z^\alpha(\{m\})$. Similarly, we first measure $\{X_1,X_2,X_4^\dagger\}$; then from Eqs. (81)–(85), we obtain

$$X_1\left(\prod_m Z_3^{\alpha_{\{m\}}^{s_2s_4}}(\{m\})X_3^\dagger\prod_m Z_3^{\dagger\alpha_{\{m\}}^{s_2s_4}}(\{m\})\right) \times\left(\prod_m Z_5^{\alpha_{\{m\}}^{s_2s_4}}(\{m\})X_5\prod_m Z_5^{\dagger\alpha_{\{m\}}^{s_2s_4}}(\{m\})\right)|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{96}$$

$$Z_1^\dagger X_2X_4^\dagger Z_5|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{97}$$

where $\{\alpha_{\{m\}}^{s_2s_4}\}$ is determined by

$$X_5^{s_2+s_4}Z_5^\alpha(\{m\})X_5^{-s_2-s_4}=\prod_{\{m\}}Z_5^{\alpha_{\{m\}}^{s_2s_4}}(\{m\}). \tag{98}$$

At the same time we make another measurement on

$$\prod_m Z_3^{\alpha_{\{m\}}^{s_2s_4}}(\{m\})X_3^\dagger\prod_m Z_3^{\dagger\alpha_{\{m\}}^{s_2s_4}}(\{m\}).$$

According to Theorem 2, we conclude that the simulated unitary transformation is indeed

$$Z^{-s_3-s_1}X^{s_2+s_4}Z_5^\alpha(\{m\}).$$

The correct result and the measurement values are also related by Eq. (95).

2. Realizations of single-qudit elements in Clifford group

As implied in Eq. (77), we only need to realize the single-qudit elements in the Clifford group. It is easy to show that all elements in the Clifford group are not required. In fact, we only need to treat the elements defined as

$$U^{mn}ZU^{mn\dagger}=\overline{Z}, \tag{99}$$

where

$$\overline{Z}=q_d^{[-(d-1)/2]mn}Z^mX^n, \tag{100}$$

with

$$(m,n)=1. \tag{101}$$

We will show that we can do all the above Clifford unitary transformations through a series of four basic types of unitary transformations. The first is defined as

$$U^{1n}ZU^{1n\dagger}=q_d^{[-(d-1)/2]n}ZX^n, \tag{102}$$

$$U^{1n}XU^{1n\dagger}=X. \tag{103}$$

The second is defined as

$$U^{n1}ZU^{n1\dagger}=q_d^{[-(d-1)/2]n}Z^nX, \tag{104}$$

$$U^{n1}XU^{n1\dagger}=Z^{\dagger}. \tag{105}$$

The third is defined as

$$VZV^{\dagger}=q_d^{-(d-1)/2}ZX, \tag{106}$$

$$VXV^{\dagger}=X. \tag{107}$$

The last is defined as

$$WZW^{\dagger}=Z, \tag{108}$$

$$WXW^{\dagger}=q_d^{-(d-1)/2}ZX. \tag{109}$$

Theorem 3. Any unitary transformation U^{mn} partially defined by Eq. (99) can be factorized into a product of a series of the above four basic unitary transformations.

Proof. Let us prove it by induction. We denote $N_f \equiv \{1,2,...,f\}$. When $m=1$ or $n=1$, U^{mn} is the first or the second types of unitary transformation. Suppose that the above theorem is valid at $m \in N_f$ or $n \in N_f$, i.e., we can do

$$U^{mn} \quad [m \text{ or } n \in N_f \quad (m,n)=1]. \tag{110}$$

For an arbitrary positive integer n there exist $i \in Z_{\infty}$ and $n' \in N_f$ such that

$$n=i(f+1)+n'. \tag{111}$$

If $(f+1,n)=1$, then

$$(f+1,n')=1. \tag{112}$$

By the induction hypothesis, we can do $U^{(f+1)n'}$. Applying V i times, we then obtain

$$V^{i}U^{(f+1)n'}=U^{(f+1)n}. \tag{113}$$

Similarly, we have

$$W^{i}U^{n'(f+1)}=U^{n(f+1)}. \tag{114}$$

Therefore the theorem is valid for m or $n \in N_{f+1}$. This completes the proof. ■

