Creation and evolution of trains of dark solitons in a trapped one-dimensional Bose-Einstein condensate

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The generation of dark solitons from large initial excitations and their evolution in a one-dimensional Bose-Einstein condensate trapped by a harmonic potential is studied analytically and numerically. We consider three different techniques of controllable creation of multisoliton structures (soliton trains) from large initial excitations and calculate their initial parameters (depths and velocities) with the use of a generalized Bohr-Sommerfeld quantization rule. Multisoliton effects are discussed.

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I. INTRODUCTION

Current experiments [1–4] on the formation of solitons in Bose-Einstein condensates (BEC's) have stimulated intensive theoretical studies devoted to generation and evolution of solitons in BEC. Special interest was attracted by the BEC with repulsive interaction between atoms where dark solitons can be generated by various methods, e.g., by inducing density defects in BEC [4] (density engineering), by imprinting spatial phase distribution [3] (phase engineering), and by collision of two condensates [5,6]. At ultralow temperatures, a trapped BEC is well described by the three-dimensional (3D) Gross-Pitaevskii (GP) equation [7], and localized excitations in BEC can be studied by its numerical solution (see, e.g., Refs. [8]). However, in highly asymmetric cigar-shape traps the 3D GP equation can be reduced under certain conditions to 1D nonlinear Schrödinger (NLS) equation (see, e.g., Refs. [9–11]) which is a well-studied mathematical model widely used for description of evolution of wave packets in various nonlinear media.

In homogeneous case, when the trap potential is dropped, the NLS equation has the property of complete integrability [12]. In this case, the parameters of solitons formed by the initial disturbance are determined by the spectrum of the Zakharov-Shabat (ZS) linear problem associated with the NLS equation. In practice, this spectrum can be calculated by such approximate methods as variational approach [13] or quasiclassical method [14,15]. If the size of the initial disturbance is much less than the size of the whole condensate, then we can consider the stage of solitons formation as taking place in homogeneous condensate and apply the methods developed for the integrable NLS equation. Just this approach was used in a recent paper [16] for the special case of steplike phase initial disturbance. In the present paper we shall consider generation of solitons from arbitrary initial disturbance by the quasiclassical method developed in Ref. [17]. In this method the spectrum of the ZS linear problem is determined by the generalized Bohr-Sommerfeld quantization rule which gives the parameters of solitons formed from the initial disturbance. If the condensate as a whole is not disturbed too much, then further propagation of dark solitons along nonuniform BEC can be described as their oscillations in BEC confined by a trap (see, e.g., Refs. [18,19]).

The organization of the paper is as follows. In Sec. II we start with outline of the reduction of the 3D GP equation to the 1D NLS equation that will provide us with the characteristic values of parameters. In Sec. III we describe generation of soliton trains in a trapped BEC from initial disturbances of different types and study their evolution. In the case of perturbation of the condensate density by a large and smooth initial pulse we find initial parameters of created solitons with the use of generalized Bohr-Sommerfeld quantization rule. We predict locations of turning points of oscillatory motion of solitons which are well confirmed by direct numerical simulations. We also discuss behavior of dark solitons generated by the phase imprinting method and creation of solitons during collisions of two condensates in the presence of the harmonic trap potential. The outcomes of the theory are summarized in Conclusion.

II. DERIVATION OF 1D NLSE FROM THE 3D GP EQUATION

We start with the 3D GP equation for the order parameter $\psi \equiv \psi(\mathbf{r},t)$:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta \psi + V_{trap}(\mathbf{r})\psi + g_0|\psi|^2\psi, \tag{1}$$

where we use the standard notation $g_0 = 4\pi\hbar^2 a_s/m$, a_s being the *s*-wave scattering length, which is considered positive, *m* is the atomic mass; $V_{trap}(\mathbf{r})$ is a trap potential, and ψ is normalized to number of particles $\mathcal{N}$ in BEC. Considering the case of a cigar-shape BEC, we take $V_{trap} = (m/2)\omega^2 x^2 + (m/2)\omega_\perp(y^2+z^2)$ where longitudinal trap frequency ω is much less than the transverse one $\omega_\perp$. Then the 3D GP equation (1) can be reduced to 1D (NLS) equation, if one can neglect excitations of higher transverse modes and only ground-state transverse motion can be taken into account. An

