

Spin squeezing via atom-cavity field coupling

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Spin squeezing via atom-field interactions is considered within the context of the Tavis-Cummings model. An ensemble of N two-level atoms interacts with a quantized cavity field. For all the atoms initially in their ground states, it is shown that spin squeezing of both the atoms and the field can be achieved provided the initial state of the cavity field has coherence between number states differing by 2. Most of the discussion is restricted to the case of a cavity field initially in a coherent state, but initial squeezed states for the field are also discussed. Optimal conditions for obtaining squeezing are obtained. An analytic solution is found that is valid in the limit that the number of atoms is much greater than unity and is also much larger than the average number of photons, α^2 , initially in the coherent state of the cavity field. In this limit, the degree of spin squeezing increases with increasing α , even though the field more closely resembles a classical field for which no spin squeezing could be achieved.

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I. INTRODUCTION

Spin squeezed states offer an interesting possibility for reducing quantum noise in precision measurements [1–3]. Spin squeezing is described in terms of spin operators that are associated with quantum-mechanical operators of two-level atoms (TLA) (we refer to atoms and spins interchangeably). In an appropriate interaction representation, combinations of atomic raising and lowering operators for atom j are associated with the x and y spin components (S_x^j and S_y^j), while the population difference operator for the two states is associated with the z spin component (S_z^j). One then defines collective operators $S_\alpha = \sum_j S_\alpha^j$ that obey the usual spin commutator relations. If one measures an average spin $|\langle \mathbf{S} \rangle| = \sqrt{\langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2}$ then the system is said to be spin squeezed if

$$\xi_\perp = \sqrt{2S\Delta S_\perp} / |\langle \mathbf{S} \rangle| < 1, \quad (1)$$

where ΔS_\perp is the uncertainty in a spin component perpendicular to $\langle \mathbf{S} \rangle$, $S = N/2$, and N is the number of atoms [1,2]. Spin squeezing is impossible for a single atom and requires the entanglement of the spins of two or more atoms. There are many ways to theoretically construct a Hamiltonian that can give rise to the necessary entanglement among N two-level atoms. Since a linear Hamiltonian merely rotates the spin components leaving the uncertainties unchanged, it is generally necessary to use Hamiltonians that are quadratic in the spin operators to generate squeezing. On the other hand, it is possible to generate squeezing using a Hamiltonian linear in the spin operators provided the spin system is coupled to another quantum system, such as a harmonic oscillator. It is then not surprising to find that a squeezed state of the oscillator can be transferred to some degree to the atoms. What may be a little more surprising is that an oscillator prepared in a coherent state and coupled to the spins can

result in spin squeezing. In this paper, we study the dynamics of the creation of squeezing in an ensemble of spins via coupling to a cavity field in the Tavis-Cummings model [4]. An ensemble of N atoms is coupled in a spatially independent manner to the field with no losses for the field and with the neglect of any spontaneous emission for the atoms. We are concerned mainly with the type of spin squeezing that can be generated by coupling to a radiation field that is initially a coherent state, but also will consider an initial state of the field that is a squeezed state. The evolution of the radiation field will also be determined. There have been a number of studies of atom-field dynamics in the Tavis-Cummings model in which the squeezing of the cavity field was calculated in various limits [5]. Some numerical solutions to the problem of spin squeezing in the Tavis-Cummings model are given in Ref. [1].

The initial condition for the atoms is taken as one in which all the atoms are in their lower energy state, corresponding to a coherent spin state. For a very large number of atoms ($N \gg 1$ and N much greater than the average number of photons in the coherent state of the field), the relevant energy levels of the spin system approach those of a simple harmonic oscillator with corrections that vanish as $N \sim \infty$. Thus it would seem that spin squeezing can never be achieved if the initial state of the cavity field is a coherent state, since one is dealing with a linear interaction between two harmonic oscillators each of them initially in a coherent state. Nevertheless, we show that for any finite N , spin squeezing occurs and the degree of spin squeezing actually increases with increasing field strength.

To follow the atom-field dynamics, we consider first a system having $N=2$. It is not difficult to obtain analytic solutions in this case, enabling us to track the dependence of ξ_\perp on field strength and N . In addition, we determine if the squeezed vacuum state results in optimal transfer of squeez-

ing from the fields to the atoms. After discussing the two-atom case, we generalize the results to N atoms.

The paper is organized as follows. In Sec. II we present the mathematical framework and obtain results that show that no squeezing can be achieved when the field is either classical, or quantized in a number state. In Sec. III, we consider the $N=2$ case and obtain analytical results for both coherent and squeezed cavity fields, in the limit that the average number of photons in the field is much less than unity. Numerical solutions for larger field strength are presented. In Sec. IV, the results are generalized to N atoms. In both Secs. III and IV, the time evolution and squeezing of the field is also calculated for the case that the field is initially in a coherent state. In Sec. V, a formal derivation of the large N limit is given using the Holstein-Primakoff transformation [6], valid for an arbitrary strength of the coherent cavity field. The Holstein-Primakoff transformation was used previously by Persico and Vetri [7] to analyze the atom-field dynamics in the limit of large N . The approach we follow differs somewhat from theirs and our results seem to have a wider range of validity than that stated by Persico and Vetri. The results are summarized in Sec. VI.

II. GENERAL CONSIDERATIONS

In dipole and rotating-wave approximations, the Hamiltonian for an ensemble of TLA (lower state $|1\rangle$, upper state $|2\rangle$, transition frequency ω) interacting with a resonant cavity field, $E(t) = Ea e^{-i\omega t} + E^* a^\dagger e^{i\omega t}$, is of the form

$$H = \hbar\omega S_z + \hbar\omega a^\dagger a + \hbar g(S_+ a + S_- a^\dagger),$$

where

$$S_z = \sum_{j=1}^N [(|2\rangle\langle 2|)_j - (|1\rangle\langle 1|)_j]/2,$$

$$S_+ = \sum_{j=1}^N (|2\rangle\langle 1|)_j e^{-i\omega t},$$

$$S_- = \sum_{j=1}^N (|1\rangle\langle 2|)_j e^{i\omega t},$$

$$S_x = (S_+ + S_-)/2,$$

$$S_y = (S_+ - S_-)/2i,$$

a and a^\dagger are annihilation and creation operators for the field, and g is a coupling constant. The spin operators have been defined in a reference frame rotating at the field frequency. Constants of the motion are $S^2 = S_x^2 + S_y^2 + S_z^2$ and $(S_z + a^\dagger a)$. If, initially, all spins are in their lower energy state, then $S^2 = N^2/4$. In order to calculate ξ_\perp from Eq. (1), one must first find $\langle \mathbf{S} \rangle$ and define two independent directions orthogonal to $\langle \mathbf{S} \rangle$, $S_{\perp 1}$ and $S_{\perp 2}$. It then follows that $\langle S_{\perp 1} \rangle = \langle S_{\perp 2} \rangle = 0$ and

$$(\Delta S_{\perp i})^2 = \frac{N}{4} + \sum_{j,j' \neq j} \langle S_{\perp i}^{(j)} S_{\perp i}^{(j')} \rangle,$$

where $i=1,2$ and $S^{(j)}$ is a spin operator for atom j .

