Quasispin model for macroscopic quantum tunneling between two coupled Bose-Einstein condensates

Chaohong Lee*
Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, Wuhan 430071, People's Republic of China and Max Planck Institute for the Physics of Complex Systems, Noethnitzer Straße 38, D-01187 Dresden, Germany

Wenhua Hai
Department of Physics, Hunan Normal University, Changsha 410081, People's Republic of China

Xueli Luo, Lei Shi, and Kelin Gao†
Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, Wuhan 430071, People's Republic of China
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The system of two coupled Bose-Einstein condensates is mapped onto a uniaxial spin with an applied magnetic field. The mean-field interaction, the coupling, and the asymmetry or the detuning correspond to the anisotropy, the transverse field, and the longitudinal field, respectively. A generalized Bloch equation is derived. In the low barrier limit for the quasispin model, the tunneling rate is analyzed with an imaginary-time path-integral method. The dependence of the tunneling rate on the system parameters is obtained. The crossover temperature T_c from the thermal regime to the quantum regime is estimated. Below T_c quantum tunneling prevails, otherwise thermal activation dominates.

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I. INTRODUCTION

The experimental realization of measuring the relative phase and the population oscillation between coupled Bose-Einstein condensates (BECs) stimulates great interest in investigating their macroscopic quantum tunneling dynamics [1–3]. There are two different types of atomic tunneling between coupled BECs, external tunneling and internal tunneling [2,3]. The former has different spatially separated single-particle states in a double-well or multiwell potential and the latter has different hyperfine internal states in a single-well potential. For external tunneling, the phase interference between BECs confined in a multiwell potential has been observed [4,5]; the experimental observation of the tunneling among BECs confined in multiwell potential has also been reported [6–8]. For internal tunneling, JILA realized a two-component BEC coupled with Raman pulses [9], MIT observed the tunneling across spin domains in BECs [10,11], and LENS reported the current-phase dynamics in two weakly coupled BECs trapped in different Zeeman states [12].

With the proceeding of the experimental exploration, a lot of theoretical investigation was performed simultaneously. Williams *et al.* demonstrated the existence of Josephson tunneling in a driven two-state single-particle BEC in a single-well trap potential [13]. Kasamatsu *et al.* investigated theoretically the existence of a metastable state and the possibility of decay to the ground state through macroscopic quantum tunneling in two-component BECs with repulsive interactions [14]. Smerzi *et al.* studied the coherent atomic tunneling and population oscillations between two zero-temperature BECs confined in a double-well potential [15–18]. Macroscopic quantum self-trapping (MQST), namely, a self-maintained population imbalance with nonzero average value of the fractional population imbalance, and π-phase oscillations in which the time-averaged value of the phase difference is equal to π were detailed in Refs. [15,16]. The authors of Ref. [17] claim that interaction with a thermal cloud will damp all different oscillations to the zero-phase mode. In addition, macroscopic quantum fluctuations have also been discussed by using second-quantization approaches [18,19]. Within the time-dependent potential, chaotic population tunneling emerges. Abdullaev and Kraenkel analyzed the nonlinear resonances and chaotic oscillations of the fractional population imbalance between two coupled BECs in a double-well trap with a time-dependent tunneling amplitude for different dampings [20]. They also considered the chaotic atomic population resonances and the possibility of stabilization of the unstable-mode regime in coupled BECs with oscillating atomic scattering length [21]. In a previous paper, we investigated the chaotic and frequency-locked population oscillation between two coupled BECs [22].

Although many papers appear in the field of the tunneling between coupled BECs, because of the nonlinearity in the Gross-Pitaevskii equation (GPE), few of them address the question of calculating the tunneling rate and the crossover temperature between different tunneling regimes. However, the tunneling rate and the crossover temperature of the spin systems have been studied systematically with the imaginary-time path-integral method, including models with applied magnetic field [24–30] and without it [31–33]. For a two-state system described with linear Schrödinger equation, it is easy to visualize the effects of coupling between two states by introducing Bloch's spin vector formalism [23].

*Corresponding author. Electronic addresses: chlee@mpipks-dresden.mpg.de; chleecn@hotmail.com
†Electronic address: klgao@wipm.ac.cn

Can we introduce a generalized Bloch vector for two coupled BECs described with the nonlinear Schrödinger equation to map it onto a spin system, and then calculate the tunneling rate and the crossover temperature with the imaginary-time path-integral method? If the coupled BECs system is equivalent to a spin system, the tunneling process is related to the decay of the metastable MQST state to the ground state. More interestingly, the crossover temperature corresponds to the transition from the classical or mean-field regime to the second quantization regime. In the following section, by introducing a generalized Bloch spin vector, the coupled BECs are mapped onto a uniaxial spin with an applied magnetic field. In Sec. III, the tunneling rate is calculated with the imaginary-time path-integral method, and the crossover temperature is estimated. In the last section, a brief discussion and summary is given.

II. QUASISPIN MODEL FOR TWO COUPLED BOSE-EINSTEIN CONDENSATES

Consider the experiments of JILA [9], two Bose-Einstein condensates in the $|F=1,m_F=-1\rangle=|1\rangle$ and $|F=2,m_F=1\rangle=|2\rangle$ spin states of $^{87}\mathrm{R}_b$ are coupled by a two-photon pulse with the two-photon Rabi frequency Ω and a finite detuning $\delta=\omega_d-\omega_{hf}$. Here, $\omega_d=\omega_1+\omega_2$ is the driven frequency of the two-photon pulses and ω_{hf} is the transition frequency between two hyperfine states. In the rotating frame, ignoring the damping and the finite-temperature effects, the coupled two-component BEC system can be described by a pair of coupled GPEs,

$$i\hbar\frac{\partial\Psi_2(r,t)}{\partial t}=\left(H_2^0+H_2^{MF}-\frac{\hbar\delta}{2}\right)\Psi_2(\vec{r},t)+\frac{\hbar\Omega}{2}\Psi_1(\vec{r},t),$$