Now we are in a position to construct the four basic unitary transformations. We will prove that the first (including the third) and the fourth can be realized on the five qudit cluster as shown in Fig. 1. From Eqs. (81)–(85), we have

$$X_1X_3^{\dagger}X_5|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{115}$$

$$Z_1^{\dagger}X_2(Z_4^{n}X_4)^{\dagger}Z_5X_5^{n}|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}. \tag{116}$$

According to Theorem 2, when the measurement pattern is $\{X_2,X_3^{\dagger},(q_d^{[-(d-1)/2]n}Z_4^{n}X_4)^{\dagger}\}$, the corresponding unitary transformation is

$$q_d^{[(d-1)/2](s_1+s_3)n}Z^{-s_1-s_3}X^{s_2+s_4-n(s_1+s_3)}U^{1n}. \tag{117}$$

Also from Eqs. (81)–(85), we obtain

$$X_1Z_3X_3^{\dagger}X_4^{\dagger}Z_5X_5|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{118}$$

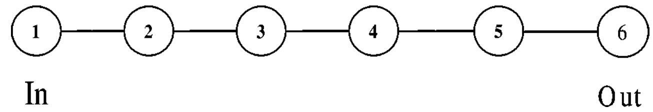

FIG. 2. Six-qudit cluster used in realization of U^{n1}. The meanings of the symbols in this figure are the same as in Fig. 1.

$$Z_1^{\dagger}X_2X_4^{\dagger}Z_5|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}. \tag{119}$$

According to Theorem 2, when the measurement pattern is $\{X_2,q_d^{[(d-1)/2]n}Z_3X_3^{\dagger},X_4^{\dagger}\}$, the corresponding unitary transformation is

$$q_d^{[(d-1)/2](s_2+s_4)}Z^{s_2-s_1-s_3}X^{s_2+s_4}W. \tag{120}$$

To realize the unitary transformation U^{n1}, we need a cluster composed of six qudits as shown in Fig. 2. The cluster state is defined by the following system of equations:

$$X_1^{\dagger}Z_2|\phi_{\mathcal{C}}\rangle=|\phi\rangle_{\mathcal{C}}, \tag{121}$$

$$Z_1X_2^{\dagger}Z_3|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{122}$$

$$Z_2X_3^{\dagger}Z_4|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{123}$$

$$Z_3X_4^{\dagger}Z_5|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{124}$$

$$Z_4X_5^{\dagger}Z_6|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{125}$$

$$Z_5X_6^{\dagger}|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}. \tag{126}$$

It follows from the above equations that

$$X_1X_3^{\dagger}X_5Z_6^{\dagger}|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{127}$$

$$Z_1^{\dagger}X_2Z_4^{n}X_4^{\dagger}X_5^{\dagger n}Z_6^{n}X_6|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}. \tag{128}$$

When the measurement pattern is $\{X_2,X_3^{\dagger},q_d^{[(d-1)/2]n}Z_4^{n}X_4^{\dagger},X_5\}$, the corresponding unitary transformation is

$$q_d^{(s_1+s_3)(s_2+s_4-ns_5+[(d-1)/2]n)}Z^{n(s_5-s_1-s_3)-s_2-s_4}X^{-s_1-s_3}U^{n1}. \tag{129}$$

B. Realization of an imprimitive two-qudit gate

Now we come to the construction of the qudit cluster to simulate two-qudit operations. The cluster composed of six qudits as shown in Fig. 3 is considered with the following system of equations:

$$X_1^{\dagger}Z_3|\phi_{\mathcal{C}}\rangle=|\phi\rangle_{\mathcal{C}}, \tag{130}$$

$$X_2^{\dagger}Z_4|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{131}$$

$$Z_1X_3^{\dagger}Z_4Z_5|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{132}$$

$$Z_2Z_3X_4^{\dagger}Z_6|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{133}$$

$$Z_3X_5^{\dagger}|\phi\rangle_{\mathcal{C}}=|\phi\rangle_{\mathcal{C}}, \tag{134}$$