A necessary condition to have $\xi_\perp < 1$ is that the different spins are entangled. To see this, take a system in which $\langle \mathbf{S} \rangle$ is

aligned along the z axis, with the x axis chosen such that ΔS_x is the minimum value of S_\perp . Using the facts that $\langle S_z \rangle = S$, $\langle S_x \rangle = \langle S_y \rangle = 0$, $\Delta S_x \Delta S_y \geq |\langle S_z \rangle|/2$, one finds

$$\xi_x = \sqrt{N} \Delta S_x / |\langle S_z \rangle| \geq \sqrt{N} / \Delta S_y = \left[1 + \sum_{j,j' \neq j} \langle S_y^{(j)} S_y^{(j')} \rangle \right]^{-1}.$$

For correlated states, the sum can be positive and one cannot rule out the possibility that $\xi_x < 1$. On the other hand, for uncorrelated states, using the fact that $\langle S_y \rangle^2 = 0$, it follows that $1 + \sum_{j,j' \neq j} \langle S_y^{(j)} S_y^{(j')} \rangle = 1 - \sum_j \langle S_y^{(j)} \rangle^2$. As a consequence, $\xi_x \geq 1$ and there is no spin squeezing for uncorrelated states.

We note two general conclusions that are valid for arbitrary N . First, if we were to replace the cavity field by a classical field, the Hamiltonian would be transformed into

$$H_{class} = \sum_j [\hbar\omega S_z^{(j)} + \hbar g' (S_+^{(j)} e^{-i\omega t} + S_-^{(j)} e^{i\omega t})],$$

where g' is a constant. Since the Hamiltonian is now a sum of Hamiltonians for the individual atoms, the wave function is a direct product of the wave functions of the individual atoms. As a consequence, there is no entanglement and no spin squeezing for a classical field. Second, if the initial state of the field is a Fock state, although there is entanglement between the atoms and the field, there is no spin squeezing. There is no spin squeezing unless the initial state of the field has coherence between at least two states differing in n by 2. For a Fock state, there is no such coherence and $\xi_\perp \geq 1$.

It is convenient to carry out the calculations in an interaction representation with the wave function expressed as

$$|\psi(t)\rangle = \sum_{m=-N/2}^{N/2} \sum_{n=0}^{\infty} c_{mk}(t) e^{-i\omega(m+n)t} |m, n\rangle, \quad (2)$$

where m labels the value of S_z and n labels the number of photons in the cavity field. In this representation, the Hamiltonian governing the time evolution of the $c_{mk}(t)$ is given by

$$H = \hbar g(S_+ a + S_- a^\dagger). \quad (3)$$

III. $N=2$

We first set $N=2$, $S=1$. If the spins are all in their lower energy state at $t=0$, the initial wave function is

$$|\psi(0)\rangle = \sum_{k=0}^{\infty} c_k |-1, k\rangle, \quad (4)$$

where the c_k are the initial state amplitudes for the field. Solving the time-dependent Schrödinger equation with initial condition (4), one finds

$$c_{-1,k}(t) = \frac{1}{(2k-1)} [k-1 + k \cos(\sqrt{4k-2}gt)] c_k, \quad (5a)$$

$$c_{0,k}(t) = -i \sqrt{\frac{k+1}{2k+1}} \sin(\sqrt{4k+2}gt) c_{k+1}, \quad (5b)$$

$$c_{1,k}(t) = \frac{\sqrt{(k+1)(k+1)}}{2k+3} [-1 + \cos(\sqrt{4k+6}gt)] c_{k+2}. \quad (5c)$$

These state amplitudes can be used to calculate all expectation values of the spin operators.

A. Coherent state

If the initial state of the cavity field is a coherent state, then

$$c_k = \alpha^k e^{-|\alpha|^2/2} / \sqrt{k!}, \quad (6)$$

and the average number n_0 of photons in the field is given by $n_0 = |\alpha|^2$. For simplicity, we take α and g to be real.

1. Solution for $|\alpha|^2 \ll 1$

Keeping terms to order α^2 , one finds from Eqs. (5) and (6) that the only state amplitudes of importance are

$$c_{-1,0}(t) = (1 - \alpha^2/2), \quad (7a)$$

$$c_{-1,1}(t) = \alpha \cos(\sqrt{2}gt), \quad (7b)$$

$$c_{-1,2}(t) = \frac{\alpha^2}{3\sqrt{2}} [1 + 2 \cos(\sqrt{6}gt)], \quad (7c)$$

$$c_{0,0}(t) = -i\alpha \sin(\sqrt{2}gt), \quad (7d)$$

$$c_{0,1}(t) = -\frac{i\alpha^2}{\sqrt{3}} \sin(\sqrt{6}gt), \quad (7e)$$

$$c_{1,0}(t) = -\frac{\alpha^2}{3} [1 - \cos(\sqrt{6}gt)]. \quad (7f)$$

The spin components' expectation values are

$$\langle S_x \rangle = 0, \quad \langle S_y \rangle = \sqrt{2}\alpha \sin(\sqrt{2}gt), \quad (8a)$$

$$\langle S_z \rangle = -[1 - \alpha^2 \sin^2(\sqrt{2}gt)]. \quad (8b)$$

The motion of the average value for the spin vector operator is in the yz plane, with the length of the vector always equal to unity, to order α^2 . Since $\langle S_x \rangle = 0$, the plane in which we look for spin squeezing is the one defined by the x axis and an axis orthogonal to both $\hat{\mathbf{x}}$ and the instantaneous direction of the spin. Making the appropriate rotation in the yz plane to define a y' axis perpendicular to $\langle \mathbf{S} \rangle$ and $\hat{\mathbf{x}}$, and afterwards choosing an arbitrary direction defined by an angle ϕ in this plane, one finds that $\xi_\phi \geq \min\{\xi_x, \xi_{y'}\}$, which implies that the best squeezing is to be found in either the x or y' directions. The analytical expressions for ξ_x , $\xi_{y'}$ are

$$\xi_x = \sqrt{2} \frac{\Delta S_x}{|\langle \mathbf{S} \rangle|} \approx 1 + \alpha^2 \left\{ \frac{1}{2} \sin^2(\sqrt{2}gt) - \frac{2}{3} \sin^2(\sqrt{6}gt/2) \right\}, \quad (9a)$$