$$i\hbar\frac{\partial\Psi_1(\vec{r},t)}{\partial t}=\left(H_1^0+H_1^{MF}+\frac{\hbar\delta}{2}\right)\Psi_1(\vec{r},t)+\frac{\hbar\Omega}{2}\Psi_2(\vec{r},t), \tag{1}$$

where the free evolution Hamiltonians $H_i^0=-\hbar^2\nabla^2/2m+V_i(\vec{r})$ $(i=1,2)$ and the mean-field interaction Hamiltonians $H_i^{MF}=(4\pi\hbar^2/m)(a_{ii}|\Psi_i[\vec{r},t)|^2+a_{ij}|\Psi_j(\vec{r},t)|^2]$ $(i,j=1,2,\ i\neq j)$. The coefficient a_{ij} is the scattering length between states i and j and it satisfies $a_{ij}=a_{ji}$. Weak coupling is defined by the Rabi frequency satisfying $\Omega/(\omega_x\omega_y\omega_z)^{1/3}=\Omega/\bar{\omega}\ll 1$, where $\bar{\omega}=(\omega_x\omega_y\omega_z)^{1/3}$ is the geometric-averaged angular frequency for the trapping potential. In this regime, we can write the macroscopic wave functions using the variational ansatz $\Psi_i(\vec{r},t)=\psi_i(t)\Phi_i(\vec{r})$ with $\psi_i(t)=\sqrt{N_i(t)}e^{i\alpha_i(t)}$ $(i=1,2)$. In the ansatz, the functions $\Phi_i(\vec{r})$ describe the spatial distribution of the i-th component, and the complex coefficient functions $\psi_i(t)$ are spatially uniform and contain all time dependence in the macroscopic quantum wave functions $\Psi_i(\vec{r},t)$. The symbols $N_i(t)$ and $\alpha_i(t)$ represent the populations and phases of the i-th condensate, respectively. Because the coupling is very weak, the spatial distributions vary slowly in time and are very close to the adiabatic solutions to the time-independent uncoupled case for GP equations (1), being slaved by the populations [13]. Thus, the complex coefficient functions $\psi_i(t)$ obey the nonlinear two-mode dynamical equations

$$i\hbar\frac{d}{dt}\psi_2(t)=\left[E_2^0-\frac{\hbar\delta}{2}+U_{22}|\psi_2(t)|^2+U_{21}|\psi_1(t)|^2\right]\psi_2(t)+\frac{K}{2}\psi_1(t),$$

$$i\hbar\frac{d}{dt}\psi_1(t)=\left[E_1^0+\frac{\hbar\delta}{2}+U_{11}|\psi_1(t)|^2+U_{12}|\psi_2(t)|^2\right]\psi_1(t)+\frac{K}{2}\psi_2(t). \tag{2}$$

The parameters satisfy $E_i^0=\int\Phi_i(\vec{r})H_i^0\Phi_i(\vec{r})d\vec{r}$, $U_{ij}=(4\pi\hbar^2a_{ij}/m)\int|\Phi_i(\vec{r})|^2|\Phi_j(\vec{r})|^2d\vec{r}=U_{ji}$, and $K=\hbar\Omega\int\Phi_1(\vec{r})\Phi_2(\vec{r})d\vec{r}$ $(i,j=1,2)$. The terms in K describe population transfer (internal tunneling) between two BEC states, whereas the terms in U_{ij}, which depend on the numbers of atoms in each BEC state, describe the mean-field interaction between atoms. When U_{21} and δ equal zero, these coupled equations can also describe the BECs in a double-well potential [15–18]. Similar to the coupled two-state system obeying the linear Schrödinger equation, we introduce a generalized Bloch spin vector (u,v,w) with the components

$$u=\psi_2^*\psi_1+\psi_2\psi_1^*,\quad v=-i(\psi_2\psi_1^*-\psi_2^*\psi_1),$$

$$w=\psi_2^*\psi_2-\psi_1^*\psi_1. \tag{3}$$

Obviously, $u^2+v^2+w^2=(N_1+N_2)^2=N_T^2$ is a conserved quantity when finite-temperature and damping effects can be ignored. Rescaling the time $t/\hbar$ to t, the Bloch spin vector satisfies

$$\frac{du}{dt}=v(\gamma+\eta w),\quad \frac{dv}{dt}=Kw-u(\gamma+\eta w),\quad \frac{dw}{dt}=-Kv, \tag{4}$$

where $\gamma=E_2^0-E_1^0+N_T(U_{22}-U_{11})/2-\hbar\delta$ and $\eta=(U_{22}+U_{11}-2U_{12})/2$. Regarding the atom in one condensate as spin-up state and the atom in the other condensate as spin-down state, the coupled BECs can be described with the quasispin $\vec{S}=\vec{u}e_x+\vec{v}e_y+\vec{w}e_z$. In this language, the longitudinal component w depicts the population difference, and the transverse components u and v characterize the coherence. Thus the effective Hamiltonian for the quasispin is

$$E=-\tfrac{1}{2}\eta S_z^2-KS_x-\gamma S_z. \tag{5}$$

The above Hamiltonian is similar to that of a uniaxial spin with an applied magnetic field [26–30]; it indicates that the mean-field interaction brings the anisotropy η, the coupling causes an effective transverse magnetic field K along axis x, and the asymmetry or the detuning induces an effective longitudinal magnetic field γ. In the symmetric case ($E_2^0 = E_1^0$, $U_{22} = U_{11}$, and $\delta = 0$), it is consistent with the one derived from the second quantized Hamiltonian in Ref. [4].