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easy estimate shows that energies of transverse motion are much greater than the nonlinear energy, if the condition

$$\frac{a_s\mathcal{N}}{l}\ll 1 \tag{2}$$

is fulfilled, where l is a longitudinal size of the condensate. In this case atoms occupy only the ground state of their transverse motion which, thus, is decoupled from the longitudinal motion. Since the transverse motion is reduced to the ground state only, we can factorize the whole condensate wave function. To make the resulting 1D evolution equation dimensionless, we introduce

$$\psi(\mathbf{r},t)=(\sqrt{2}\pi a_\perp^2 a_s)^{-1/2}\exp\left(-i\omega_\perp t-\frac{y^2+z^2}{2a_\perp^2}\right)\Psi(x,t), \tag{3}$$

where $a_\perp^2=\hbar/m\omega_\perp$, and make a change of independent variables $x=2^{-1/4}a_\perp x'$, $t=2^{1/2}t'/\omega_\perp$. This results in the canonical form of the NLS equation with a parabolic potential:

$$i\frac{\partial\Psi}{\partial t}+\frac{\partial^2\Psi}{\partial x^2}-2|\Psi|^2\Psi=\frac{1}{2}\nu^2x^2\Psi, \tag{4}$$

where $\nu=\omega/\omega_\perp\ll 1$ and the primes are suppressed. The dimensionless BEC wave function $\Psi(x,t)$ is normalized according to

$$\int_{-\infty}^{\infty}|\Psi(x,t)|^2dx=2^{3/4}\frac{\mathcal{N}a_s}{a_\perp}. \tag{5}$$

A stationary solution $\Psi(x,t)=\Psi(x)\exp(-i\mu t)$, corresponding to the ground state of BEC, is given by $\Psi(x)$ satisfying the equation

$$\frac{d^2\Psi}{dx^2}+\mu\Psi-2|\Psi|^2\Psi=\frac{1}{2}\nu^2x^2\Psi, \tag{6}$$

subject to the zero boundary condition at $|x|\to\infty$ and having no other zeros. The eigenvalue μ (chemical potential) is determined by the normalization condition (5). In dimensionless units the longitudinal size of the condensate is of order of magnitude $\mu^{1/2}/\nu$ and if the kinetic energy of longitudinal motion is much less than μ, which gives the condition $\mu \gg \nu$, then considerable part of the condensate can be described by Thomas-Fermi (TF) approximation in which the term with the second space derivative in Eq. (6) can be neglected almost everywhere, so that

$$\rho_0(x)\equiv|\Psi_{TF}(x)|^2=F(x)=\tfrac{1}{2}(\mu-\tfrac{1}{2}\nu^2x^2) \tag{7}$$

with normalization condition

$$\int_{-\sqrt{2\mu}/\nu}^{\sqrt{2\mu}/\nu}|\Psi_{TF}(x)|^2dx=\frac{(2\mu)^{3/2}}{3\nu}. \tag{8}$$

By equating Eqs. (5) and (8) we determine the value of the dimensionless chemical potential μ in terms of experimentally measurable parameters

$$\mu=2^{-1/2}\left(\frac{3\nu\mathcal{N}a_s}{a_\perp}\right)^{2/3}. \tag{9}$$

Then the condition $\mu\gg\nu$ yields the criterion

$$\nu^{1/2}\ll\mathcal{N}a_s/a_\perp \tag{10}$$

of applicability of the TF approximation in physical units. Note that the condition that the characteristic width of the soliton solution of the NLS equation must be much less than condensate's length l yields again the same inequality (10). The initial dimensionless axial length of the condensate is equal to $l=\sqrt{2\mu}/\nu$, or in dimensional units

$$l=(3\mathcal{N}a_s a_\perp^2/\nu^2)^{1/3}. \tag{11}$$

Substitution of this expression into Eq. (2) gives the condition of applicability of 1D reduction of the 3D GP equation in the form $a_s\mathcal{N}/a_\perp\ll 1/\nu$, so that combining this inequality with Eq. (10), we arrive at criteria of applicability of the present theory:

$$\nu^{1/2}\ll\frac{a_s\mathcal{N}}{a_\perp}\ll\frac{1}{\nu}. \tag{12}$$

In what follows we deal mainly with two dimensionless parameters ν and μ, which completely define the initial distribution of the condensate density.

