Elliptic Rydberg states as direction indicators

Netanel H. Lindner, Asher Peres, and Daniel R. Terno*

Department of Physics, Technion—Israel Institute of Technology, 32 000 Haifa, Israel

(Received 28 May 2003; published 9 October 2003)

The orientation in space of a Cartesian coordinate system can be indicated by the two vectorial constants of motion of a classical Keplerian orbit: the angular momentum and the Laplace-Runge-Lenz vector. In quantum mechanics, the states of a hydrogen atom that mimic classical elliptic orbits are the coherent states of the SO(4) rotation group. It is known how to produce these states experimentally. They have minimal dispersions of the two conserved vectors and can be used as direction indicators. We compare the fidelity of this transmission method with that of the idealized optimal method.

DOI: 10.1103/PhysRevA.68.042308 PACS number(s): 03.67.Hk, 03.65.Ta, 03.65.Ud

I. UNSPEAKABLE QUANTUM INFORMATION

Information theory usually deals with the transmission of a sequence of discrete symbols, such as 0 and 1. Even if the information to be transmitted is of continuous nature, such as the position of a particle, it can be represented with arbitrary accuracy by a string of bits. However, there are situations where information cannot be encoded in such a way. For example, the emitter (conventionally called Alice) wants to indicate to the receiver (Bob) a direction in space. If they have a common coordinate system to which they can refer, or if they can create one by observing distant fixed stars, Alice simply communicates to Bob the components of a unit vector **n** along that direction, or its spherical coordinates θ and ϕ. But if no common coordinate system has been established, all she can do is to send a real physical object, such as a gyroscope, whose orientation is deemed stable.

In the quantum world, the role of the gyroscope is played by a system with large spin. For example, Alice can send angular momentum eigenstates satisfying $\mathbf{n}\cdot\mathbf{J}|\psi\rangle = j|\psi\rangle$. This is essentially the solution proposed by Massar and Popescu [1] who took N parallel spins, polarized along **n**. This, however, is not the most efficient procedure: for two spins, a higher accuracy is achieved by preparing them with opposite polarizations [2]. For more than two spins, optimal results are obtained with entangled states [3,4].

The above discussion can be generalized to the transmission of a Cartesian frame. If N spins are available, one can encode a Cartesian frame in an entangled state of these spins, as in Ref. [5]. However, a more accurate transmission is then obtained if Alice uses half of the spins to indicate the x axis, and the other half for her y axis [6]. In this case the x and y directions found by Bob may not be exactly perpendicular and some adjustment will be needed to obtain Bob's best estimate of the x and y axes. Finally, the z axis can be inferred from the estimates of the x and y axes.

However, it is not possible to proceed in this way if a *single* quantum messenger is available. The optimal transmission of a Cartesian frame by a hydrogen atom (formally, a spinless particle in a Coloumb potential) was derived by Peres and Scudo [7]. The results of Ref. [5] can also be used for that case if one considers the angular-momentum eigenstates to be those of the atom, rather than those of N spins.

In this paper we show how to transmit a Cartesian frame by using elliptic Rydberg states. These are the quantum-mechanical analogs of a classical Keplerian orbit, and it is known how to produce these states experimentally. Elliptic Rydberg states, just as their classical counterparts, define three orthogonal directions in space, and thus are natural candidates for encoding a Cartesian frame.

In a real experimental situation, relativistic effects cannot be neglected: the spin-orbit coupling and spin-spin coupling remove the degeneracy of the Rydberg energy levels and cause the ellipse to precess appreciably within milliseconds [8]. This precession may perhaps be controlled. However, coupling to the radiation field causes the Rydberg levels to decay irreversibly and sets an absolute time limit on such an experiment. Fortunately, the mean lifetime of the levels is much longer than the precession time [9],

$$T_{\mathrm{rad}}/T_{\mathrm{p}} = 8\pi e^2/3\hbar c \approx 16.40, \tag{1}$$

and the experiment, although difficult, may be feasible.