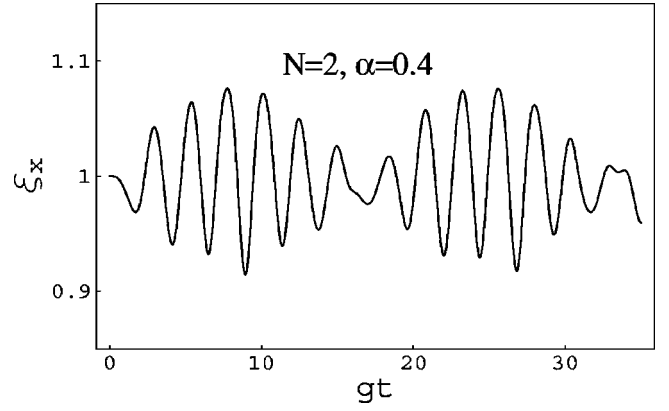


FIG. 1. Spin squeezing ξ_x as a function of gt for $\alpha=0.4$ and $N=2$.

$$\xi_{y'} = \sqrt{2} \frac{\Delta S_{y'}}{|\langle \mathbf{S} \rangle|} \approx 1 + \alpha^2 \left\{ -\frac{1}{2} \sin^2(\sqrt{2}gt) + \frac{2}{3} \sin^2(\sqrt{6}gt/2) \right\}. \quad (9b)$$

The lowest possible value for the squeezing occurs in the x direction and is equal to

$$\xi_{\min} = 1 - \frac{2}{3} \alpha^2 \quad (10)$$

at a time when $\sin(\sqrt{2}gt) = 0$ and $\cos(\sqrt{6}gt) = -1$. The squeezing ξ_x as a function of gt for $\alpha=0.4$ is plotted in Fig. 1.

2. Numerical results for all values of α

General expressions for the spin expectation values and variances can be obtained and used for numerical simulations for any values of α . With α real, the expectation value of the x component of the spin vanishes and, with the notation $\overline{c_{0,n}} = c_{0,n}/i$,

$$\langle S_y \rangle = \sqrt{2} \sum_{n=0}^{\infty} \overline{c_{0,n}} (c_{1,n} - c_{-1,n}),$$

$$\langle S_z \rangle = \sum_{n=0}^{\infty} (|c_{1,n}|^2 - |c_{-1,n}|^2).$$

The variances are

$$(\Delta S_x)^2 = \langle S_x^2 \rangle = \frac{1}{2} + \sum_{n=0}^{\infty} \left\{ \frac{1}{2} |c_{0,n}|^2 + c_{1,n} c_{-1,n} \right\}, \quad (11a)$$

$$\begin{aligned} (\Delta S_y)^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2 = & \frac{1}{2} + \sum_{n=0}^{\infty} \left\{ \frac{1}{2} |c_{0,n}|^2 - c_{1,n} c_{-1,n} \right\} \\ & - \langle S_y \rangle^2. \end{aligned} \quad (11b)$$

The variance in the x component of the spin cannot be less than $1/2$ unless $c_{1,n} c_{-1,n} < 0$. Since $c_{1,n} c_{-1,n}$ is proportional to $c_{k+2} c_k$, where the c_k s are initial state amplitudes for the cavity field, spin squeezing can be induced by a field only if

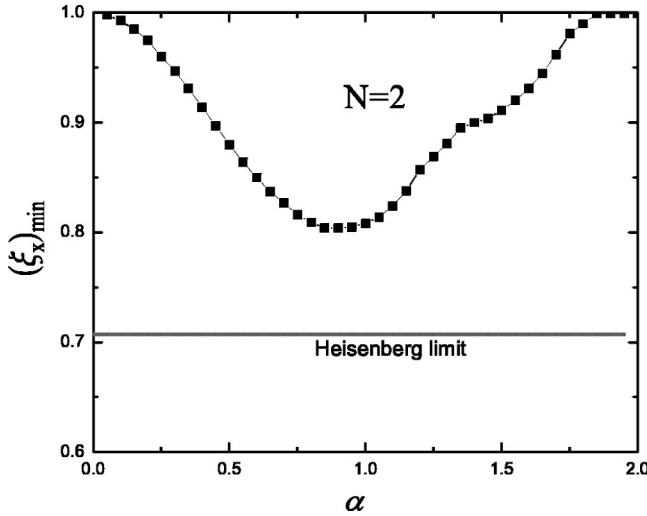


FIG. 2. Optimal spin squeezing $(\xi_x)_{\min}$ as a function of α for $N=2$. The time range out to $gt=5000$ was explored in obtaining the minimal squeezing. In this and other plots, the points represent actual values for which the squeezing was calculated. A line is drawn through these points.

the field has at least one nonvanishing off-diagonal density matrix element $\rho_{kk'}$ for which $|k-k'|=2$.

The values for the spin averages and uncertainties are calculated in terms of α and gt . For $\alpha^2 \ll 1$ the numerical and analytical results agree. For larger values of α , no analytical solution is available. The numerical results indicate that the optimal squeezing is obtained in the \hat{x} direction. As α is increased, the spin squeezing increases and then decreases for $\alpha \geq 0.9$, as shown in Fig. 2. With increasing α , the optimal squeezing occurs at increasingly large values of gt . For example, with $\alpha=1.6$, there is effectively no spin squeezing for $gt < 333$ and the optimal spin squeezing occurs for $gt=2439$. The squeezing data in this and subsequent graphs are the optimal squeezing that is obtained for gt less than some arbitrary cutoff that we have chosen. In the limit of $\alpha \gg 1$, the field closely resembles a classical field and $(\xi_x)_{\min}$ approaches unity. Formally, this result could be derived by using a transformation proposed by Mollow [8] in which the transformed Hamiltonian is that of a classical field having amplitude α plus a fluctuating field. Any spin squeezing that is produced depends on the ratio of the fluctuations to the average field strength and must decrease with increasing α , provided the average number of photons in the field is much larger than N .

B. Squeezing in the radiation field

Although the field is initially in a coherent state, it is squeezed as a result of its interaction with the atoms [5]. In terms of quadrature operators \hat{P} and \hat{Q} defined as

$$\hat{Q} = \frac{1}{\sqrt{2}}(a + a^+), \quad \hat{P} = -\frac{i}{\sqrt{2}}(a - a^+)$$

with $[\hat{Q}, \hat{P}] = i$, squeezing of the field occurs if the variance of one of these two operators is smaller than the value it

would have for the vacuum field. Initially the field is in a coherent state of real amplitude α with $\langle \hat{Q} \rangle = \sqrt{2}\alpha$ and $\langle \hat{P} \rangle = 0$, and variances $(\Delta \hat{Q})^2 = (\Delta \hat{P})^2 = \frac{1}{2}$ satisfying the minimum uncertainty condition

$$(\Delta \hat{Q})^2 (\Delta \hat{P})^2 = \frac{1}{4} |[\hat{Q}, \hat{P}]|^2 = \frac{1}{4}. \quad (12)$$