Let us estimate these values for typical experimental parameters [20] of the condensate with $\mathcal{N}=3\times10^3$ atoms of ^{87}Rb with scattering length $a_s\approx5$ nm. Taking $a_\perp\approx3$ μm (which corresponds to the frequency $\omega_\perp\sim5\times10^2$ Hz) and the frequency of longitudinal trap $\omega\sim10$ Hz, one can obtain $\nu=0.02$, the initial condensate length $l\sim0.12$ mm. The parameter $a_s\mathcal{N}/a_\perp$ is equal to ~5 and conditions (12) of applicability of this theory are satisfied quite well: $0.14\ll5\ll50$.

III. FORMATION OF SOLITONS FROM LARGE INITIAL EXCITATION

The problem of evaluation of parameters of dark solitons formed from a large initial excitation on a constant background is formally solved by the inverse scattering method [12]. In the framework of this method, the NLS equation is associated with the ZS linear spectral problem [12], and soliton parameters are related with the eigenvalues of this problem calculated for a given initial condensate wave function $\Psi(x,0)$. If the initial disturbance is large enough, so that the linear spectral problem possesses many eigenvalues, then a well-known quasiclassical method can be applied for their calculation. As was shown in a recent paper [17], a generalized Bohr-Sommerfeld quantization rule is very convenient for this aim.

To formulate this rule, it is convenient to introduce a new small parameter ε, $\varepsilon\ll1$, into Eq. (4) by means of replacements $x=x'/\varepsilon$, $t=t'/\varepsilon$, and $\nu=\nu'\varepsilon$, so that the equation transforms to

$$i\varepsilon\frac{\partial\Psi}{\partial t}+\varepsilon^2\frac{\partial^2\Psi}{\partial x^2}-2|\Psi|^2\Psi=\frac{1}{2}\nu^2x^2\Psi, \tag{13}$$

where we have omitted for simplicity the primes in the new variables. Then the limit $\varepsilon\ll 1$ corresponds to formation of a large number of solitons from an initial disturbance with parameters of order of magnitude $O(1)$ (see Ref. [17]). In framework of the inverse scattering transform method the NLS equation, i.e., Eq. (13) with the zero right-hand side, is treated as a compatibility condition of two linear equations for auxiliary function χ, which we write down in the form

$$\varepsilon^2\chi_{xx}=\mathcal{A}\chi,\quad \chi_t=-\tfrac{1}{2}\mathcal{B}\chi_x+\mathcal{B}_x\chi \tag{14}$$

(equivalent to the ZS problem; see Refs. [21,22]), where

$$\mathcal{A}=-\left(\lambda-\frac{i\varepsilon\Psi_x}{2\Psi}\right)^2+|\Psi|^2-\varepsilon^2\left(\frac{\Psi_x}{2\Psi}\right)_x, \tag{15}$$

$$\mathcal{B}=2\lambda+\frac{i\varepsilon\Psi_x}{\Psi}. \tag{16}$$

The first equation (14) may be considered as a second-order scalar spectral problem with a given "potential" Ψ and λ playing the role of the spectral parameter. When $\Psi(x,t)$ evolves according to Eq. (13) with $\nu=0$, the eigenvalues λ_n of this problem do not change with time t, and each eigenvalue corresponds to a soliton created from the initial pulse. To investigate the limit $\varepsilon\ll 1$, let us represent the condensate wave function in the form

$$\Psi(x,t)=\sqrt{\rho(x,t)}\exp\left(\frac{i}{\varepsilon}\int^x v(x',t)dx'\right), \tag{17}$$

where $\rho(x,t)$ is the condensate density and $v(x,t)$ is the hydrodynamic velocity. Indeed, substitution of Eq. (17) into Eq. (13) yields the system

$$\tfrac{1}{2}\rho_t+(\rho v)_x=0, \tag{18}$$

$$\tfrac{1}{2}v_t+vv_x+\rho_x+\varepsilon^2[(\rho_x^2-2\rho\rho_{xx})/8\rho^2]_x=-\tfrac{1}{2}\nu^2x,$$

which for $\varepsilon\to 0$ takes the form of hydrodynamic equations. For smooth enough functions $\rho(x,t)$ and $v(x,t)$, when