III. TUNNELING RATE AND CROSSOVER TEMPERATURE

In conventional spherical coordinates, the spin components can be written as $S_x = N_T \sin\theta\cos\phi$, $S_y = N_T \sin\theta \sin\phi$, and $S_z = N_T \cos\theta$ (see Fig. 1). Thus, the corresponding effective Hamiltonian is formulated as

$$E = -\eta N_T^2 \left(\frac{1}{2}\cos^2\theta + \frac{K}{\eta N_T}\sin\theta\cos\phi + \frac{\gamma}{\eta N_T}\cos\theta \right). \tag{6}$$

Based upon the analysis of a spin in a uniaxial magnetic field [26–30], we know that there are stationary states if some angles (θ_0, ϕ_0) satisfy $\partial E/\partial\phi|_{\phi=\phi_0}^{\theta=\theta_0} = 0$ and $\partial E/\partial\theta|_{\phi=\phi_0}^{\theta=\theta_0} = 0$. The condition $\partial E/\partial\phi|_{\phi=\phi_0}^{\theta=\theta_0} = 0$ locates the stationary states in the XOZ plane ($\sin\phi_0 = 0$). The existence of multiple stationary states in this quasispin system is equivalent to the existence of multiple metastable MQST states in the coupled BECs. Near the metastable states the potential describes a "canyon" satisfying

$$E_\theta = E(\theta,\phi_0)/(\eta N_T^2) = -\tfrac{1}{2}\cos^2\theta - P\cos(\theta-\theta_P). \tag{7}$$

The parameters obey $P = \sqrt{K^2+\gamma^2}/|\eta N_T|$, $\sin\theta_P = K\cos\phi_0/\sqrt{K^2+\gamma^2}$, and $\cos\theta_P = \gamma/\sqrt{K^2+\gamma^2}$. As stated in the preceding section, the parameter $K \propto \Omega > 0$, therefore $\sin\theta_P > 0$ and $\sin\theta_P < 0$ correspond to the equal-phase mode ($\phi_0 = 0$) and the antiphase mode ($\phi_0 = \pi$) in the coupled two-component BECs, respectively. In the case of $E_2^0 - E_1^0 + N_T(U_{22} - U_{11})/2 = 0$, the parameter γ is just the negative detuning $-\delta$, thus $\cos\theta_P > 0$ and $\cos\theta_P < 0$ correspond to the red detuning and the blue detuning of the coupling laser, respectively. The $\partial E/\partial\theta|_{\phi=\phi_0}^{\theta=\theta_0} = 0$ is equivalent to $\partial E_\theta/\partial\theta|_{\phi=\phi_0}^{\theta=\theta_0} = 0$, that is, $\sin 2\theta_0 + 2P\sin(\theta_0 - \theta_P) = 0$. For some critical points where both the first and the second derivatives of E_θ equal zero, an appreciable tunneling rate appears. This gives

$$\sin 2\theta_C + 2P_C\sin(\theta_C - \theta_P) = 0,$$

$$\cos 2\theta_C + P_C\cos(\theta_C - \theta_P) = 0, \tag{8}$$

where θ_C and P_C are critical values for θ and P, respectively. Solving the above equations, one can obtain $\tan^3\theta_C = -\tan\theta_P$ and $P_C = (\sin^{2/3}\theta_P + \cos^{2/3}\theta_P)^{-3/2}$. The system has an instanton solution at the critical point $P = P_C$, i.e., $[K/(\eta N_T)]^{2/3} + [\gamma/(\eta N_T)]^{2/3} = 1$. This critical point stands on the separatrix between the single-stable regime and the multiple-stable regime. It separates the metastable multi-MQST behavior between the single-stable population oscillation in the coupled two-component BEC.

According to the dependence of E_θ on θ, we obtain that the condition for the existence of multiple stationary states is $P < P_C$, i.e., $[K/(\eta N_T)]^{2/3} + [\gamma/(\eta N_T)]^{2/3} < 1$. One can easily find the small oscillations around these stationary states with nonzero time-averaged values for S_z and $\sqrt{S_x^2+S_y^2}$. These oscillations correspond to the phase-locked MQST states with time-averaged relative phase 0 or π and multiple stationary states correspond to multiple metastable MQST states with fixed nonzero population difference and relative phase 0 or π. The appearance of multiple stationary states indicates, only for some proper parameters, that multiple metastable MQST states exist. For simplicity we only consider the case where the parameter P is slightly lower than the fixed critical value P_C, $P = P_C(1-\varepsilon)$, $\varepsilon \ll 1$. This requires that the Rabi frequency, the detuning, the scattering lengths, and the total atomic number in the coupled BEC system must cooperate with each other to approach the critical values for the emergence of multiple metastable MQST states. One way to maintain the critical value P_C unchanged is fixing the values of the ratio γ/K and other correlated parameters (η and N_T), that is, keeping the angle θ_P unchanged. By introducing a new positive variable $\xi = \theta - \theta_0$, potential (6) can be expanded into

$$E_\theta(\theta) = E_\theta(\theta_0) + \tfrac{1}{4}[\sqrt{6\varepsilon}\,\xi^2 - \xi^3 + O(\xi^4)]\sin(2\theta_C). \tag{9}$$

With the definition in Refs. [26,27,29,30], the tunneling rate Γ obeys $N_p(t) = N_P(0)\exp(-\Gamma t)$ and it can be written as $\Gamma = A\exp(-B)$ for the quantum tunneling regime. Here, $N_P(t)$ is the population occupying the metastable state at time t and the tunneling exponent B ($\geqslant 0$) is determined by the imaginary time action of the instanton solution. Similar to Ref. [26], the tunneling exponent follows from the path integral $\int D\{\phi(\tau)\}\int D\{\cos\theta(\tau)\}\exp(I/\hbar)$ over the continuum of trajectories which start and end at (θ_0, ϕ_0) and are close to the instanton solution, where τ is the imaginary time it and I is the imaginary time action $I = \int d\tau[iN_T(1-\cos\theta)d\phi/d\tau + E(\theta,\phi)]$. Integrating the imaginary time action by parts, one can gain the tunneling exponent

$$\begin{aligned} B &= N_T\int_{-\infty}^{+\infty} d\tau \left\{ \frac{(d\xi/d\tau)^2 \sin\theta_C}{2P_C\sin\theta_P} \right. \\ &\quad \left. + \frac{1}{4}\sin(2\theta_C)[\sqrt{6\varepsilon}\,\xi^2 - \xi^3 + O(\xi^4)] \right\}, \\ &= 16\times 6^{1/4} N_T \varepsilon^{5/4} |\cot\theta_P|^{1/6}/5 \\ &= 16\times 6^{1/4} N_T \varepsilon^{5/4} |\gamma/K|^{1/6}/5. \end{aligned} \tag{10}$$