Using the wave function (2), one finds

$$\begin{aligned} \langle \hat{Q} \rangle &= \sum_{m=-1}^1 \sum_{k=0}^{\infty} [\sqrt{k} c_{m,k-1}^*(t) \\ &\quad + \sqrt{k+1} c_{m,k+1}^*(t)] c_{m,k}(t) / \sqrt{2}, \\ \langle \hat{P} \rangle &= \sum_{m=-1}^1 \sum_{k=0}^{\infty} [\sqrt{k} c_{m,k-1}^*(t) \\ &\quad + \sqrt{k+1} c_{m,k+1}^*(t)] c_{m,k}(t) / \sqrt{2}, \\ \langle \hat{Q}^2 \rangle &= \frac{1}{2} + \frac{1}{2} [\sqrt{k(k-1)} c_{m,k-2}^*(t) c_{m,k}(t) \\ &\quad + \sqrt{(k+1)(k+2)} c_{m,k+2}^*(t) c_{m,k}(t) \\ &\quad + 2k \sqrt{k+1} |c_{m,k}(t)|^2], \\ \langle \hat{P}^2 \rangle &= \frac{1}{2} - \frac{1}{2} [\sqrt{k(k-1)} c_{m,k-2}^*(t) c_{m,k}(t) \\ &\quad + \sqrt{(k+1)(k+2)} c_{m,k+2}^*(t) c_{m,k}(t) \\ &\quad - 2k \sqrt{k+1} |c_{m,k}(t)|^2]. \end{aligned}$$

To order α^2 , for the field initially in a coherent state, one finds squeezing parameters

$$\xi_Q = \sqrt{2} \Delta \hat{Q} \approx 1 - \alpha^2 \{ \cos^2(\sqrt{2}gt) - \frac{1}{3} [1 + 2 \cos(\sqrt{6}gt)] \},$$

$$\xi_P = \sqrt{2} \Delta \hat{P} \approx 1 + \alpha^2 \{ \cos^2(\sqrt{2}gt) - \frac{1}{3} [1 + 2 \cos(\sqrt{6}gt)] \}.$$

With this definition, squeezing occurs for $\xi_Q < 1$ or $\xi_P < 1$. To second order in α the state of the field evolves in time as a minimum uncertainty state but with squeezing transfer between the two quadratures. The minimum value for the squeezing parameters that can be obtained is

$$(\xi_Q)_{\min} = 1 - \frac{4}{3} \alpha^2, \quad (12')$$

$$(\xi_P)_{\min} = 1 - \alpha^2. \quad (12'')$$

A continuous transfer of squeezing between the Q quadrature and the x component of the spin, and also between the P quadrature and the y component of the spin is taking place. The maximum field squeezing as a function of α is shown in Fig. 3.

C. Squeezed initial cavity field

From Eq. (11) one can see that initial state coherence between photon field states differing by 2 is needed for squeezing. The squeezed vacuum is a superposition of even Fock states; therefore, it is a good choice for inducing the

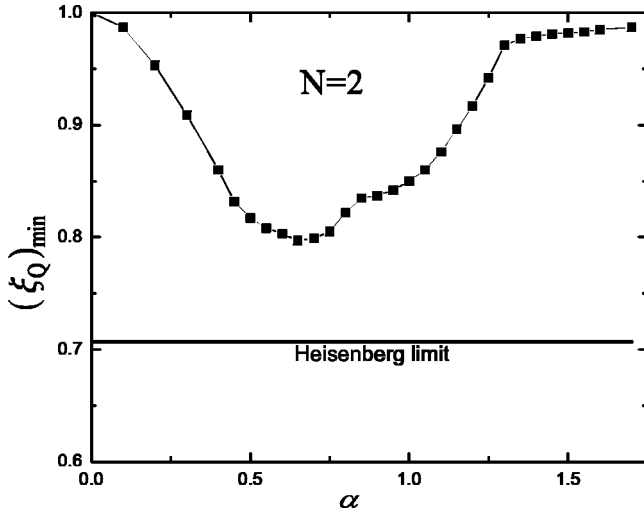


FIG. 3. Optimal field squeezing $(\xi_Q)_{\min}$ as a function of α for $N=2$. The time range out to $gt=5000$ was explored in obtaining the minimal squeezing.

necessary coherences in the atomic system. Analytical results are available for a small squeezing parameter of the field, and numerical results can be obtained for larger values. For a squeezed vacuum the c_k s are given by

$$c_0 = \frac{1}{\sqrt{\cosh r}},$$

$$c_k = \frac{(k-1)!! (-1)^{k/2} \tanh^{k/2} r}{\sqrt{k! \cosh r}} \text{ for } k \text{ even,}$$

$$c_k = 0 \text{ for } k \text{ odd,}$$

where r is the squeezing parameter, assumed real. For any field containing only even expansion coefficients, $\langle S_x \rangle = \langle S_y \rangle = 0$. For $r \ll 1$, one obtains for the spin squeezing

$$\xi_x = \sqrt{2} \frac{\Delta S_x}{|\langle S \rangle|} \approx 1 + \frac{4}{3} r \sin^2 \left(\sqrt{\frac{3}{2}} gt \right),$$

$$\xi_y = \sqrt{2} \frac{\Delta S_y}{|\langle S \rangle|} \approx 1 - \frac{4}{3} r \sin^2 \left(\sqrt{\frac{3}{2}} gt \right).$$

To the first order in r , the resulting state is a minimum uncertainty state, and the minimum squeezing that can be achieved is the same for both components. Squeezing as a function of r is shown in Fig. 4. With increasing r , ξ_y decreases to minimum value of 0.78 for $r \approx 0.7$, and then increases with increasing r . This result is consistent with the general conclusion that optimal squeezing is obtained when the average number of photons in the field is much less than N .

One might think that the squeezed vacuum produces optimal squeezing, but field states that more closely approach the Heisenberg limit $\xi_y = 1/\sqrt{2}$ can be constructed. One such state is

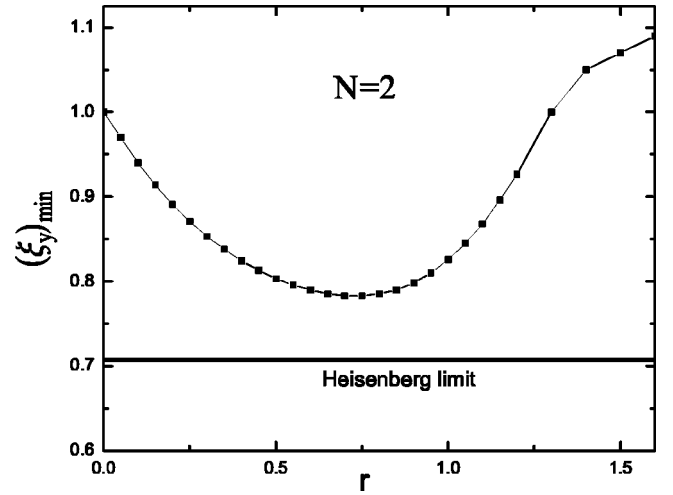


FIG. 4. Optimal spin squeezing $(\xi_y)_{\min}$ as a function of the squeezing parameter r for an initially squeezed cavity field for $N=2$. The time range out to $gt=5000$ was explored in obtaining the minimal squeezing.

$$|\psi(0)\rangle = -0.79|0\rangle - 0.594|2\rangle + 0.15|4\rangle + 0.021|6\rangle$$

for which a minimum value $\xi_y = 0.724$ is achieved. We have not been able to formulate a general proof as to the minimum squeezing one can obtain for an arbitrary initial state of the field.

