Thermal entanglement in the two-qubit Heisenberg *XY* model under a nonuniform external magnetic field

Yang Sun, Yuguang Chen, and Hong Chen
Pohl Institute of Solid State Physics, Tongji University, Shanghai 200092, China
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We investigate the thermal entanglement in the two-qubit Heisenberg *XY* model with a nonuniform magnetic field. Concurrence, the measurement of entanglement, is calculated. The behavior of concurrence is present at three different cases. Contrary to the uniform magnetic field case, we find that the entanglement and the critical temperature T_C may be enhanced under a nonuniform magnetic field.

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Entanglement has been intensively studied in recent years because it implies a nonclassical nature through which we can investigate the conceptual foundations and interpretation of quantum mechanics, and, more importantly, it provides a fundamental resource in realizing quantum information and quantum computers [1]. In the preparation of entangled states, the external control can be performed through changes in temperature. And for any real physical system, the temperature should be considered since in most cases the thermal fluctuation may suppress quantum effects. Therefore, it is necessary to reveal the behaviors of thermal entanglement under external magnetic field conditions [2–5].

In this paper we investigate thermal entanglement using spin notations. Spin itself cannot only be used as qubits in some real physical systems but also in many other systems, such as a superconductor, quantum dots, and a trapped ion; spins can represent their freedoms used as qubits. Within condensed matter physics, the basic spin-spin interaction is the Heisenberg interaction [6]. Naturally, Heisenberg interaction may be a suitable candidate to simulate the relation between qubits in a quantum computer, which is the most important in quantum information studies [7]. But for some systems, only Heisenberg interaction is insufficient to describe the correlation of qubits. Therefore, other kinds of spin interaction are introduced into the model Hamiltonian [4]. In the frame of quantum information, there are many works on the usual Heisenberg *XX* model [5], the *XY* model [4,5], the *XXZ* model [5], *et al.* Within these studies, the central matter is the entanglement between qubits, especial the thermal entanglement.

It should be noted that only the uniform field case is carefully studied in the above-mentioned papers. The nonuniform case is rarely taken into account. But for perform quantum computing, it is necessary to control the magnetic field at each spin separately [8]. So in the theoretical analysis, the nonuniform external magnetic field should be included in the model Hamiltonian. This is the main motivation of this paper.

In this paper we would like to study the thermal entanglement in the Heisenberg model with a nonuniform external magnetic field. For simplification, only the *XY* model is considered in this paper. We mainly concentrate on the effect of non-uniform external magnetic field on thermal entanglement. In common sense, none of the thermal fluctuation and external magnetic fields is in favor of entanglement between qubits. Our results show that although temperature is destructive to entanglement, this destructive has a different effect in a different magnetic field region and through modulating magnetic field on each spin, respectively, we can make the external magnetic field be a favorable external factor for entanglement at a given temperature. Our paper is arranged as follows: first we will give the definition of concurrence, the measurement of entanglement. After giving the model Hamiltonian, we will present our calculation results by several figures. Finally, the discussion and conclusion remarks will be given.

First we would like to give the measurement of entanglement, the concurrence. For a pair of qubits, the concurrence is given by [9]

$$C_{12}=\max\{\lambda_1-\lambda_2-\lambda_3-\lambda_4,0\}, \tag{1}$$

where λ_i ($i=1,2,3,4$) are the square roots of the eigenvalues of the operator

$$\varrho_{12}=\rho_{12}(\sigma_1^y\otimes\sigma_2^y)\rho_{12}^*(\sigma_1^y\otimes\sigma_2^y), \tag{2}$$

with $\lambda_1\geqslant\lambda_2\geqslant\lambda_3\geqslant\lambda_4$, and ρ_{12} is the density matrix of the pair qubits; σ_1^y and σ_2^y are the normal Pauli operators. The concurrence $C_{12}=0$ corresponds to an unentangled state and $C_{12}=1$ corresponds to a maximally entangled state.