From the definition of ε, one can obtain

$$\begin{aligned} \varepsilon &= 1 - P/P_C = 1 - (1 + |\gamma/K|^2) \\ &\quad \times (1 + |\gamma/K|^{2/3})^{-3/2} |K/(\eta N_T)|. \end{aligned} \tag{11}$$

Thus the tunneling exponent can be expressed as

$$\begin{aligned} B &= 16\times 6^{1/4} N_T [1 - (1 + |\gamma/K|^2) \\ &\quad \times (1 + |\gamma/K|^{2/3})^{-3/2} |K/(\eta N_T)|]^{5/4} |\gamma/K|^{1/6}/5. \end{aligned} \tag{12}$$

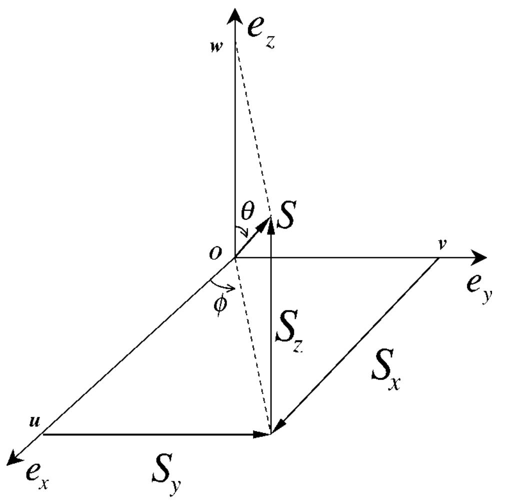

FIG. 1. The quasispin $\vec{S}$ and its components (u,v,w) in conventional spherical coordinates.

To control the tunneling, one has to select proper values for parameters γ, K, and η. In the experiments performed in a double-well potential [4,5], it can be realized by modifying the barrier position, the barrier height, and the magnetic field (using Feshbach resonances to adjust the scattering lengths [35]), respectively. In the experiments with two-component BECs in a single-well potential [9] , it can be realized by adjusting the laser detuning, the laser intensity, and the magnetic field, respectively. For fixed values of η and γ/K, the tunneling exponent B decreases with the increasing of the intensity of the coupling laser. In Fig. 2, we show how the tunneling exponent B depends on the angle θ_P. In the region between 0 and π, the ratio $B(\theta_P)/B(\pi/4)$ decreases from positive infinity to zero when the angle θ_P equals $\pi/2$, which corresponds to the symmetric case $(\gamma=0)$, and then increases to positive infinity when the angle θ_P is close to π. It is almost flat when the angle θ_P is not close to 0, $\pi/2$, and π. This angular dependence indicates, in the case of fixed value of ε, that the tunneling exponent increases with increasing $|\gamma/K|$.

The result for the angle θ_P close to $\pi/2$, which corresponds to the symmetric case $\gamma=0$, should be taken with great caution because the coefficient $\sin(2\theta_C)$ in the Taylor expansion series (9) is equal to zero. In this case, the problem corresponds to the tunneling between two equivalent minima which correspond to the angle θ_P equal to 0 and π. Thus the potential can be expanded into the form of $\xi^2-\xi^4$ and the tunneling exponent B is expressed as $B=4S\varepsilon^{3/2}=4N_T\varepsilon^{3/2}$. Therefore, the tunneling exponents (10) and (12) only hold for the asymmetric case where $\gamma\neq 0$.

To confirm our prediction from the quasispin model, we perform a numerical simulation of Eq. (2). A qualitative change in the stationary-state behavior occurs at $|K/(\eta N_T)|=1$. When $|K/(\eta N_T)|>1$, there are no metastable states for any effective detuning γ. However, when $|K/(\eta N_T)|<1$, metastable states exist in the region $[-\gamma_c,+\gamma_c]$ for proper relative phase, where γ_c satisfies $[K/(\eta N_T)]^{2/3}+[\gamma_c/(\eta N_T)]^{2/3}=1$. See the left column of Fig. 3. Two stationary states, indicated as S_1 and S_2 in the figure, are stable and the other one (U) is unstable. Adiabatically changing the effective detuning γ from $\gamma_c-\varepsilon$ to $\gamma_c+\varepsilon$ (ε is a very small positive number), in the space of the fractional population difference $z=(N_2-N_1)/N_T$ and the relative phase $\phi=\alpha_2-\alpha_1$, a trajectory in the vicinity of S_2 becomes a large orbit C encircling S_1. From the views of instanton method, the tunneling exponent is determined by the canonical action of the orbit, i.e., B follows from the path integral $\int D\{z(\tau)\}\int D\{\phi(\tau)\}\exp(I_c/\hbar)$ over the continuum of trajectories which are close to the instanton solution. At different bifurcation points γ_c, the numerical results show $B(|\gamma/K|)/B(|\gamma/K|=1)\propto|\gamma/K|^{0.163\pm 0.002}\approx|\gamma/K|^{1/6}$, this confirms our previous prediction from the quasispin model (see the right column of Fig. 3).