IV. N ATOMS

As the number of atoms N increases, the spin squeezing that can be achieved depends critically on the initial state of the cavity field. If the field is in a coherent state, one might expect that the squeezing vanishes as N goes to infinity since the atomic spin Hamiltonian approaches that of a simple harmonic oscillator in this limit. A formal proof of this result is given below. On the other hand, for finite N , there are times for which spin squeezing occurs, and $(\xi_x)_{\min}$ decreases with increasing field strength, provided N is much larger than the average number of photons in the field. If the initial state of the field is a squeezed state such as the squeezed vacuum, the field squeezing can be transferred to the atoms. In this manner, one can generate a high degree of spin squeezing $\xi_x \ll 1$, but still considerably less than that predicted by the Heisenberg limit $\xi_x = 1/\sqrt{N}$.

For arbitrary N , the cavity field can, in principle, couple $(N+1)$ collective states corresponding to the angular momentum manifold $S=N/2$. In practice, the number of states coupled is on the order of the average number of photons in the initial field. The equations of motion for the state amplitudes, obtained from Eqs. (2) and (3) are

$$\dot{c}_{mn} = -ig \left\{ \sqrt{\left(\frac{N}{2} + m\right)\left(\frac{N}{2} - m + 1\right)} (n+1) c_{m-1, n+1} + \sqrt{\left(\frac{N}{2} - m\right)\left(\frac{N}{2} + m + 1\right)} n c_{m+1, n-1} \right\} \quad (13)$$

with initial condition $c_{m,n}(0) = c_n \delta_{m,-N/2}$. This equation represents a set of coupled equations, starting from $m = -N/2$ and reaching some maximum value to $-N/2$ plus n_{\max} , where n_{\max} is the smallest n where the initial field state amplitude c_n is negligibly small.

A. Coherent cavity field

1. Analytical solution for $|\alpha|^2 \ll 1$

For $\alpha^2 \ll 1$, the lowest order nonvanishing amplitudes obtained from Eqs. (13) and (6) are

$$c_{-S,0}(t) = (1 - \alpha^2/2), \quad (14a)$$

$$c_{-S,1}(t) = \alpha \cos(\sqrt{N}gt), \quad (14b)$$

$$c_{-S+1,0}(t) = -i\alpha \sin(\sqrt{N}gt), \quad (14c)$$

$$c_{-S,2}(t) = \frac{\alpha^2 \sqrt{2}}{4N-2} [N-1 + N \cos(\sqrt{4N-2}gt)], \quad (14d)$$

$$c_{-S+1,1}(t) = -i \frac{\alpha^2 \sqrt{N}}{\sqrt{4N-2}} \sin(\sqrt{4N-2}gt), \quad (14e)$$

$$c_{-S+2,0}(t) = - \frac{\alpha^2 \sqrt{2N(N-1)}}{4N-2} [1 - \cos(\sqrt{4N-2}gt)]. \quad (14f)$$

In the large N limit, the average spin components calculated using these amplitudes are

$$\begin{aligned} \langle S_x \rangle &= 0, & \langle S_y \rangle &= \sqrt{N}\alpha \sin(\sqrt{N}gt), \\ \langle S_z \rangle &= -\frac{N}{2} + \alpha^2 \sin^2(\sqrt{N}gt), \end{aligned} \quad (15)$$

such that $|\langle \mathbf{S} \rangle| = S = N/2$ to order α^2 .

The squeezing parameter, calculated using Eqs. (11), is given by

$$\begin{aligned} \xi_x &= \sqrt{N} \frac{\Delta S_x}{|\langle \mathbf{S} \rangle|} \approx 1 + \alpha^2 \left\{ \frac{N-1}{N} \sin^2(\sqrt{N}gt) \right. \\ &\quad \left. - \frac{2(N-1)}{2N-1} \sin^2[\sqrt{(2N-1)/2}gt] \right\}. \end{aligned} \quad (16)$$

In the limit of large N this reduces to

$$\xi_x \sim 1 + \alpha^2 \sin \left[\left(2\sqrt{N} - \frac{1}{4\sqrt{N}} \right) gt \right] \sin \left(\frac{gt}{4\sqrt{N}} \right). \quad (17)$$

As N approaches infinity, the squeezing vanishes; however, for any finite N , there is a time of order $2\pi\sqrt{N}/g$ where spin squeezing with $\xi_x \sim 1 - \alpha^2$ occurs. Note that, for small $gt \ll N^{-1/2}$, ξ_x from Eq. (18) varies as $[1 - \alpha^2(gt)^4(N-1)/6]$ while ξ_x from Eq. (19) varies as $[1 + \alpha^2(gt)^2/2]$, which have different functional forms; however, the *difference* between these two results varies as $\alpha^2/N \ll 1/N \ll 1$.

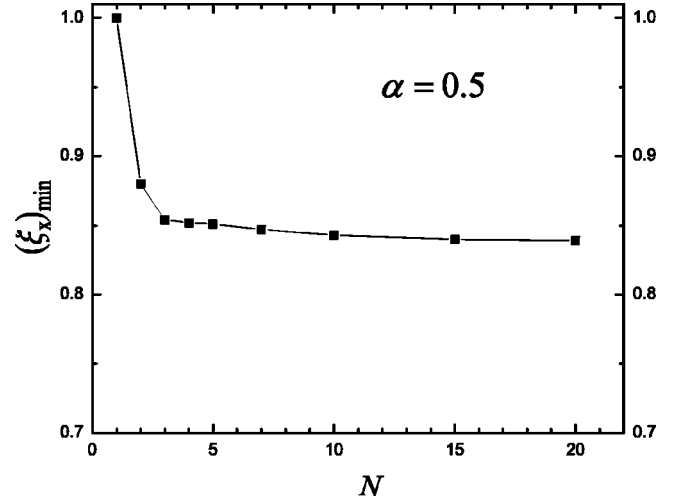


FIG. 5. Optimal spin squeezing $(\xi_x)_{\min}$ as a function of N for $\alpha = 0.5$.