The model Hamiltonian studied in this paper is given by

$$H=J(S_1^xS_2^x+S_1^yS_2^y)+B_1S_1^z+B_2S_2^z,$$

where $S^\alpha=\sigma^\alpha/2(\alpha=x,y,z)$ are the spin-$\frac{1}{2}$ operator, σ^α are the normal Pauli operator, and J is the strength of Heisenberge interaction. B_1 and B_2 are the external magnetic fields. In this paper, by changing B_1 and B_2 separately we want to study the effects of magnetic field on the thermal entanglement in a very general way. The eigenvalues and eigenvectors of H can be obtained as

$$H|00\rangle=-(B_1+B_2)|00\rangle,$$

$$H|11\rangle=(B_1+B_2)|11\rangle,$$

$$H|\Psi^\pm\rangle=\pm\sqrt{D}|\Psi^\pm\rangle, \tag{3}$$

in which

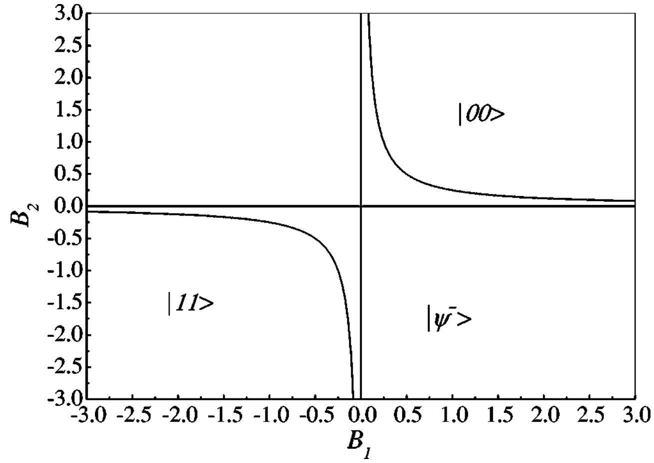

FIG. 1. The phase diagram of a two qubits *XY* Heisenberg model. The parameter J is set to one.

$$D=(B_1-B_2)^2+J^2,$$

$$|\Psi^{\pm}\rangle=\frac{1}{N_{\pm}}\left[|01\rangle+\frac{(B_1-B_2)\pm\sqrt{D}}{J}|10\rangle\right].$$

In the standard basis, $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the density matrix $\rho(T)$ can be written as

$$\rho(T)=\frac{1}{Z}\begin{pmatrix} e^{(B_1+B_2)/kT} & 0 & 0 & 0\\ 0 & m+n & -s & 0\\ 0 & -s & m-n & 0\\ 0 & 0 & 0 & e^{-(B_1+B_2)/kT}\end{pmatrix}, \tag{4}$$

where

$$m=\cosh\left(\frac{\sqrt{D}}{kT}\right),$$

$$n=\frac{(B_1-B_2)}{\sqrt{D}}\sinh\left(\frac{\sqrt{D}}{kT}\right),$$

$$s=\frac{J\sinh\left(\frac{\sqrt{D}}{kT}\right)}{\sqrt{D}},$$

and $Z=T_r\exp(-H/kT)$ is the partition function. In following calculation we will select J as the energy unit, and set $k=1$. By Eqs. (1), (2), (4), we can obtain the concurrence of the present two qubits system.

Before considering the thermal entanglement, we would like to spend a little space to discuss the ground state of the system at absolute zero temperature since the two qubits system is exactly solved and the eigenvalue and eigenvector have been given in Eqs. (3). After carefully comparing the eigenvalue of each state, a phase diagram is given in Fig. 1. A hyperbola divides the whole space into three parts. As $B_1B_2>\frac{1}{4}$ and B_1, $B_2>0$, the ground state is $|00\rangle$. And