There are two important aspects which must be noted. The one is that these results for tunneling are only valid in the low barrier limit for the quasispin model, i.e., $\varepsilon\ll 1$. This means that the above results only hold in the region which approaches the critical point of emergence of multiple metastable MQST states. The parametric dependence of the general case is still an open problem. The other is the validity of the Wentzel-Kramers-Brillouin (WKB) semiclassical approximation. The semiclassical approach can only be used in the case of small tunneling probability, that is, $B\gg 1$. In this low barrier limit, from the Taylor expansion series (9) one can obtain the following tunneling amplitude by using the theory developed by Caldeira and Leggett [34]:

$$
\begin{aligned}
A&=(15B/8\pi)^{1/2}\omega,\\
&=\eta N_T\left(\frac{15B}{2\pi}\right)^{1/2}\left(\frac{3\varepsilon}{8}\right)^{1/4}|\cot\theta_P|^{1/6}/(1+\cot^{2/3}\theta_P), \qquad (13)\\
&=\eta N_T\left(\frac{15B}{2\pi}\right)^{1/2}\left(\frac{3\varepsilon}{8}\right)^{1/4}|\gamma/K|^{1/6}/(1+|\gamma/K|^{2/3}).
\end{aligned}
$$

Here, ω is the angular frequency of small oscillations near the bottom of the inverse potential. Apparently, when the angle θ_P is close to the $k\pi$ $(k=0,1)$, which corresponds to small Rabi frequency or large detuning of the coupling laser between two BECs, the tunneling amplitude A approaches to zero, see Fig. 4. As presented above, in the case of θ_P close to $\pi/2$ which corresponds to the symmetric case $\gamma=0$, the potential is not in form of $\xi^2-\xi^3$ but in form of $\xi^2-\xi^4$ because the coefficient $\sin(2\theta_C)$ in the Taylor expansion series (9) equals zero. Therefore, the above formula for the tunneling amplitude only holds for the asymmetric case $\gamma\neq 0$. Generally, contrary to the tunneling exponent B, the tunneling amplitude A is sensitive to the structure of quantum levels in the potential. Therefore, for the case of the full potential (6) and (7), the estimation of A is still an open problem.

Population transfer between two states in a bistable system can occur either due to classical thermal activation which depends on the system temperature or due to quantum tunneling which does not depend on the system temperature. There exists a phase transition from the thermal regime to the quantum regime which occurs at the crossover tempera-

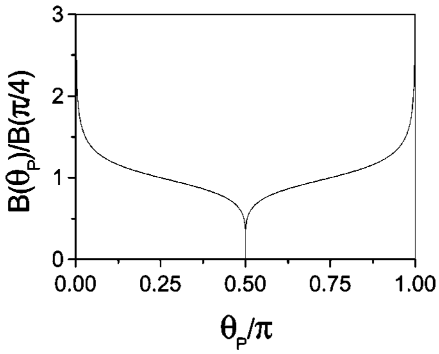

FIG. 2. The tunneling exponent ratio $B(\theta_P)/B(\theta_P=\pi/4)$ vs different θ_P. Here, the angle θ_P characterizes the angle between the effective magnetic field $\vec{B}_{eff}=K\vec{e}_x+\gamma\vec{e}_z$ and the axis z.

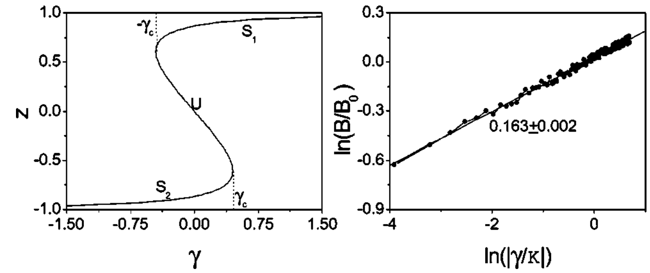

FIG. 3. In the left column, the stationary states for $|K|<|\eta N_T|$ are shown. There are two metastable states S_1, S_2 and one unstable state U. In the right column, the tunneling exponent ratio B/B_0 vs different $|\gamma/K|$ is presented, where $B_0=B(|\gamma/K|=1)$. The black dots show the numerical data and the straight line represents the linear fit for the logarithmic data.

ture T_C. Above T_C, quantum effects are very small and the population transfer rate follows the Arrhenius law,

$$\Gamma_{thermal}=\Gamma_0\exp\left(-\frac{U_B}{k_BT}\right). \tag{14}$$

Here, U_B is the height of the energy barrier between two states and k_B is the Boltzmann constant. Below T_C, the population transfer is purely quantum,

$$\Gamma_{quantum}=A\exp(-B), \tag{15}$$

with B independent of the system temperature. Thus the transition occurs when $\Gamma_{thermal}=\Gamma_{quantum}$. Neglecting the prefactors and equating the exponents, the crossover temperature can be estimated as

$$T_C=U_B/(k_BB). \tag{16}$$

The transition region is approximately the temperature interval $[T_C(1-B^{-1}),T_C(1+B^{-1})]$. This crossover resembles a first-order phase transition of the tunneling rate Γ because it is accompanied with the discontinuity of $d\Gamma/dT$ at T_C [29].

There is another regime for tunneling, the thermally assisted tunneling (TAT), in which the particle strides over the barrier to the bottom of the potential with lowering temperature [29,30]. The transition from the classical regime to the TAT regime resembles a second-order classical-quantum phase transition of the tunneling rate Γ because it is accompanied with a discontinuity of $d^2\Gamma/dT^2$ and no discontinuity of $d\Gamma/dT$ at the crossover temperature. The corresponding transition temperature can be estimated as

$$T_C'=\hbar/(\tau_0k_B)=\hbar\omega/(2\pi k_B), \tag{17}$$

where τ_0 and ω are the period and the angular frequency of small oscillations near the bottom of the inverse potential, respectively [25–27,29,30]. In the low barrier limit ($\varepsilon\gg1$), from the Taylor expansion series (9) one can obtain the barrier height

$$U_B=\eta N_T^2(2\varepsilon/3)^{3/2}|\sin(2\theta_C)|=\frac{5\pi}{18}(\hbar/\tau_0)B, \tag{18}$$

where

$$|\sin(2\theta_C)|=2|\gamma/K|_C/(1+|\gamma/K|_C^2). \tag{19}$$

Comparing both crossover temperatures, one can easily find that they differ by a factor $T_C/T_C'=5\pi/18=1/1.15$, which means that they are of the same order of magnitude and can both be used to estimate the crossover temperature.