Similar situations also arise with other types of atoms, Yeazell and Stroud [10] excited a sodium atom into an angularly localized wave packet and subsequently had its localization direction probed by an ionizing field. The states excited in that experiment are good approximations of the SO(4) coherent states, and the precession of the Laplace-Runge-Lenz (LRL) vector was dominated by relativistic effects just as it is in hydrogen.

In the following section we discuss the properties of quantum elliptic states. Section III deals with the transmission of one direction by means of them, and in Secs. IV we use them to transmit two orthogonal axes (and thus a Cartesian frame). In Sec. III and IV the detection procedure is based on SO(3) coherent states as in Ref. [7]. SO(4) coherent states are employed in Sec. V to produce a positive operator valued measure (POVM) which enables the use of elliptic states for the transmission of two directions that are not orthogonal. Even when these states are used by Alice to transmit two orthogonal axes, the two directions found by Bob are not necessarily perpendicular and further adjustment is needed, as explained above. As shown in the Appendix, these adjustments increase the fidelity of transmission of two or-

*Also at Perimeter Institute, Waterloo, Ontario, Canada.

thogonal axes. However, a higher fidelity is achieved with a POVM based on the SO(3) rotation group, especially when the energy quantum number n is large.

II. CONSTRUCTION OF AN ELLIPTIC STATE

A classical bounded Keplerian orbit in a potential $-k/r$ can be defined by its constants of motion: the energy $E<0$, the angular momentum $\mathbf{L}$ which is an axial vector perpendicular to the plane of the orbit, and the Laplace-Runge-Lenz (LRL) vector [11]

$$\mathbf{K}=(-2H)^{-1/2}(\mathbf{p}\times\mathbf{L}-\mu k\mathbf{r}/r), \tag{2}$$

where μ is the particle's reduced mass, and we introduced a prefactor $(-2H)^{-1/2}$ for later convenience. This prefactor, which is a constant of motion, does not appear in the usual definition of the LRL vector [11]. The Hamiltonian is $H=\mathbf{p}^2/2\mu-k/r$. We consider only bounded motion for which the energy E, which is the numerical value of H, is negative. The classical orbit is then an ellipse, and the LRL vector is a polar vector directed along its major axis. It satisfies

$$\mathbf{L}\cdot\mathbf{K}=0, \tag{3}$$

and

$$\mathbf{K}^2+\mu\mathbf{L}^2=-\mu^2k^2/2E. \tag{4}$$

Because of these relations, only five out of the seven constants are independent. They uniquely determine the shape and orientation of the ellipse. Its eccentricity [11] is

$$e=|\mathbf{K}|\sqrt{-2E}/\mu k. \tag{5}$$

We now turn to the quantum version. We use natural units, $\mu=k=\hbar=1$, so that the energy levels for bound states are $E=-1/2n^2$, where n is an integer. The operator $\mathbf{K}$ is defined by

$$\mathbf{K}=(-2H)^{-1/2}[\tfrac{1}{2}(\mathbf{p}\times\mathbf{L}-\mathbf{L}\times\mathbf{p})-\mathbf{r}/r]. \tag{6}$$

Note that H commutes with $[\frac{1}{2}(\mathbf{p}\times\mathbf{L}-\mathbf{L}\times\mathbf{p})-\mathbf{r}/r]$. The commutation relations for the operators $\mathbf{L}$ and $\mathbf{K}$ [12] are

$$[L_i,K_j]=i\epsilon_{ijk}K_k, \tag{7}$$

$$[K_i,K_j]=i\epsilon_{ijk}L_k. \tag{8}$$

Together with $[L_i,L_j]=i\epsilon_{ijk}L_k$, these are the commutation rules of infinitesimal rotations in four-dimensional Euclidean space, which leave the nth energy-level subspace invariant [13]. The coherent states of SO(4), i.e., the states for which the dispersion of $\mathbf{L}^2+\mathbf{K}^2$ is minimal, can be built from the coherent states of SO(3), since $\mathrm{SO}(4)=\mathrm{SO}(3)\times\mathrm{SO}(3)$.