2. Numerical results for all values of α

Since the average number of photons in a coherent state is α^2 , one needs to solve Eq. (13) up to terms with $n \gg \alpha^2$. As α grows, the numerical solution becomes somewhat unwieldy. In Fig. 5, the optimal squeezing is plotted as a function of N for $\alpha = 0.5$. The quantity $(\xi_x)_{\min}$ diminishes with increasing N , eventually reaching an asymptotic value of 0.86. This result represents the general trend that the squeezing saturates for $N \gg \alpha^2$. Spin squeezing as a function of α for fixed $N = 20$ is shown in Fig. 6 for $0.6 \leq \alpha \leq 3.5$. The values of ξ_x in Fig. 6 do not necessarily represent the optimal spin squeezing; rather they give first minimum of the *envelope* of a graph of ξ_x versus gt . It is possible that better spin squeezing occurs at higher values of gt than those considered (e.g., for $\alpha = 0.6$, the first envelope minimum at $gt = 9.03$ gives $\xi_x = 0.906$, while the second envelope mini-

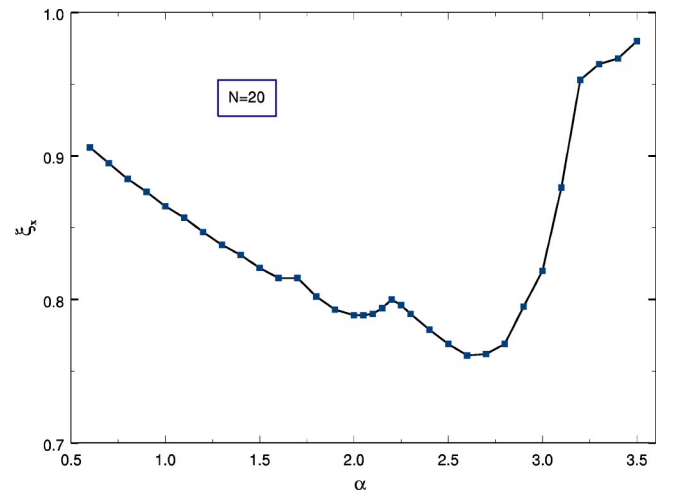


FIG. 6. Optimal spin squeezing $(\xi_x)_{\min}$ as a function of α for $N = 20$ and $0 \leq gt \leq 10$. Since only a restricted range of gt was considered, the values plotted may not represent the global optimal squeezing, but still reflect the qualitative variation of $(\xi_x)_{\min}$ with α .

mum at $gt=28.1$ gives $\xi_x=0.817$); the computation time that would be needed to determine $(\xi_x)_{\min}$ for all values of gt grows rapidly with increasing α . Spin squeezing improves with increasing α up to $\alpha \approx 2.7 \approx O(\sqrt{20})$ and then decreases with increasing α , following the general trend noted above.. Spin squeezing for larger values of α and $N \gg \alpha^2$ are better treated by the method given in Sec. V.

3. Squeezing in the field

For $\alpha \ll 1$, one finds squeezing parameters

$$\xi_Q = \sqrt{2\Delta} \hat{Q} \approx 1 - \alpha^2 \cos^2(\sqrt{N}gt) + \frac{\alpha^2}{2N-1} \\ \times [N-1 + N \cos(\sqrt{4N-2}gt)],$$

$$\xi_P = \sqrt{2\Delta} \hat{P} \approx 1 + \alpha^2 \cos^2(\sqrt{N}gt) - \frac{\alpha^2}{2N-1} \\ \times [N-1 + N \cos(\sqrt{4N-2}gt)],$$

implying that

$$(\xi_Q)_{\min} = 1 - \frac{2N}{2N-1} \alpha^2, \quad (18a)$$

$$(\xi_P)_{\min} = 1 - \alpha^2. \quad (18b)$$

The best squeezing is obtained for $N=2$. With increasing α , the field squeezing mirrors the spin squeezing.

B. Squeezed initial cavity field

The spin squeezing one can achieve increases dramatically if the initial state of the cavity field is a squeezed state. For a squeezed vacuum with squeezing parameter r , the initial squeezing in one quadrature component of the field is $\xi_Q = e^{-r}$. In the limit that $N \gg \sinh^2 r + \sqrt{2} \sinh r \cosh r$ (average plus standard deviation of the number of photons in the original cavity field), one can show [1] that this squeezing can be transferred totally to the spins $\xi_x = e^{-r}$. For large r , this represents substantial squeezing, but since $N \gg \sinh^2 r + \sqrt{2} \sinh r \cosh r$, it follows that $\xi_x = e^{-r} \gg (1 + \sqrt{2})/2\sqrt{N}$. Thus, one is still far from the Heisenberg limit. It may be possible to construct an original cavity field state that leads more closely to the Heisenberg limit $\xi_x = 1/\sqrt{N}$, but we have not explored this possibility in the large N limit.

V. ASYMPTOTIC SOLUTION FOR LARGE N

For an ensemble having a number of atoms much larger than unity and much larger than the average number of photons in the field, the interaction between the atoms and the cavity field can be seen as an interaction between a harmonic oscillator (the field) and an imperfect oscillator (the atoms). To attempt to map this problem into one of interacting harmonic oscillators, which will be valid as the number of atoms N approaches infinity, one defines boson operators for the atoms via

$$S_z = -N/2 + b^\dagger b, \quad (19a)$$

$$S^+ = e^{-i\omega t} N^{1/2} b^\dagger (1 - b^\dagger b/N)^{1/2} \\ \approx e^{-i\omega t} \left(\sqrt{N} b^\dagger - \frac{1}{2\sqrt{N}} b^\dagger b^\dagger b \right), \quad (19b)$$

$$S^- = e^{i\omega t} N^{1/2} (1 - b^\dagger b/N)^{1/2} b \\ \approx e^{i\omega t} \left(\sqrt{N} b - \frac{1}{2\sqrt{N}} b^\dagger b b \right). \quad (19c)$$

The boson occupation states (Fock states) $|m\rangle = (b^\dagger)^m / \sqrt{m!} |0\rangle$ correspond to the different projections onto the collective angular momentum states and, in effect, represent excitations above the lowest state having $S_z = -N/2$. The transformation to the b bosons (Holstein-Primakoff transformation [6]) is exact. The approximations in Eqs. (21a) and (21b) are valid provided the relative variations of the spin projection are small,

$$\langle b^\dagger b \rangle / N \ll 1; \quad (20)$$

in other words, the average spin remains aligned very close to the z axis. The key point in this calculation is that all changes in the eigenkets of order $1/\sqrt{N}$ are neglected. Changes in the eigenenergies of order $1/\sqrt{N}$ lead to significant changes in the *phases* of the time-dependent wave function for any finite N . Such changes in the phase can result in spin squeezing.