Below, from the experimental parameters in the experiments of JILA [9], we will give a quantitative estimation for the tunneling rate and the crossover temperature. In those experiments, the atomic mass $m_1=m_2=m_{Rb}=1.45\times10^{-25}$ kg, the time-averaged orbiting potential (TOP) magnetic trap has an axial frequency $v_z=59$ Hz and a radial frequency $v_{x,y}=v_r=v_z/\sqrt{8}=21$ Hz, the s-wave scatter lengths $a_{11}=5.36$ nm, $a_{12}=a_{21}=5.53$ nm, and $a_{22}=5.70$ nm, and the total atomic number $N_T\approx5\times10^5$. To obtain the numerical values conveniently, we choose the natural units of the problem, in which, time is in units of $1/(\omega_x\omega_y\omega_z)^{1/3}=1/\bar{\omega}$, length is in units of the size of the geometric-averaged harmonic-oscillator length $\bar{d}=\sqrt{\hbar/[(\omega_x\omega_y\omega_z)^{1/3}m_{Rb}]}=\sqrt{\hbar/(\bar{\omega}m_{Rb})}$, energy is in units of the geometric-averaged trap level spacing $\hbar(\omega_x\omega_y\omega_z)^{1/3}=\hbar\bar{\omega}$, and mass is in units of Rb atomic mass m_{Rb}.

Due to gravity acting besides the TOP, the centers of two condensates will displace along the vertical direction and the two equilibrium displacements are generally not the same. Thus, if the interparticle interaction is absent, the lowest single-particle state has the familiar wave function

$$\Phi_{0i}(\vec{r})=\frac{1}{\pi^{3/4}(d_xd_yd_z)^{1/2}}\times\exp\left(-\frac{x^2}{2d_x^2}-\frac{y^2}{2d_y^2}-\frac{(z-\mathcal{F}_iz_0)^2}{2d_z^2}\right). \tag{20}$$

Here, $\mathcal{F}_1=+1$, $\mathcal{F}_2=-1$, $2z_0$ is the offset between two potential centers along the vertical axis, and $d_k=\sqrt{\hbar/(\omega_km_{Rb})}$ ($k=x,y,z$) are the oscillator lengths. The offset $2z_0$ between two condensates can be varied by adjusting the magnitude of the rotating magnetic field. In the presence of interatomic

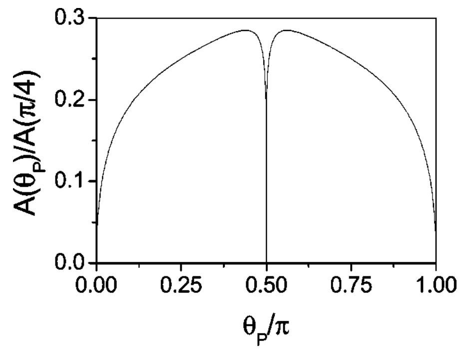

FIG. 4. The tunneling amplitude ratio $A(\theta_P)/A(\theta_P=\pi/4)$ vs different θ_P, where the angle θ_P characterizes the angle between the effective magnetic field $\vec{B}_{eff}=K\vec{e}_x+\gamma\vec{e}_z$ and the axis z.

interaction, the dimensions of the condensates are changed. The spatial parts of the macroscopic quantum wave functions are in the shape of

$$\Phi_i(\vec{r})=\frac{1}{\pi^{3/4}(b_{ix}b_{iy}b_{iz})^{1/2}} \times \exp\left(-\frac{x^2}{2b_{ix}^2}-\frac{y^2}{2b_{iy}^2}-\frac{(z-\mathcal{F}_i z_0)^2}{2b_{iz}^2}\right). \quad (21)$$

The variational parameters b_{ik} ($k=x,y,z$; $i=1,2$) depend on the scattering length, the total atom number, and the trapping potential and they have almost the same numerical values as d_k. For proper values of the offset $2z_0$, the numerical results of Ref. [13] show that the spatial distributions $\Phi_i(\vec{r})$ and their overlap only weakly depend on the total atom numbers in each condensate. For simplicity, in the following calculations, the variational parameters b_{ik} are replaced by the oscillator lengths d_k. Therefore, the parameters E_i^0, U_{ij}, and K are determined by

$$E_1^0=E_2^0=\hbar(\omega_x+\omega_y+\omega_z)/2,$$

$$U_{ii}=4\pi\hbar^2 a_{ii}/[(\sqrt{2\pi})^3 d_x d_y d_z m_{Rb}] \quad (i=1,2),$$

$$U_{12}=4\pi\hbar^2 a_{12}\exp(-2z_0^2/d_z^2)/[(\sqrt{2\pi})^3 d_x d_y d_z m_{Rb}], \quad (22)$$

$$K=\hbar\Omega\exp(-z_0^2/d_z^2).$$

So the corresponding parameters in the quasispin model (5) can be written as $\gamma=\hbar^2 N_T(a_{22}-a_{11})/(\sqrt{2\pi}d_x d_y d_z m_{Rb})-\hbar\delta$ and $\eta=\hbar^2[a_{22}+a_{11}-2a_{12}\exp(-2z_0^2/d_z^2)]/(\sqrt{2\pi}d_x d_y d_z m_{Rb})$. In the case of complete overlap ($2z_0=0$), the anisotropy parameter η equals zero, thus the metastable multi-MQST behavior will never appear, but some running-phase MQST states may still exist. This indicates that, to ensure the existence of multiple metastable MQST states, a finite offset must be kept between two condensates. Furthermore, the appearance of this kind of MQST requires $K^{2/3}+\gamma^{2/3}<(\eta N_T)^{2/3}$. Because $K\propto\Omega$ and $\gamma\propto\delta$, this inequality indicates that the Rabi frequency and the detuning of the coupling pulses must be relatively small. Choosing the total atom number $N_T=2.0\times10^4$, the half offset $z_0=0.20d_z$, the Rabi frequency $\Omega=2\pi\times10$ Hz, and the detuning $\delta=-179$ Hz, one can get $\eta N_T=6.70\times10^{-32}$, $\gamma=4.57\times10^{-32}$, $K=6.90\times10^{-33}$, and $\varepsilon=9.78\times10^{-3}$. Thus, the corresponding tunneling exponent B and crossover temperature T_C are around 4.22×10^2 and 3.54×10^{-2} nK, respectively. Obviously, the crossover temperature T_C, which corresponds to a phase transition from classical tunneling to quantum tunneling, is far below the critical temperature $T_0\approx150$ nK for Bose-Einstein condensation in a dilute gas of ^{87}Rb.