Define

$$\mathbf{J}_1=\tfrac{1}{2}(\mathbf{L}-\mathbf{K}) \quad\text{and}\quad \mathbf{J}_2=\tfrac{1}{2}(\mathbf{L}+\mathbf{K}). \tag{9}$$

These two operators have the commutation relations of two independent three-dimensional angular momenta:

$$[J_{1i},J_{1j}]=i\epsilon_{ijk}J_{1k}, \tag{10}$$

$$[J_{2i},J_{2j}]=i\epsilon_{ijk}J_{2k}, \tag{11}$$

$$[J_{1i},J_{2j}]=0. \tag{12}$$

Instead of the classical equations (3) and (4), we now have [14]

$$\mathbf{L}\cdot\mathbf{K}=\mathbf{K}\cdot\mathbf{L}=0 \tag{13}$$

and

$$\mathbf{L}^2+\mathbf{K}^2=-1-1/2H=n^2-1, \tag{14}$$

where the last form of the equality holds for energy eigenstates. In the classical limit, $n\gg1$, Eq. (14) reduces to the classical one (4).

In the rest of this paper we consider only energy eigenstates. Owing to Eq. (9), we have

$$j_1(j_1+1)=j_2(j_2+1)=\tfrac{1}{4}(n^2-1), \tag{15}$$

where j_1 and j_2 are the quantum numbers referring to the operators $\mathbf{J}_1^2$ and $\mathbf{J}_2^2$, respectively. It follows that j_1 and j_2 are equal: $j_1=j_2\equiv j$ and $j(j+1)=\frac{1}{4}(n^2-1)$, so that

$$j=\tfrac{1}{2}(n-1). \tag{16}$$

The coherent states of a three-dimensional angular momentum will be denoted by $|J,\mathbf{u}\rangle$. They obey $\mathbf{u}\cdot\mathbf{J}|J,\mathbf{u}\rangle=j|J,\mathbf{u}\rangle$ for an arbitrary classical unit vector $\mathbf{u}$. For the coherent states the dispersion $\Delta\mathbf{J}=(\langle\mathbf{J}^2\rangle-\langle\mathbf{J}\rangle^2)^{1/2}$ is minimal: $(\Delta\mathbf{J})^2=j$. In particular, $\Delta J_u=0$ and $\Delta J_\perp=\sqrt{j/2}$, where $J_u=\mathbf{J}\cdot\mathbf{u}$ and $J_\perp=\mathbf{J}\cdot\mathbf{v}$ with $\mathbf{v}\perp\mathbf{u}$. The coherent states of SO(3) are obtained by a rotation of a fiducial coherent state $|J,\mathbf{z}\rangle$,

$$|J,\mathbf{u}_{\theta\phi}\rangle=e^{-iL_z\phi}e^{-iL_y\theta}|J,\mathbf{z}\rangle. \tag{17}$$

The coherent states of SO(4) are now obtained as direct products of coherent states for each of the SO(3) subgroups,

$$|n,\mathbf{u}_1\mathbf{u}_2\rangle=|J_1,\mathbf{u}_1\rangle\otimes|J_2,\mathbf{u}_2\rangle, \tag{18}$$

where the unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are again classical. The coherent state $|J_1,\mathbf{u}_1\rangle$ obeys

$$\mathbf{u}_1\cdot\mathbf{J}_1|J_1,\mathbf{u}_1\rangle=j|J_1,\mathbf{u}_1\rangle=\tfrac{1}{2}(n-1)|J_1,\mathbf{u}_1\rangle, \tag{19}$$

and likewise, we have

$$\mathbf{u}_2\cdot\mathbf{J}_2|J_2,\mathbf{u}_2\rangle=j|J_2,\mathbf{u}_2\rangle=\tfrac{1}{2}(n-1)|J_2,\mathbf{u}_2\rangle. \tag{20}$$