$$|\varepsilon\rho_x/\rho|\ll\rho \quad\text{and}\quad |\varepsilon v_x|\ll\rho, \tag{19}$$

which corresponds to neglecting the space derivatives in $\mathcal{A}$, spectral problem (14) transforms into

$$\varepsilon^2\chi_{xx}=[-(\lambda+v/2)^2+\rho]\chi. \tag{20}$$

It is to be mentioned here that conditions (19) are nothing but the conditions of applicability of the well-known hydrodynamical approach when due to a relatively high density the two-body interactions are strong enough and one can neglect the "quantum pressure" (see, e.g., Ref. [7]).

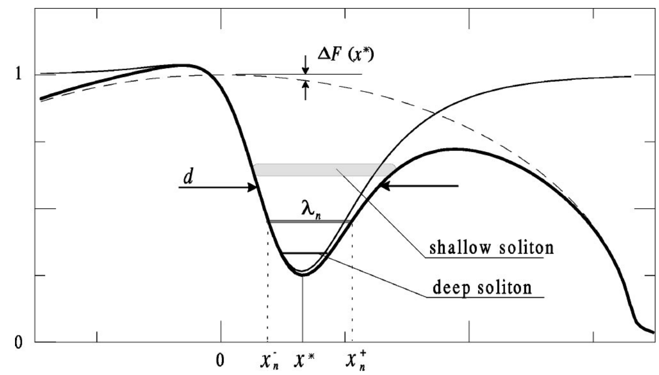

FIG. 1. Schematic plot of the Riemann invariant λ^+ given by Eq. (21) (thick solid line); the thin solid line shows the Riemann invariant for disturbance with respect to uniform background; the dashed line shows background without disturbance. x^* denotes the position of localized disturbance and $\Delta F(x^*)=1-F(x^*)$ is the change of the condensate density because of condensate nonuniformity. d is the characteristic scale of the initial disturbance. The horizontal lines of different width indicate positions of eigenvalues λ_n and the width of each line characterizes the lifetime of the corresponding solitons (the thicker a line is, the smaller is the lifetime). Dimensionless variables are defined in the text.

Equation (20) has a formal analogy with a stationary Schrödinger equation for a quantum-mechanical motion of a particle in the "energy-dependent" potential, i.e., the potential depending on the spectral parameter λ. According to the above-mentioned independence of the eigenvalues λ_n on time, they can be calculated with the use of the initial distributions $\rho(x,0)$ and $v(x,0)$. Since we are interested in such initial data which give rise to creation of large number of solitons, this means that functions $\rho(x,0)$ and $v(x,0)$ correspond to problem (20) with a large number of eigenvalues. Then the quasiclassical approach can be used for their calculation [17]. To clarify this method, we have shown schematically in Fig. 1 a plot of the "Riemann invariant"

$$\lambda^+=-v(x,0)/2+\sqrt{\rho(x,0)}, \tag{21}$$

which plays a role of the "quantum-mechanical potential" for problem (20). (The second Riemann invariant $\lambda^-=-v(x,0)/2-\sqrt{\rho(x,0)}$ can be considered in the same way.) The Riemann invariant for the same disturbance but with respect to uniform background $F(x)=1$ is shown for comparison by thin solid line. We see that in both cases there is a "potential well," but for the nonuniform Thomas-Fermi background case the eigenvalues acquire imaginary part ("decay width") due to tunneling effect. This means that dark solitons in confined condensate have a finite lifetime. Nevertheless, it makes sense to speak about solitons in a confined condensate, if their lifetimes τ_n are much greater than the period $\sim 1/\nu$ of their oscillations. It is clear that "shallow" solitons with small τ_n do not survive in the confined condensate, so that λ_n with values close to the top of the "potential barrier" do not correspond to any real solitons. On the contrary, in the case of the uniform background discussed in Ref. [17] all eigenvalues correspond to real solitons appearing eventually from the initial pulse. In multisoliton problem there is also one more scale of time equal to time of formation of solitons from the initial pulse. For deep enough solitons it can be estimated by the order of magnitude as time necessary for solitons with velocity $|V_n|=2|\lambda_n|$ (see below) to pass the distance equal to the width d of the initial problem. For the problems under consideration, this time $\sim d/2|\lambda_n|$ must be much less than the period $\sim 1/\nu$. Thus, we are interested in the eigenvalues λ_n which satisfy inequalities

$$\tau_n \gg \frac{1}{\nu} \gg \frac{d}{2|\lambda_n|}. \tag{22}$$

It is clear that there are no such reservations in the case of uniform background [17] where formally $\tau_n=\infty$ and one can wait long enough to observe formation of soliton with any value of λ_n.