IV. DISCUSSION AND SUMMARY

The generalized Bloch equation (4) and its stability analysis will help to control the population transfer and realize the single-qubit operation with BECs qubit. Theoretically, any two-state quantum system can serve as a qubit, many of them have been realized experimentally. To make use of two quantum states, the coherence and superposition between them is the most essential qualification. The experimental observation of coherence and superposition between two BECs indicates the possibility of encoding two coupled BECs as a qubit. However, because of the mean-field interaction among Bosonic condensed atoms, the qubit operations become very difficult to perform. To accomplish a single-qubit operation, it must be possible rotated arbitrarily in the Hilbert space. This requires that the atomic populations can be transferred arbitrarily. From the Bloch equations (4), we find that MQST prevents the arbitrary rotation of the state vector, and even if there is no MQST, when $\eta\neq0$, the complete population inversion cannot be accomplished with linear operations. Thus, to accomplish a linear qubit operation, one has to adjust the parameter η to zero by varying the atomic scattering length with a Feshbach resonance [35]. In this case, the mean-field interaction gives a density shift to the original energy levels and, according to Rabi's theory, the arbitrary rotation of the state vector can be performed easily. Thus, if one encodes the qubit states $|0\rangle$ and $|1\rangle$ as the condensate wave functions for two condensates in a double-well potential or two hyperfine-state condensates coupled with Raman pulses [36], an arbitrary one-bit linear operation can be realized when the anisotropy is absent ($\eta=0$) and an arbitrary one-bit nonlinear operation can be realized when the metastable multi-MQST behavior is absent ($|K|>|\eta N_T|$). This means that, to perform an arbitrary one-bit transformation, it at least needs choosing proper parameters to avoid the emergence of the metastable multi-MQST behavior.

The tunneling of the quasispin model described by Hamiltonian (5) has also been investigated by mapping it onto a particle moving in an asymmetric double-well potential [27–29]. Using this approach, Garanin *et al.* have explored some new fascinating features of this uniaxial spin model in the strongly biased limit [29]. They find that there exist two different regimes for the classical-quantum transition of the tunneling rate and the kind of transition depends on both the strength and the direction of the magnetic field. In this paper, we directly analyze the tunneling in the low barrier limit for the quasispin model, which corresponds to the effective magnetic fields near their critical values for appearance of metastable states. This requires that all physical parameters of the coupled BECs collaborate with each other to approach the critical point of appearance of multiple metastable MQST states. The symmetric case ($\gamma=0$) of the coupled BECs corresponds to the unbiased case ($H_z=0$) of the anisotropic spin model, which has been investigated in details by mapping it onto a particle moving in a symmetric double-well potential [25].

The macroscopic quantum tunneling of two-component BECs has also been investigated by Kasamatsu and Coworkers. Using a numerical approach, they have analyzed the tunneling between two kinds of metastable stationary states, a symmetry-breaking state and a symmetry-preserving state, in uncoupled two-component BECs [14]. To improve the usual Gaussian variational method, they have introduced a collective coordinate approach and then calculated the tunneling rate within the WKB approximation. In that system, the populations of the two components cannot be converted into each other because of the absence of coupling. This means, the tunneling does not occur between two components but between stationary states with different spatial configurations. Thus, this kind of tunneling originates from the quantized spatial structure of the Hamiltonian. In our model, due to the coupling, the population can be transferred from one component to the other. Furthermore, we assume the coupling is very weak, thus both components stay in their ground stationary states through the full process. The metastability (metastable MQST) is the result of the cooperation between the coupling and the mean-field interaction (including both the intracomponent and the intercomponent interactions). Correspondingly, the tunneling from the metastable self-trapped state to its ground state of the coupled two-component BECs is caused by the quantized structure of their field operators.

In conclusion a system of coupled BECs (two BECs in a double-well potential or two internal state BECs coupled with laser pulses) has been mapped to a spin in a magnetic field by introducing a generalized Bloch vector. The mean-field interaction, the coupling, and the asymmetry or the detuning are relevant to the anisotropy, the transverse magnetic field, and the longitudinal magnetic field, respectively. The corresponding generalized Bloch equation is obtained. The analysis of this generalized Bloch equation will be propitious to control the population transfer and realize the quantum computation with coupled BECs. Based upon experience from the well-studied tunneling of spin systems, the detailed information about the tunneling between two metastable MQST states in coupled two-component BECs can be obtained with the imaginary-time path-integral method. The crossover temperature T_C at the critical point for a transition from the classical thermal regime to the quantum regime was obtained. When the system temperature decreases through T_C, the population conversion goes from classical thermal activation regime to purely quantum tunneling regime. This means, below the crossover temperature T_C, the quantum fluctuations in the atomic fields take the dominant position. We also find that the tunneling rate can be adjusted by varying the coupling and the trapping magnetic field.

Note added in proof. Recently some interesting work in the field of macroscopic quantum tunneling in BECs has appeared [37].