As from now, we shall omit the symbols n,J_1, and J_2 in state vectors, since the quantum numbers n,j_1,j_2 have fixed values, related by Eq. (15). For example, the state $|nlm\rangle$ which obeys $H|nlm\rangle=n|nlm\rangle$, $\mathbf{L}^2|nlm\rangle=l(l+1)|nlm\rangle$, and $L_z|nlm\rangle=m|nlm\rangle$ will be written simply as $|lm\rangle$, $|J,\mathbf{u}\rangle$ becomes $|\mathbf{u}\rangle$, and $|J,\mathbf{z}\rangle$ becomes $|jj\rangle$, etc. The symbol j will always denote the fixed value $j=\frac{1}{2}(n-1)$.

Owing to Eq. (9), the dispersion of $\mathbf{L}^2+\mathbf{K}^2$ is minimal for coherent states [14]:

$$(\Delta \mathbf{L})^2+(\Delta \mathbf{K})^2=2[(\Delta \mathbf{J}_1)^2+(\Delta \mathbf{J}_2)^2]=2(n-1). \quad (21)$$

To obtain the expansion of the coherent state $|n,\mathbf{u}_1\mathbf{u}_2\rangle$ in the familiar nlm basis, we first expand each of the $|\mathbf{u}_i\rangle$ in Eq. (18):

$$|\mathbf{u}_i\rangle=\sum_{m=-j}^{j} D_m^j(\theta_i\,\phi_i)|jm\rangle, \quad i=1,2, \quad (22)$$

where the $D_m^j(\theta_i\,\phi_i)$ are related to the usual rotation matrices [15]:

$$D_m^j(\theta\phi)\equiv\mathcal{D}^{(j)}(\phi\theta 0)_{m\,j}, \quad (23)$$

$$=\binom{2j}{j+m}^{1/2}\left(\cos\frac{\theta}{2}\right)^{j+m}\left(\sin\frac{\theta}{2}\right)^{j-m} e^{-im\phi}. \quad (24)$$

Substitution into Eq. (18) gives

$$|\mathbf{u}_1\mathbf{u}_2\rangle=\sum_{m_1=-j}^{j}\sum_{m_2=-j}^{j} D_{m_1}^j(\theta_1\phi_1)D_{m_2}^j(\theta_2\phi_2)|jm_1\rangle\otimes|jm_2\rangle. \quad (25)$$

We then use the angular-momentum addition formula

$$|jm_1\rangle\otimes|jm_2\rangle=\sum_{l=0}^{2j}\sum_{m=-l}^{l} C_{m_1m_2\,m}^{jj\,l}|lm\rangle, \quad (26)$$

where $C_{m_1m_2\,m}^{j_1j_2\,l}$ is the Clebsch-Gordan coefficient [16] which vanishes for $m\neq m_1+m_2$. Combining Eqs. (25) and (26), we finally get

$$|\mathbf{u}_1\mathbf{u}_2\rangle=\sum_{l=0}^{2j}\sum_{m=-l}^{l}\left(\sum_{m_1=-j}^{j}\sum_{m_2=-j}^{j} D_{m_1}^j(\theta_1\phi_1)\times D_{m_2}^j(\theta_2\phi_2)C_{m_1m_2\,m}^{jj\,l}\right)|lm\rangle. \quad (27)$$

The classical orbit that corresponds to the coherent state $|\mathbf{u}_1\mathbf{u}_2\rangle$, in the limit of large n, can be obtained as follows. From Eq. (9), we have

$$\mathbf{L}=(\mathbf{J}_1+\mathbf{J}_2) \quad \text{and} \quad \mathbf{K}=(\mathbf{J}_2-\mathbf{J}_1). \quad (28)$$