To calculate the real parts of the eigenvalues λ_n corresponding to deep solitons, one can use the generalized Bohr-Sommerfeld quantization rule

$$\frac{1}{\varepsilon}\int_{x_n^-}^{x_n^+}\sqrt{\left(\lambda_n+\frac{1}{2}v(x,0)\right)^2-\rho(x,0)}\,dx=\pi\left(n+\frac{1}{2}\right),$$

$$n=0,1,2,\ldots,M, \tag{23}$$

with given initial distributions $\rho(x,0)$ and $v(x,0)$. We suppose here that the integrand has only one maximum and x_n^+ and x_n^- are the points where the integrand function vanishes: they depend on λ_n and are chosen such that relationship (23) is satisfied (see Fig. 1). Analytical form of each emerging soliton in an asymptotic region where it is well separated from other solitons (i.e., in the limit $t\to\infty$) is expressed in terms of λ_n as follows:

$$\rho_s^{(n)}(x,t)=\rho_0-\frac{\rho_0-\lambda_n^2}{\cosh^2[\sqrt{\rho_0-\lambda_n^2}\,(x-2\lambda_n t)/\varepsilon]}, \tag{24}$$

$$v_s^{(n)}(x,t)=\lambda_n(\rho_0/\rho_s^{(n)}-1). \tag{25}$$

As is clear, formulas (24) and (25) represent the one-soliton solution which parameters such as density ρ_0 and phase difference ϑ_n at $\pm\infty$ are connected by the relation $\lambda_n=\sqrt{\rho_0}\cos(\vartheta_n/2)$ [17,19]. Thus, the last formula allows one to find initial values of ϑ_n for solitons emerging from the dark excitation of the condensate with given initial distributions of $\rho(x,0)$ and $v(x,0)$ against a constant uniform background.

If the background is not uniform, then solution of Eq. (13) can be searched in the form of an initial excitation $\Phi(x,t)$ against an inhomogeneous background $F(x)$ [19], i.e., in the form

$$\Psi(x,t)=F(x)\Phi(x,t), \tag{26}$$

where $\Phi(x,0)$ is given by Eq. (17).

In order to use ansatz (26) it is natural to choose $F(x)$ such that the resulting equation for $\Phi(x,t)$ would be close to the NLS equation [19]. This can be achieved by requiring $F(x)$ to be an eigenfunction of the nonlinear spectral problem

$$\varepsilon^2F_{xx}+\left(\varepsilon\omega_b-\frac{(\nu x)^2}{2}\right)F-2\rho_0F^3=0, \tag{27}$$

$$\lim_{x\to\pm\infty}F(x)=0, \tag{28}$$

which satisfies the following normalization conditions:

$$F(0)=1,\quad F_x(0)=0. \tag{29}$$

In Eq. (27) ω_b is an eigenvalue.

In the case when an inhomogeneous background changes in space in the intervals of integration in Eq. (23), we can apply the same method with ρ_0 replaced by the value of the background density $F^2(x^*)$ at the place x^* of the localized initial excitation (see Fig. 1).

We have used this approach for finding soliton parameters for different types of initial excitations: (i) excitations of the density $\rho(x)$, (ii) excitation of the hydrodynamic velocity $v(x)$ (“phase imprinting method”), and (iii) collision of condensates.