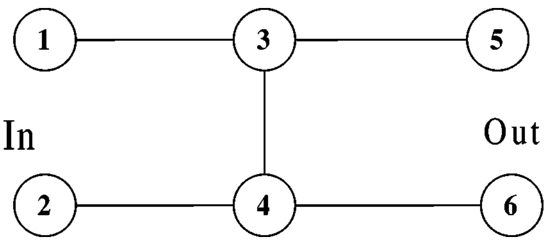

FIG. 3. Six-qudit cluster used in the realization of an imprimitive two-qudit gate T. The meanings of the symbols in this figure are the same as in Fig. 1.

$$Z_4X_6^{\dagger}|\phi\rangle_C=|\phi\rangle_C. \quad (135)$$

It follows from the above equations that

$$X_1X_5^{\dagger}|\phi\rangle_C=|\phi\rangle_C, \quad (136)$$

$$X_2X_6^{\dagger}|\phi\rangle_C=|\phi\rangle_C, \quad (137)$$

$$Z_1^{\dagger}X_3Z_5^{\dagger}X_6^{\dagger}|\phi\rangle_C=|\phi\rangle_C, \quad (138)$$

$$Z_2^{\dagger}X_4Z_6^{\dagger}X_5^{\dagger}|\phi\rangle_C=|\phi\rangle_C. \quad (139)$$

By measuring the system according to the measurement pattern $\{X_1X_2X_3X_4Z_5Z_6\}$, the simulated two-qudit gate T satisfies, and is also defined by,

$$TX_5T^{\dagger}=X_5^{\dagger}, \quad (140)$$

$$TX_6T^{\dagger}=X_6^{\dagger}, \quad (141)$$

$$TZ_5T^{\dagger}=Z_5^{\dagger}X_6^{\dagger}, \quad (142)$$

$$TZ_6T^{\dagger}=Z_6^{\dagger}X_5^{\dagger}. \quad (143)$$

According to Theorem 2, the above measurement pattern realizes the following unitary gate:

$$TZ_5^{-s_1}X_5^{s_3}Z_6^{-s_2}X_6^{s_4}=q_d^{s_1s_2}Z_5^{s_1}X_5^{s_2-s_3}Z_6^{s_2}X_6^{s_1-s_4}T. \quad (144)$$

The next task is to prove that T is an imprimitive two-qudit operation. Reference [16] tells us that a two gate V is primitive if and only if $V=S_1\otimes S_2$ or $V=(S_1\otimes S_2)P$. Here, S_1 and S_2 are different single-qudit operators, P is the interchanging operator obeying $P|x\rangle\otimes|y\rangle=|y\rangle\otimes|x\rangle$. Based on this fact, we can easily conclude that a primitive operator always maps a single-qudit operator to another single-qudit operator. Obviously, the above two-qudit operator T is imprimitive. Another way to prove it is to evaluate the unitary transformation directly. Then we can find that it maps all $Z\otimes Z$—bases to the maximally entangled states.

As demonstrated in this section, any single-qudit unitary gate and one imprimitive two-qudit gate can be realized on qudit clusters. Therefore, the measurement-based quantum computing on qudit clusters is universal.

VI. CONCLUSIONS

We have introduced the concept of a qudit cluster state in terms of finite-dimensional representations of QPA. Based on these qudit cluster states, we have built all the elements of qudit clusters needed for implementation of universal measurement-based quantum computations. With generalizations of cluster states and measurement patterns, most of the results in qubit cluster can work well for qudit clusters in parallel ways. We also show that there still exists the celebrated theorem guaranteeing the availability of qudit cluster states for quantum computations. To prove the universality of this quantum computation, we show that we can implement all single-qudit unitary transformations and one imprimitive two-qudit gate on specific qudit clusters. In addition, we point out that the high-dimensional ''Ising'' model can be used to generate the concerned cluster states dynamically in building a one-way universal quantum computer with qudit cluster states.

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