Let ζ be half the angle between $\mathbf{u}_1$ and $\mathbf{u}_2$, i.e., $\sin\zeta\equiv|\mathbf{u}_1\times\mathbf{u}_2|/|\mathbf{u}_1+\mathbf{u}_2|$, and define three orthogonal classical unit vectors

$$\boldsymbol{\ell}\equiv\frac{\mathbf{u}_1+\mathbf{u}_2}{|\mathbf{u}_1+\mathbf{u}_2|}, \quad \mathbf{k}\equiv\frac{\mathbf{u}_2-\mathbf{u}_1}{|\mathbf{u}_2-\mathbf{u}_1|}, \quad (29)$$

and $\mathbf{w}\equiv\boldsymbol{\ell}\times\mathbf{k}$. Denoting by $\mathbf{u}_{1\perp}$ an arbitrary vector orthogonal to $\mathbf{u}_1$, we have

$$\mathbf{J}_1=(\mathbf{J}_1\cdot\mathbf{u}_1)\mathbf{u}_1+(\mathbf{J}_1\cdot\mathbf{u}_{1\perp})\mathbf{u}_{1\perp}, \quad (30)$$

$$\mathbf{J}_1\cdot\mathbf{u}_2=(\mathbf{J}_1\cdot\mathbf{u}_1)(\mathbf{u}_1\cdot\mathbf{u}_2)+(\mathbf{J}_1\cdot\mathbf{u}_{1\perp})(\mathbf{u}_{1\perp}\cdot\mathbf{u}_2). \quad (31)$$

Then from

$$\langle\mathbf{u}_1|(\mathbf{u}_1\cdot\mathbf{J}_1)|\mathbf{u}_1\rangle=j \quad (32)$$

and

$$\langle\mathbf{u}_1|\mathbf{u}_{1\perp}\cdot\mathbf{J}_1|\mathbf{u}_1\rangle=0, \quad (33)$$

we get

$$\langle\mathbf{u}_1|(\mathbf{u}_2\cdot\mathbf{J}_1)|\mathbf{u}_1\rangle=\langle\mathbf{u}_2|(\mathbf{u}_1\cdot\mathbf{J}_2)|\mathbf{u}_2\rangle, \quad (34)$$

$$=j\,\mathbf{u}_1\cdot\mathbf{u}_2=j\cos 2\zeta. \quad (35)$$

Noting that

$$|\mathbf{u}_1+\mathbf{u}_2|=2|\cos\zeta| \quad (36)$$

and

$$|\mathbf{u}_1-\mathbf{u}_2|=2|\sin\zeta|, \quad (37)$$

we obtain from Eq. (35) the expectation values of the components of $\mathbf{K}$ and $\mathbf{L}$ along the directions of $\mathbf{k}$, $\boldsymbol{\ell}$, and $\mathbf{w}$, for the coherent state $|\mathbf{u}_1\mathbf{u}_2\rangle$,

$$\langle K_k\rangle=(n-1)\sin\zeta, \quad (38)$$

$$\langle L_\ell\rangle=(n-1)\cos\zeta, \quad (39)$$

where $K_k\equiv\mathbf{k}\cdot\mathbf{K}$, etc. In the perpendicular directions the expectation values vanish:

$$\langle K_\ell\rangle=\langle K_w\rangle=\langle L_k\rangle=\langle L_w\rangle=0. \quad (40)$$

From Eqs. (38)–(40) we see that in the limit of large n, the coherent state $|\mathbf{u}_1\mathbf{u}_2\rangle$ corresponds to a classical elliptic trajectory in the **k**-**w** plane with the LRL vector in the **k** direction, the angular momentum in the $\boldsymbol{\ell}$ direction, and eccentricity $e=\langle K_k\rangle/(n-1)=\sin\zeta$, which is the quantum-mechanical analog to Eq. (5). The unit vector **w** is parallel to the minor axis of the ellipse.