TABLE I. Parameters of solitons created from initial intensity disturbance.

n	$\lambda^{(n)}$	$\vartheta^{(n)}$	$a_{theor}^{(n)}$	$a_{num}^{(n)}$
0	0.41	2.29	2.74	2.48
1	0.65	1.72	4.35	4.23
2	0.80	1.28	5.33	5.54

A. Density disturbance

For illustration of the process of solitons formation from the density disturbance, the initial data were taken in the form

$$\rho(x,0)=\left(1-\frac{\alpha}{\cosh(x)}\right)^2,\quad v(x,0)=0, \tag{30}$$

where the parameter α measures the strength of the disturbance. We have chosen the following values of the parameters: $\alpha=0.8$, $\nu=0.3$, and $\varepsilon=0.3$. The values of λ_n for the three deepest solitons calculated with the use the Bohr-Sommerfeld rule (23) are shown in Table I together with the corresponding values of ϑ_n and amplitudes $a_{theor}^{(n)}=2\lambda^n/\nu$ of their oscillatory motion (see Ref. [19]). In the last column the amplitudes of the oscillatory motion found from numerical solution of Eq. (4) with the initial data (30) are shown. The discrepancy is less than 10% and is caused, apparently, by the fact that in our case the created solitons did not yet reach the asymptotic values of their velocities $V_n=2\lambda_n$. Besides that, numerical calculations show that the “initial coordinates” of solitons created from the initial pulse cannot be identified exactly with $X(0)=0$. This is the reason why solitons created from one initial pulse do not reverse simulta-

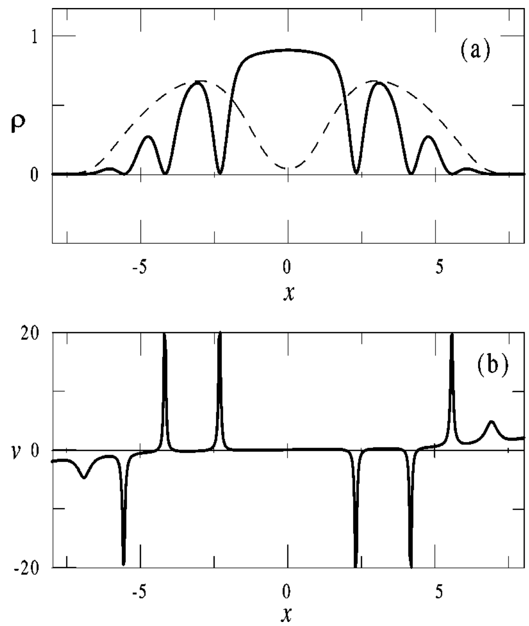

FIG. 2. Space distribution of the density of a BEC (a) and of its hydrodynamic velocity (b) in the harmonic trapped potential with $\nu=0.3$ at time $t=4.24$ (solid line) with initial excitation taken in the form of pulse (30) (dashed line) with $\rho_0=1$, $\alpha=0.8$, and $\varepsilon=0.3$. Dimensionless variables are defined in the text.

neously their directions of motion even during the first period of oscillations in the confined condensate. This phenomenon is illustrated in Fig. 2, where the density $\rho(x,t)$ and hydrodynamic velocity $v(x,t)$ of the condensate are shown as functions of x at the moments when two solitons in each train have already reversed the direction of their propagation and are moving to the center, while the other two solitons are still moving from the center. This process of "solitons' reflection from the potential well" takes about 20% of the whole period of their oscillations.

B. Phase disturbance

For illustration of the process of soliton formation by the phase imprinting method, we have chosen the initial conditions in the form

$$\rho(x,0)=F^2(x), \quad v(x,0)=\frac{\alpha}{\cosh^2(\kappa x)}. \tag{31}$$

Again the soliton parameters can be calculated with the use of the Bohr-Sommerfeld quantization rule (23) and their values correspond well to numerical simulation. In particular, the Bohr-Sommerfeld rule gives correct number of solitons and signs of their initial velocities. As one can see in Fig. 3, at first the deepest soliton moves to the right but the hydrodynamic velocity distribution corresponds to its motion to the left. Only when the local-density minimum touches the x axis, the velocity distribution makes a flip and after that it corresponds to the predicted direction of the soliton propagation (see Fig. 4). All solitons predicted by the Bohr-Sommerfeld quantization rule can be observed during some