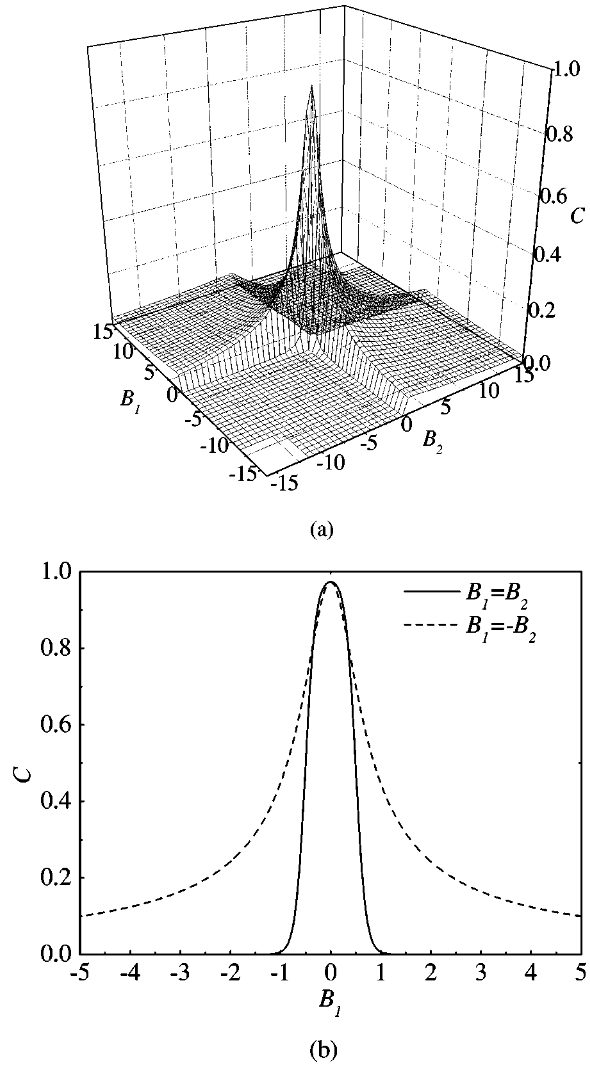

FIG. 2. (a) The concrescence versus B_1 and B_2. The temperature $T=0.2$. The parameter J is set to one. (b) The concurrence C under two special external fields. $B_1=B_2$ (solid line). $B_1=-B_2$ (dashed line). The parameter J is set to one.

$B_1B_2>\frac{1}{4}$ and B_1, $B_2<0$, $|11\rangle$ is the ground state. In these two states, there is no entanglement. But for $B_1B_2<\frac{1}{4}$, $|\Psi^-\rangle$ is the ground state, and the entanglement has nonzero value. Especially, as $B_1=B_2$ the state has maximal entanglement $C=1$. In the following discussion, for emphasizing the effects of a nonuniform external field, we will give the compression between the two cases: $B_1=B_2$ and $B_1=-B_2$.

Skipping details, we give our calculation results in Figs. 2, 3, and 4, in which the concurrence is plotted in the whole parameter space at a given temperature, and three typical cases are shown. As the temperature is low (in Fig. 2, $T=0.2$), we may find that there are two features. First, there is only one sharp peak. The center of this peak locates at $B_1=0$, $B_2=0$, where the concurrence is about 1. As we increase the external field B_1 and B_2, C rapidly decays. If the two external field have the same direction $(B_1B_2>0)$, C will decrease to zero as $B_1B_2\gtrsim\frac{1}{4}$. On the contrary, if $B_1B_2<0$, C decreases more slowly than in the case of $B_1B_2>0$. This is the second feature, which means that in the strong field re-