III. TRANSMISSION OF ONE DIRECTION

We now turn to the use of elliptic wave functions as direction indicators. Consider two observers (Alice and Bob) who do not have a common reference frame. Alice wants to indicate to Bob her z axis by using an elliptic Rydberg state. In the following section, we shall likewise discuss the transmission of two orthogonal axes. We use as much as possible the same notations in both sections. Alice's signal is

$$|A\rangle=\sum_{l=0}^{2j}\sum_{m=-l}^{l} a_{lm}|lm\rangle, \quad (41)$$

where

$$\sum_{l=0}^{2j}\sum_{m=-l}^{l}|a_{lm}|^2=1. \quad (42)$$

Bob's detectors have labels $\psi\theta\phi$ which indicate the unknown Euler angles relating his Cartesian axes to those used by Alice. The mathematical representation of his apparatus is a POVM,

$$\int dE(\psi\theta\phi)=\mathbb{1}, \tag{43}$$

where

$$dE(\psi\theta\phi)=d_{\psi\theta\phi}U(\psi\theta\phi)|B\rangle\langle B|U^{\dagger}(\psi\theta\phi), \tag{44}$$

and $d_{\psi\theta\phi}=\sin\theta d\psi d\theta d\phi/8\pi^2$ is the SO(3) Haar measure for Euler angles [15]. As usual, $U(\psi\theta\phi)$ is the unitary operator for a rotation by Euler angles $\psi\theta\phi$, and $|B\rangle$ is Bob's fiducial vector defined as in Ref. [7],

$$|B\rangle=\sum_{l=0}^{2j}\sqrt{2l+1}\sum_{m=-l}^{l}b_{lm}|lm\rangle, \tag{45}$$

where for each l,

$$\sum_{m=-l}^{l}|b_{lm}|^2=1. \tag{46}$$

Note that Eq. (41) was written with Alice's notation, while Eq. (45) is in Bob's notation (recall that they use different coordinate systems).

Optimizing the transmission fidelity, defined by Eq. (49) below, leads [7] to

$$b_{lm}=a_{lm}\left(\sum_{n=-l}^{l}|a_{ln}|^2\right)^{-1/2} \tag{47}$$

for each l. Since $|B\rangle$ is a direct sum of vectors, one for each value of l, then likewise $U(\psi\theta\phi)$ is a direct sum with one term for each irreducible representation:

$$U(\psi\theta\phi)=\sum_{l}\oplus\mathcal{D}^{(l)}(\psi\theta\phi), \tag{48}$$

where the $\mathcal{D}^{(l)}(\psi\theta\phi)$ are the usual irreducible unitary rotation matrices [15]. A generalization of Schur's lemma [17] confirms that Eq. (44) is indeed satisfied, owing to the coefficients $\sqrt{2l+1}$ in Eq. (45).

The fidelity of the transmission of a single direction is defined as usual:

$$F=\langle\cos^2(\omega/2)\rangle=\tfrac{1}{2}(1+\langle\cos\omega\rangle), \tag{49}$$

where ω is the angle between the direction indicated by Alice and the one that is estimated by Bob. If Alice indicates her z axis, we thus want to maximize $\langle\cos\omega_z\rangle$. Following the method of Peres and Scudo [7], we define Euler angles $\alpha\beta\gamma$ whose effect is rotating Bob's Cartesian frame into his *estimate* of Alice's frame, and then rotating back the result by the *true* angles from Alice's to Bob's frame. The angles $\alpha\beta\gamma$ thus indicate Bob's measurement error. Since in this case Bob's estimate refers to Alice's z axis only, the angle ω_z is identical to the second Euler angle β:

$$\langle\cos\omega_z\rangle=\int d_{\alpha\beta\gamma}|\langle A|U(\alpha\beta\gamma)|B\rangle|^2\cos\beta. \tag{50}$$

Let us examine two extreme cases. First, we take Alice's vector to be a circular state, with null eccentricity ($\sin\zeta=0$), i.e.,

$$|A\rangle=|ll\rangle, \tag{51}$$

with

$$l=2j=n-1. \tag{52}$$

Bob's vector is obtained from Eq. (47), which in this case gives

$$|B\rangle=\sqrt{2n-1}|ll\rangle. \tag{53}$$

We then [15] have

$$\langle A|U(\alpha\beta\gamma)|B\rangle=\sqrt{2n-1}e^{i(n-1)(\alpha+\gamma)}\cos^{2(n-1)}\frac{\beta}{2}. \tag{54}$$