The total Hamiltonian (in an interaction representation) is written as $H = H_0 + H'$, with

$$H_0 = \hbar \sqrt{N} g (b^\dagger a + a^\dagger b), \quad (21a)$$

$$H' = -\frac{\hbar g}{2\sqrt{N}} (b^\dagger b^\dagger b a + a^\dagger b^\dagger b b). \quad (21b)$$

We now diagonalize H_0 and treat H' as a perturbation. The Hamiltonian H_0 can be written as

$$H_0 = \omega_+ \Gamma^\dagger \Gamma + \omega_- \gamma^\dagger \gamma, \quad \omega_\pm = \pm \sqrt{N} g,$$

with

$$\Gamma^\dagger = \frac{a^\dagger + b^\dagger}{\sqrt{2}}, \quad \gamma^\dagger = \frac{a^\dagger - b^\dagger}{\sqrt{2}},$$

$$a^\dagger = \frac{\Gamma^\dagger + \gamma^\dagger}{\sqrt{2}}, \quad b^\dagger = \frac{\Gamma^\dagger - \gamma^\dagger}{\sqrt{2}},$$

while perturbation H' has the form

$$H' = \frac{\hbar g}{4\sqrt{N}} [\gamma^\dagger \gamma^\dagger \gamma \gamma - \Gamma^\dagger \Gamma^\dagger \Gamma \Gamma + \Gamma^\dagger \Gamma^\dagger \gamma \gamma + \gamma^\dagger \gamma^\dagger \Gamma \Gamma \\ + 2\{\Gamma^\dagger \gamma^\dagger (\gamma \gamma - \Gamma \Gamma) + \text{H.c.}\}].$$

Only the first two terms in this expression, conserving the total number of excitations, contribute in first order.

The eigenkets of H_0 are

$$|n\rangle_+ |m\rangle_- = \frac{(\Gamma^\dagger)^n}{\sqrt{n!}} |0\rangle \frac{(\gamma^\dagger)^m}{\sqrt{m!}} |0\rangle \quad (22)$$

with energies

$$\epsilon^{(0)}(n,m) = \hbar(\omega_+ n + \omega_- m).$$

The first order correction to the energies of these states is

$$\epsilon^{(1)}(n,m) = \frac{\hbar g}{4\sqrt{N}}(n - n^2 - m + m^2) \equiv \epsilon_+^{(1)}(n) + \epsilon_-^{(1)}(m),$$

and we define

$$\epsilon_\pm(n) = \hbar\omega_\pm n + \epsilon_\pm^{(1)}(n).$$

To this order states (24) are unmodified.

In order to neglect higher order correction to the energies, it is necessary that the phase produced by such corrections must be much less than unity. This translates into the condition $[g^2/N(\omega_+ - \omega_-)]t = gt/2N^{3/2} \ll 1$, which can always be satisfied for sufficiently large N , but would be violated for $N=2$. There is no restriction on the value of the phase $|\epsilon_\pm(n+1) - \epsilon_\pm(n)|t/\hbar \approx ngt/\sqrt{N}$, provided $gt/(2N^{3/2}) \ll 1$. In fact, such phases are responsible for the finite N corrections calculated below.

Coherent cavity field

For an initial state in which the cavity field is in a coherent state and $S_z = -N/2$, one has

$$|\Psi(t=0)\rangle = e^{-\tilde{\alpha}^2} e^{\sqrt{2}\tilde{\alpha}a^\dagger} |0\rangle = e^{-\tilde{\alpha}^2} e^{\tilde{\alpha}(\Gamma^\dagger + \gamma^\dagger)} |0\rangle,$$

$$|\Psi(t)\rangle = e^{-\tilde{\alpha}^2} \left(\sum_m \frac{\tilde{\alpha}^m}{\sqrt{m!}} e^{-i\epsilon_+(m)t/\hbar} e^{-im\omega t} |m\rangle_+ \right) \times \left(\sum_n \frac{\tilde{\alpha}^n}{\sqrt{n!}} e^{-i\epsilon_-(n)t/\hbar} e^{-in\omega t} |n\rangle_- \right),$$

$$\begin{aligned} \langle \Psi(t) | b | \Psi(t) \rangle &= \frac{e^{-\tilde{\alpha}^2}}{\sqrt{2}} \left\{ \sum_m \langle m-1 | \Gamma | m \rangle_+ \frac{\tilde{\alpha}^{m-1} \tilde{\alpha}^m}{\sqrt{m!(m-1)!}} \right. \\ &\quad \times e^{-i\omega t} e^{-i[\epsilon_+(m) - \epsilon_+(m-1)]t/\hbar} \\ &\quad - \sum_n \langle n-1 | \gamma | n \rangle_- \frac{\tilde{\alpha}^{n-1} \tilde{\alpha}^n}{\sqrt{n!(n-1)!}} \\ &\quad \left. \times e^{-i\omega t} e^{-i[\epsilon_-(n) - \epsilon_-(n-1)]t/\hbar} \right\} \\ &= \tilde{\alpha} e^{-i\omega t} e^{-\tilde{\alpha}^2} \sqrt{2} i \sum_n \frac{\tilde{\alpha}^{2(n-1)}}{(n-1)!} \\ &\quad \times \sin[(\lambda_1 - n\lambda_2)]t \end{aligned}$$

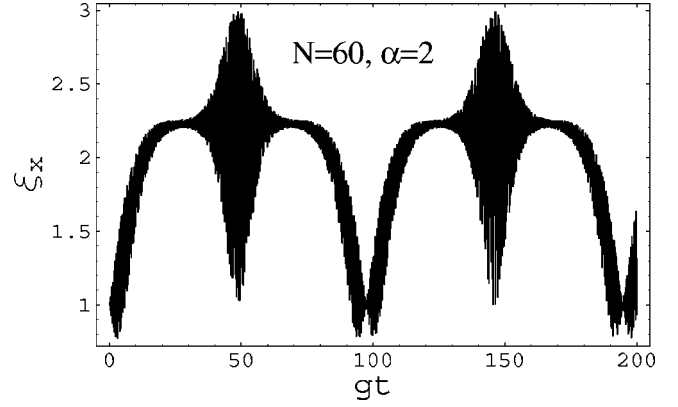


FIG. 7. Spin squeezing ξ_x as a function of gt for $\alpha=2$ and $N=60$.

with $\tilde{\alpha} = \alpha/\sqrt{2}$ and

$$\lambda_1 = \left(\sqrt{N} + \frac{1}{2\sqrt{N}} \right) g, \quad \lambda_2 = \frac{1}{2\sqrt{N}} g.$$

Note that

$$\langle S_x \rangle = \sqrt{N} [\langle b e^{i\omega t} \rangle + \langle b^\dagger e^{-i\omega t} \rangle] = 0.$$

In order to compute the squeezing, we need the following averages:

$$\begin{aligned} \langle \Gamma^\dagger \Gamma^\dagger \rangle &= \tilde{\alpha}^2 e^{i2\omega t} e^{-\tilde{\alpha}^2} e^{i(2\lambda_1 - 3\lambda_2)t} \sum_n \frac{(\tilde{\alpha}^2 e^{-2i\lambda_2 t})^n}{n!} \\ &= \tilde{\alpha}^2 e^{i2\omega t} e^{i(2\lambda_1 - 3\lambda_2)t} e^{\tilde{\alpha}^2 (e^{-2i\lambda_2 t} - 1)}, \\ \langle \gamma^\dagger \gamma^\dagger \rangle &= \tilde{\alpha}^2 e^{i2\omega t} e^{-i(2\lambda_1 - 3\lambda_2)t} e^{\tilde{\alpha}^2 (e^{2i\lambda_2 t} - 1)}, \\ \langle \Gamma^\dagger \rangle &= \tilde{\alpha} e^{i\omega t} e^{i\sqrt{N}gt} e^{\tilde{\alpha}^2 (e^{-i\lambda_2 t} - 1)}, \\ \langle \gamma^\dagger \rangle &= \tilde{\alpha} e^{i\omega t} e^{-i\sqrt{N}gt} e^{\tilde{\alpha}^2 (e^{i\lambda_2 t} - 1)}. \end{aligned}$$