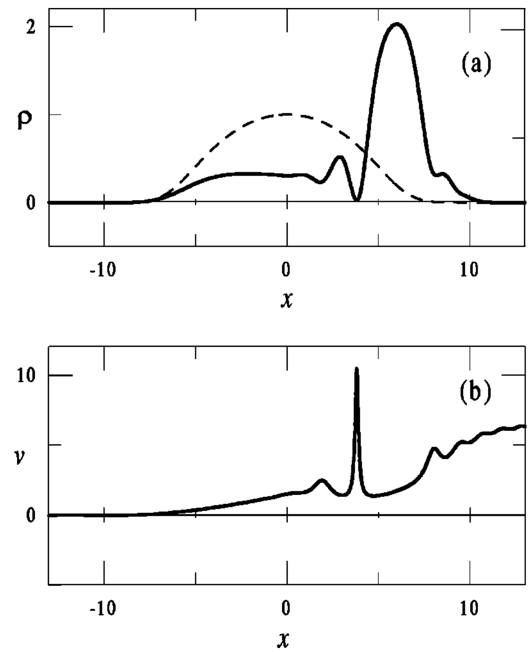

FIG. 3. Space distribution of density of BEC (a) and its hydrodynamic velocity (b) in the harmonic trapped potential with $\nu=0.3$ at time $t=1$ without (dashed line) and with (solid line) initial excitation that is taken as a phase step with $\rho_0=1$, $\alpha=4$, $\kappa=0.4$, and $\varepsilon=1$. Dimensionless variables are defined in the text.

time interval after their formation (see Figs. 4 and 5). However, in the case of initial data (31) we also observe strong nonsoliton contribution to excitation of the condensate (see Fig. 3) which leads to much more complicated picture of its evolution. One may say that in this case solitons move along background varying with time and the influence of this time dependence is not small contrary to the previous case of the density disturbance. As a result, the motion of solitons cannot be described as (almost) harmonic oscillations along con-

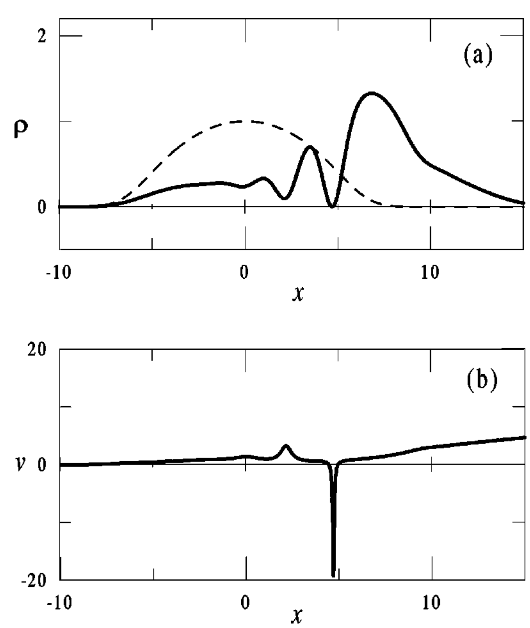

FIG. 4. The same as in Fig. 3, at $t=1.5$.

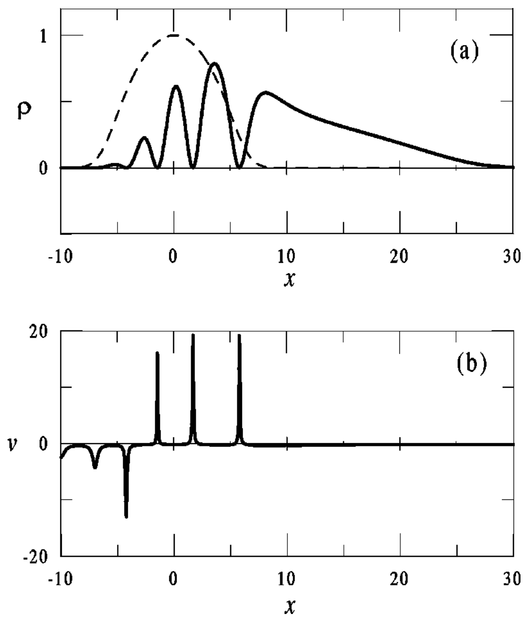

FIG. 5. The same as in Fig. 3, at $t=4$.