Inserting the last equation into Eq. (50) gives

$$\langle\cos\omega_z\rangle=(n-\tfrac{1}{2})\int_0^{\pi}\sin\beta d\beta\cos^{4(n-1)}(\beta/2)\cos\beta, \tag{55}$$

$$=(n-1)/n. \tag{56}$$

The "infidelity" $1-F$, whose typical meaning is Bob's mean-square error [3], is

$$\tfrac{1}{2}(1-\langle\cos\omega_z\rangle)=1/2n. \tag{57}$$

This result is identical to the one obtained by Massar and Popescu [1], when the number of their parallel spins is taken to be $N=2l=2n-2$.

The other extreme case corresponds to a classical elliptical orbit with unit eccentricity, so that $\mathbf{L}=0$. Let $\mathbf{K}$ lie in the $\mathbf{z}$ direction. The corresponding quantum state, denoted by $|K,\mathbf{z}\rangle$, is an extreme Stark state with $\langle L_z\rangle=0$ and maximal $\langle K_z\rangle$:

$$|K,\mathbf{z}\rangle\equiv|-\mathbf{z}\rangle\otimes|\mathbf{z}\rangle=|j,-j\rangle\otimes|jj\rangle. \tag{58}$$

This state satisfies $L_z|K,\mathbf{z}\rangle=0$ and it is also an eigenstate of K_z:

$$K_z|K,\mathbf{z}\rangle=(J_{2z}-J_{1z})|j,-j\rangle\otimes|jj\rangle, \tag{59}$$

$$=(n-1)|K,\mathbf{z}\rangle, \tag{60}$$

owing to

$$(J_{2z}-J_{1z})|j,-j\rangle\otimes|jj\rangle=-J_{1z}|j,-j\rangle\otimes J_{2z}|jj\rangle. \tag{61}$$

In the nlm basis we have

$$|K,\mathbf{z}\rangle=\sum_{l=0}^{2j}\sum_{m=-l}^{l}C_{m_1m_2m}^{jj\,l}|lm\rangle=\sum_{l=0}^{2j}C_{-jj0}^{jj\,l}|l0\rangle, \tag{62}$$

since $m_1=-j$ and $m_2=j$. The fidelity of transmission by this state will be evaluated at the end of this section.

Both the circular state and the extreme Stark state are coherent states of SO(4), but only the circular state is also an angular-momentum coherent state. Moreover, the circular state is symmetric, $\langle l\,l|\mathbf{r}|l\,l\rangle=0$, while the extreme Stark state is not. This can be seen [19] from:

$$\langle nlm|\mathbf{r}|nlm\rangle=\tfrac{2}{3}\langle nlm|\mathbf{K}|nlm\rangle. \tag{63}$$

Let us examine which one of these states gives better results when used by Alice to transmit the directions of her z axis. The overlap between two angular-momentum coherent states [18] is

$$|\langle\mathbf{u}_1|\mathbf{u}_2\rangle|^2=\cos^{4j}(\chi/2), \tag{64}$$

where χ is the angle between the directions of $\mathbf{u}_1$ and $\mathbf{u}_2$. It is noteworthy that the overlap between two extreme Stark states is the same, as we will see shortly. First, a rotation of the $|K,\mathbf{z}\rangle$ state by angles $(\theta\phi)$ gives

$$|K,\mathbf{u}_{\theta\phi}\rangle=e^{-iL_z\phi}e^{-iL_y\theta}|K,\mathbf{z}\rangle, \tag{65}$$

where again the operator $e^{-iL_z\phi}e^{-iL_y\theta}$ performs an active rotation of the vector $|K,\mathbf{z}\rangle$. Using Eq. (28) we have

$$|K,\mathbf{u}_{\theta\phi}\rangle=e^{-i(J_{1z}+J_{2z})\phi}e^{-i(J_{1y}+J_{2y})\theta}|-\mathbf{z}\rangle\otimes|\mathbf{z}\rangle, \tag{66}$$