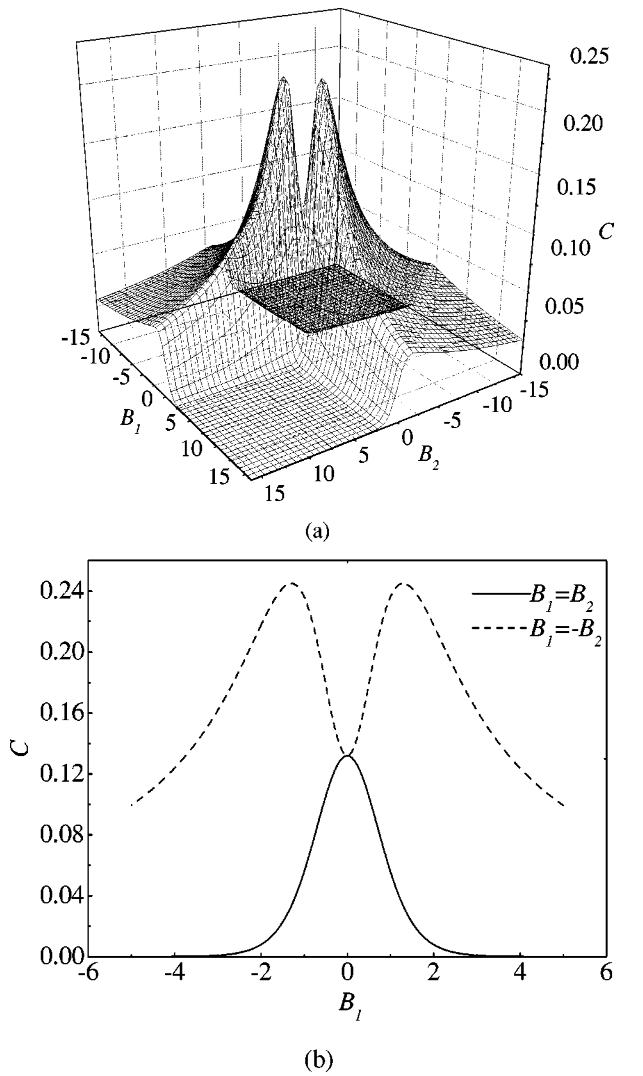

FIG. 3. (a) The concurrence versus B_1 and B_2. The temperature $T=0.9$. The parameter J is set to one. (b) The concurrence C under two special external fields. $B_1=B_2$ (solid line). $B_1=-B_2$ (dashed line). The parameter J is set to one.

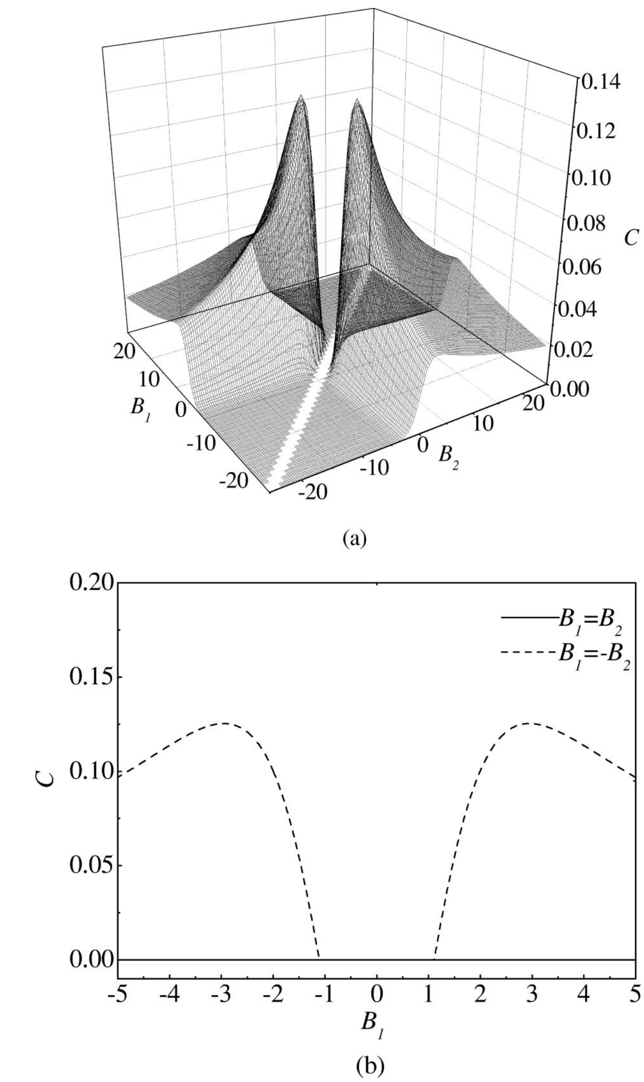

FIG. 4. (a) The concurrence versus B_1 and B_2. The temperature $T=1.5$. The parameter J is set to one. (b) The concurrence C under two special external fields. $B_1=B_2$ (solid line). $B_1=-B_2$ (dashed line). The parameter J is set to one.

gion, the nonuniform field and the uniform field demonstrate obviously different effects to entanglement. For clearly showing this point in Fig. 2(b), we give a two-dimensional (2D) plot for the two special cases: $B_1=B_2$, and $B_1=-B_2$. In this figure, one can find that in most parameter space, the concurrence of the field with opposite direction is much larger than that of uniform field.