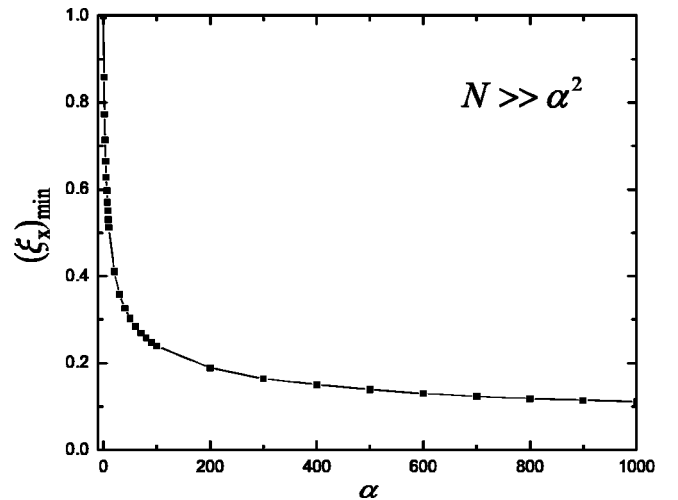


FIG. 8. Optimal spin squeezing $(\xi_x)_{\min}$ as a function of α for $N \gg \alpha^2$.

The value of $\langle S \rangle$ remains equal to $N/2$, with corrections of order α^2/N , and the squeezing $\xi_x \approx (2/\sqrt{N})\Delta S_x$ is calculated as

$$\begin{aligned} \xi_x = & \sqrt{\langle (b^\dagger e^{-i\omega t} + b e^{i\omega t})^2 \rangle} = (1 + \alpha^2 \{ e^{-\alpha^2 \sin^2(\lambda_2 t)} \cos[(2\sqrt{N}g \\ & - \lambda_2)t - (\alpha^2/2)\sin(2\lambda_2 t)] - e^{-2\alpha^2 \sin^2(\lambda_2 t/2)} + 1 \\ & - e^{-2\alpha^2 \sin^2(\lambda_2 t/2)} \cos[2\sqrt{N}gt - \alpha^2 \sin(\lambda_2 t)] \})^{1/2}. \end{aligned} \quad (23)$$

This expression agrees with Eq. (19) in the limit that $\alpha \ll 1$; however, it extends that result to all values of α for which condition (22) remains valid and for which $gt/(2N^{3/2}) \ll 1$ [9]. Persico and Vetri [7] employ a somewhat different approach in solving this problem using the Holstein-Primakoff transformation and obtain a validity range, $gt < \sqrt{N}/\alpha^2$. Since $2N^{3/2} \gg \sqrt{N}/\alpha^2$, the validity range for Eq. (23) should be much greater than that of Persico and Vetri. To test this hypothesis, we compared the term of order α^4 in the exact solution with the α^4 term of Eq. (23). The two results agreed for times $gt/(2N^{3/2}) \ll 1$, as expected. It might be noted that Eq. (23) agrees with the exact result to order α^2 , independent of gt , provided $N \gg 1$. This is why we had to compare the α^4 terms.

For $\alpha \ll 1$, there is a slow modulation having period $gt = 4\pi\sqrt{N}$, in addition to the rapid oscillations having period $gt = \pi/\sqrt{N}$. With increasing α , and $N \gg \alpha^2$, the overall period is $gt = 4\pi\sqrt{N}$, with a subharmonic having period $gt = 2\pi\sqrt{N}$, and the rapid oscillations having period $gt = \pi/\sqrt{N}$. These features are seen clearly in Fig. 7, drawn for $\alpha = 2$ and $N = 60$. Similar curves were obtained by Kozierowski and Chumakov [5] for the field squeezing. With increasing α , $(\xi_x)_{\min}$ decreases slowly as is shown in Fig. 8, where the condition $N \gg \alpha^2$ is maintained as α is varied. In contrast to the $\alpha \ll 1$ case, the optimal squeezing for $\alpha \gg 1$, always occurs at a time $gt \approx \sqrt{N}/\alpha^{3/2} \ll \sqrt{N}$. In the limit that $\alpha \gg 1$ and $z \equiv \alpha^2 gt/2\sqrt{N} \ll \sqrt{\alpha}$, one can show that Eq. (23) can be approximated as

$$\xi_x \approx \{ 1 + z \sin(\sigma z - z) + z^2 \sin^2[(\sigma z - z)/2] \}^{1/2},$$

where $\sigma = 4N/\alpha^2$. From this expression it is possible to show that the squeezing parameter goes to zero with increasing α , but that the approach to zero is slower than $\alpha^{-1/2}$ (the actual dependence seems to be close to $\alpha^{-0.31}$). Even though the field is getting more classical with increasing α , quantum

fluctuations in the field still lead to increased squeezing with increasing α . Of course, if we explore the range $\alpha^2 > N$, we would find an increase in $(\xi_x)_{\min}$ with increasing α , as we found for the case $N = 2$.

VI. SUMMARY

It has been shown that a linear interaction Hamiltonian between a coherent state cavity field and an ensemble of two-level atoms can produce spin squeezing. Analytical solutions for small values of the amplitude of the field state were derived, showing a reduction in the squeezing parameter quadratic in α . Computer simulations were used to find the best value for squeezing, when α is varied over a range of real, positive values. The limit of a large number of atoms was also examined. For an initial coherent state for the cavity field, it was found that the squeezing parameter approaches zero with increasing α . This might seem like a remarkable result since the coherent state closely resembles a classical field for large α . Even though α is large, the number of atoms is assumed to be much larger than α^2 ; as such the field can be totally depleted. The entanglement of the field and the spins can produce significant phase shifts that can lead to spin squeezing. Although $(\xi_x)_{\min}$ approaches zero with increasing α , the ratio $\xi_x = (\xi_x)_{\min}/\sqrt{N}$ that relates the squeezing to the Heisenberg limit, *decreases* with increasing α . If squeezing relative to the Heisenberg limit is used as a measure, the best squeezing is obtained for $N = 2$. This is in marked contrast to the optimal squeezing that can be obtained with nonlinear spin interactions [1,2].

The interaction with a squeezed cavity field was also investigated. While a squeezed vacuum field has the potential to transfer significant spin squeezing to the atoms, the degree of spin squeezing produced is still well above the Heisenberg limit. By constructing alternative squeezed states, we were able to improve the squeezing relative to that of a spin-squeezed vacuum, but the ultimate degree of spin squeezing that can be transferred to the atoms via an interaction with a cavity field remains an open question.

ACKNOWLEDGMENTS

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