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[1] J.R. Anglin and W. Ketterle, Nature (London) **416**, 211 (2002).
[2] A.J. Leggett, Rev. Mod. Phys. **73**, 307 (2001).
[3] P. Anderson, M. A. Kasevich, and F. Sols, in *Bose-Einstein Condensation in Atomic Gases*, Proceedings of International School of Physics ''Enrico Fermi'', edited by M. Inguscio, S. Stringari, and C. E. Wieman (IOS Press, Amsterdam, 1999), Vol. 140.
[4] C. Orzel, A.R. Tuchman, M.L. Fenselau, M. Yasuda, and M.A. Kasevich, Science **291**, 2386 (2001).
[5] M.R. Andrews, C.G. Townsend, H.-J. Miesner, D.S. Durfee, D.M. Kurn, and W. Ketterle, Science **275**, 637 (1997).
[6] M. Greiner, O. Mandel, T. Esslinger, T.W. Hansch, and I. Bloch, Nature (London) **415**, 39 (2002).
[7] F.S. Cataliotti, S. Burger, C. Fort, P. Maddaloni, F. Minardi, A. Trombettoni, A. Smerzi, and M. Inguscio, Science **293**, 843 (2001).
[8] B.P. Anderson and M.A. Kasevich, Science **282**, 1686 (1998).
[9] D.S. Hall, M.R. Matthews, J.R. Ensher, C.E. Wieman, and E.A. Cornell, Phys. Rev. Lett. **81**, 1539 (1998); D.S. Hall, M.R. Matthews, C.E. Wieman, and E.A. Cornell, *ibid.* **81**, 1543 (1998).
[10] D.M. Stamper-Kurn, H.-J. Miesner, A.P. Chikkatur, S. Inouye, J. Stenger, and W. Ketterle, Phys. Rev. Lett. **83**, 661 (1999).
[11] J. Stenger, S. Inouye, D.M. Stamper-Kurn, H.-J. Miesner, A.P. Chikkatur, and W. Ketterle, Nature (London) **396**, 345 (1998).
[12] A. Smerzi, A. Trombettoni, T. Lopez-Arias, C. Fort, P. Maddaloni, F. Minardi, and M. Inguscio, Eur. Phys. J. B **31**, 457 (2003).
[13] J. Williams *et al.*, Phys. Rev. A **59**, R31 (1999).
[14] K. Kasamatsu, Y. Yasui, and M. Tsubota, Phys. Rev. A **64**, 053605 (2001).
[15] A. Smerzi, S. Fantoni, S. Giovanazzi, and S.R. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997).
[16] S. Raghavan, A. Smerzi, S. Fantoni, and S.R. Shenoy, Phys. Rev. A **59**, 620 (1999).
[17] I. Marino, S. Raghavan, S. Fantoni, S.R. Shenoy, and A. Smerzi, Phys. Rev. A **60**, 487 (1999).
[18] A. Smerzi and S. Raghavan, Phys. Rev. A **61**, 063601 (2000).

[19] L.-M. Kuang and Z.-W. Ouyang, Phys. Rev. A **61**, 023604 (2000); Y. Wu and X. Yang, *ibid.* **68**, 013608 (2003); Y. Wu, X. Yang, and Y. Xiao, Phys. Rev. Lett. **86**, 2200 (2001).

[20] F.Kh. Abdullaev and R.A. Kraenkel, Phys. Rev. A **62**, 023613 (2000).

[21] F. Kh. Abdullaev and R. A. Kraenkel, e-print cond-mat/0005445.

[22] C. Lee, W. Hai, L. Shi, X. Zhu, and K. Gao, Phys. Rev. A **64**, 053604 (2001); W. Hai, C. Lee, G. Chong, and L. Shi, Phys. Rev. E **66**, 026202 (2002).

[23] L. Allen and J. H. Eberly, *Optical Resonance and Two-State Atoms* (Wiley, New York, 1975).

[24] E.M. Chudnovsky and L. Gunther, Phys. Rev. Lett. **60**, 661 (1988).

[25] E.M. Chudnovsky and D.A. Garanin, Phys. Rev. Lett. **79**, 4469 (1997).

[26] M.-C. Miguel and E.M. Chudnovsky, Phys. Rev. B **54**, 388 (1996); E. M. Chudnovsky, *Macroscopic Quantum Tunneling of the Magnetic Moment* (Cambridge University Press, New York, 1998).

[27] O.B. Zaslavskii, Phys. Rev. B **42**, 992 (1990).

[28] G. Scharf, W.F. Wreszinski, and J.L. van Hemmen, J. Phys. A **20**, 4309 (1987).

[29] D.A. Garanin, X.M. Hidalgo, and E.M. Chudnovsky, Phys. Rev. B **57**, 13 639 (1998).

[30] D.A. Garanin and E.M. Chudnovsky, Phys. Rev. B **56**, 11 102 (1997).

[31] J.-Q. Liang, H.J.W. Muller-Kirsten, D.K. Park, and F. Zimmerschied, Phys. Rev. Lett. **81**, 216 (1998); J.-Q. Liang, Y.-B. Zhang, H.J.W. Muller-Kirsten, J.G. Zhou, F. Zimmerschied, and F.-C. Pu, Phys. Rev. B **57**, 529 (1998).

[32] J. von Delft and C.L. Henley, Phys. Rev. Lett. **69**, 3236 (1992).

[33] D. Loss, D.P. DiVincenzo, and G. Grinstein, Phys. Rev. Lett. **69**, 3232 (1992).

[34] A.O. Caldeira and A.J. Leggett, Phys. Rev. Lett. **46**, 211 (1981); Ann. Phys. (N.Y.) **149**, 374 (1983).

[35] S. Inouye, M.R. Andrews, J. Stenger, H.-J. Miesner, D.M. Stamper-Kurn, and W. Ketterle, Nature (London) **392**, 151 (1998); Ph. Courteille, R.S. Freeland, D.J. Heinzen, F.A. vanAbeelen, and B.J. Verhaar, Phys. Rev. Lett. **81**, 69 (1998); S.L. Cornish, N.R. Claussen, J.L. Roberts, E.A. Cornell, and C.E. Wieman, *ibid.* **85**, 1795 (2000).

[36] Y. Shi, Int. J. Mod. Phys. B **15**, 3007 (2001).

[37] N. T. Ng, C.K. Law, and P.T. Leung, Phys. Rev. A **68**, 013604 (2003); Y. Zhou, H. Zhai, R. Lu, Z. Xu, and L. Chang, *ibid.* **67**, 043606 (2003); R.A. Barankov and S.N. Burmistrov, *ibid.* **67**, 013611 (2003); J. Liu, B. Wu, and Q. Niu, Phys. Rev. Lett. **90**, 170404 (2003).