As the temperature is increased, the feature of C will be changed. Figure 3 gives the second case. In this plot, $T=0.9$, and two peaks appear. That is to say, the center peak in Fig. 2(a) splits into two peaks with smaller values as we increase temperature. It should be noted that the culmination of these two peaks locate in the range: $B_1B_2<0$, which may imply that the opposite direction external field is more helpful for entanglement than the uniform field in the same temperature, which is also shown in Fig. 2 with lower temperature. And in Fig. 3(b) the plot for two special cases, $B_1=B_2$ and $B_1=-B_2$, also clearly shows this point. In this plot, for the uniform field case, the maximum C appears at $B_1=B_2=0$. But for the case of $B_1=-B_2$, at this point C is a minimum point. This means that at this temperature, if we apply an opposite field on two qubits, C can be enhanced about two times, from 0.12 to 0.24. That is to say, a proper external field can partially weaken the destructive effect of thermal fluctuation and enhance the entanglement. This result is very interesting since both the thermal fluctuation and the external field are against entanglement. As the temperature has a certain value, a proper external field is helpful for entanglement.

Above the main result can also be seen as the temperature is further increased, which is shown in Fig. 4 with $T=1.5$. In this figure the two peaks can be completely separated, that is, between these two peaks, there is a region with $C=0$, where the entanglement is entirely destroyed. In Fig. 4(b), as $B_1=B_2$, $C=0$. For $B_1=-B_2$, as $|B_1|<1.1$, $C=0$. When we increase the value of $|B_1|$, first the concurrence will be increased to a peak value. After this peak, C will decrease monotonously. These results shows that the external field can be used as a switch to turn on or off the entanglement. For this properties, possible applications are expected.

From the results shown in Figs. 2, 3, and 4, people may find that the nonuniform field may enhance the concurrence whose reason will be analyzed in the following. At zero temperature, the ground state is $|00\rangle$ ($|11\rangle$) or $|\Psi^-\rangle$. The eigenvalue for the maximally entangled state $|\Psi^-\rangle$ is $-\sqrt{D}$. When the external field is uniform, $\sqrt{D}=J$. But for the nonuniform field $\sqrt{D}=\sqrt{(B_1-B_2)^2+J^2}$. That is to say, at zero temperature, if the external field is nonuniform $(B_1-B_2 \neq 0)$, the ground-state energy of the entangled state $|\Psi^-\rangle$ will be lowed. At finite temperature, the occupation probability of this state will be enhanced since it is in proportion to $\exp(\sqrt{D}/kT)$, which can be increased as the nonuniform external field is introduced. This is the mean reason that the nonuniform field can enhance the concurrence.

As one compares the above results under different temperatures, it can be found that the concurrence C will decrease with increasing the temperature. That is to say, as the thermal fluctuation is introduced into the system, the states will be mixed. In our system, the maximally entangled states will be mixed with the unentangled states. This effect will make the concurrence decrease. So there may exist a critical temperature T_C. When $T>T_C$, C has a zero value and entanglement will completely disappear. In Fig. 5 we plot the T_C in the whole parameter space. By this figure, one may find that as the external field is uniform, the field B cannot change T_C. However, T_C will be enhanced by nonuniform external field. Especially, as the $B_1=-B_2$, this phenomenon is most strong. This point implies that the nonuniform external field may be a possible way to retain entanglement under finite temperature.

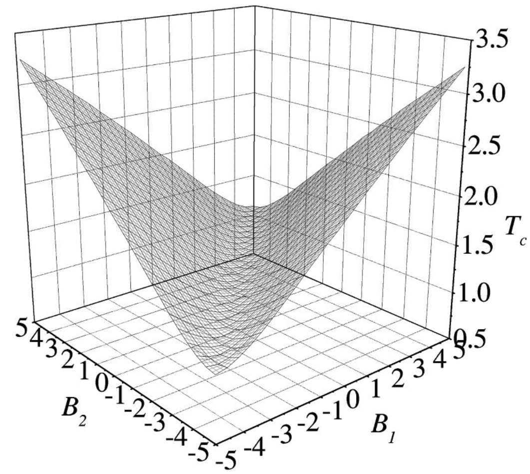

FIG. 5. The critical temperature T_C versus B_1 and B_2. The parameter J is set to one.

In conclusion, in this paper we investigate the effects of a nonuniform magnetic field on the thermal entanglement in the two-qubit Heisenberg *XY* model. At absolute zero temperature, we give the phase diagram of ground state. The effects of the nonuniform magnetic field are studied at different temperatures. Three typical results are shown in the figures. In the whole parameter space, we find that the entanglement may be enhanced under nonuniform magnetic field. And the critical temperature T_C can also be increased. Our results imply that entanglement may be effectively controlled through nonuniform magnetic field.

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