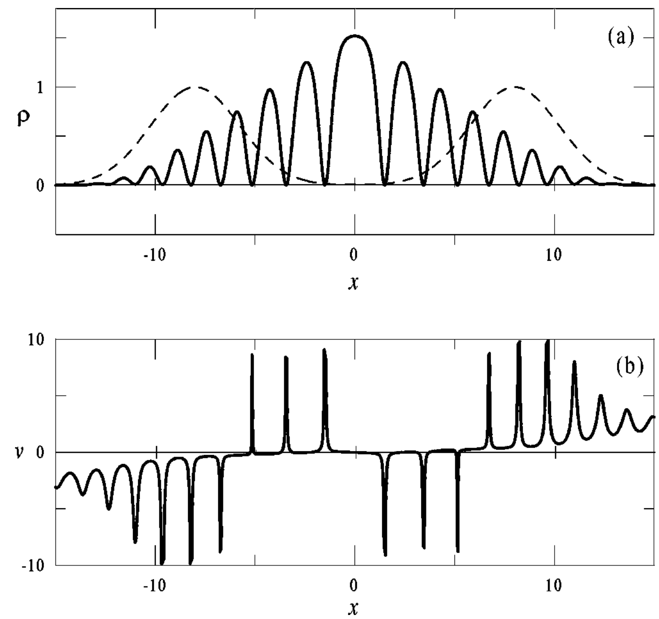

FIG. 6. Formation of dark solitons (solid line) during the collision of two initially separated condensates (dashed line) corresponding to the initial data (32) with parameters $\kappa=0.5$, $\xi=8$, and $\varepsilon=0.5$ in harmonic trap potential $\nu=0.2$ at $t=4$. (a) The density of BEC and (b) hydrodynamic velocity. Dimensionless variables are defined in the text.

stant nonuniform background. But even in this case quasiclassical method of finding the solitons' parameters yields quite accurate description of solitons' motion for times less than $2\pi/\nu$.

C. Collision of two separated condensates

Formation of dark solitons can be observed also during collision of two separated condensates that move under influence of the trap potential [6]. Taking the initial conditions in the form

$$\rho(x,0)=\exp[-\kappa(x-\xi)^2]+\exp[-\kappa(x+\xi)^2],$$

$$v(x,0)=0, \tag{32}$$

we have solved Eq. (4) numerically. It was found that these two condensates oscillate in the trap potential and interact with each other in quite complicated way when they overlap with each other. Since the initial data (32) lead to a number of eigenvalues $\lambda^{(n)}$, one may expect that during the collision of two condensates the corresponding number of dark solitons must be observed. This is indeed the case as one can see in Fig. 6, where the density and the hydrodynamic velocity distributions are shown at the moment of maximal overlap of two condensates whose initial density distributions are indicated by dashed lines. The number of solitons matches very well with that predicted by the Bohr-Sommerfeld quantization rule, but their motion cannot be presented as propagation with slowly changing parameters along constant nonuniform background.

The plot in Fig. 6(a) can be considered as an interference pattern of two coherent condensates after their overlap [23]. From Eq. (23) we can estimate the eigenvalues as $\lambda_n d \simeq \varepsilon\pi n$, where d is the initial distance between condensates. Then velocities of solitons $v_n=2\lambda_n$ differ from one another by $\Delta v=2\varepsilon\pi/d$ and distances between interference fringes are of order of magnitude

$$\Delta x \simeq \frac{2\pi\varepsilon}{d}t. \tag{33}$$

After transformation to dimensional units we reproduce the estimate $\Delta x=2\pi\hbar t/(md)$ of Ref. [23].

Thus, the quasiclassical approach provides simple and effective method for calculation of the parameters of solitons arising from large enough initial disturbance. This method can be used for estimation of these parameters in the present day experiments with BEC solitons.

IV. DISCUSSION AND CONCLUSION

In the present paper we have investigated the evolution of localized excitations in a 1D BEC with a positive scattering length confined by a harmonic trap potential. It has been shown that the existence of an inhomogeneous background becomes especially important when initially multisoliton pulses are under consideration. In comparison to the integrable case with constant background, new temporal scales appear in the problem. They are associated with the harmonic-oscillator frequency and finite lifetime of solitons.

The lifetime decreases with the soliton amplitude, which leads to rather rapid disappearance of shallow dark solitons. Generally speaking, a soliton with a small amplitude can even lose its meaning when it is considered against a nonuniform background.

The generalized Bohr-Sommerfeld quantization rule provides simple method for evaluation of parameters of solitons emerging eventually from large initial disturbance. It permits one to find number of solitons as well as their initial amplitudes and velocities in agreement with numerical simulations.

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