$$=e^{-iJ_{1z}\phi}e^{-iJ_{1y}\theta}|-\mathbf{z}\rangle\otimes e^{-iJ_{2z}\phi}e^{-iJ_{2y}\theta}|\mathbf{z}\rangle, \tag{67}$$

owing to Eqs. (7) and (8). Thus the rotated extreme Stark state is just

$$|K,\mathbf{u}_{\theta\phi}\rangle=|-\mathbf{u}_{\theta\phi}\rangle\otimes|\mathbf{u}_{\theta\phi}\rangle, \tag{68}$$

where the SO(3) coherent states $|\mathbf{u}_{\theta\phi}\rangle$ are defined as in Eq. (22). This Stark state is an eigenstate of $\mathbf{u}\cdot\mathbf{K}$ with the maximal eigenvalue $n-1$, and it satisfies $\mathbf{k}=\mathbf{u}$, as can be seen from Eq. (29). The overlap between two such states $|\langle K,\mathbf{u}'|K,\mathbf{u}''\rangle|^2$ is

$$|\langle-\mathbf{u}'|-\mathbf{u}''\rangle|^2|\langle\mathbf{u}'|\mathbf{u}''\rangle|^2, \tag{69}$$

which, by using Eq. (64), is just

$$\cos^{4j_1}(\chi/2)\cos^{4j_2}(\chi/2)=\cos^{4(n-1)}(\chi/2), \tag{70}$$

where χ is the angle between the vectors $\mathbf{u}'$ and $\mathbf{u}''$.

Such a simple expression cannot hold for the overlap of two generic elliptic states whose eccentricities are not 0 or 1. Let a generic elliptic state

$$|\mathbf{u}_1\mathbf{u}_2\rangle=|\mathbf{u}_1\rangle\otimes|\mathbf{u}_2\rangle \tag{71}$$

be an elliptic state with eccentricity $0<e<1$. Unlike the $e=1$ and $e=0$ cases, this state does not define one direction, but two independent ones, $\mathbf{u}_1$ and $\mathbf{u}_2$. If it is rotated by Euler angles $\alpha\beta\gamma$, the result is

$$e^{-iL_z\alpha}e^{-iL_y\beta}e^{-iL_z\gamma}|\mathbf{u}_1\mathbf{u}_2\rangle=U_1|\mathbf{u}_1\rangle\otimes U_2|\mathbf{u}_2\rangle, \tag{72}$$

where

$$U_1=e^{-iJ_{1z}\alpha}e^{-iJ_{1y}\beta}e^{-iJ_{1z}\gamma}, \tag{73}$$

and likewise for U_2. To obtain this result we have used Eq. (28) and the commutation relations (7) and (8). The rotation $e^{-iL_z\alpha}e^{-iL_y\beta}e^{-iL_z\gamma}$ opens an angle χ_1 between the classical vectors $\mathbf{u}_1$ and $R(\alpha\beta\gamma)\mathbf{u}_1$, and an angle χ_2 (which is generally different from χ_1) between $\mathbf{u}_2$ and $R(\alpha\beta\gamma)\mathbf{u}_2$. Here $R(\alpha\beta\gamma)$ denotes the classical rotation matrix [11]. It follows that

$$\langle\mathbf{u}_1\mathbf{u}_2|e^{-iL_z\alpha}e^{-iL_y\beta}e^{-iL_z\gamma}|\mathbf{u}_1\mathbf{u}_2\rangle=\left(\cos\frac{\chi_1}{2}\cos\frac{\chi_2}{2}\right)^{2(n-1)}. \tag{74}$$

Generally, both χ_1 and χ_2 are different from the angle between the directions $\mathbf{k}$ and $\mathbf{k}'=R(\alpha\beta\gamma)\mathbf{k}$, or between the directions ℓ and $\ell'=R(\alpha\beta\gamma)\ell$.

We now calculate the transmission fidelity for the case where Alice sends an extreme Stark state $|K,\mathbf{z}\rangle$. Since $|A\rangle$ contains only $m=0$ terms, so does Bob's fiducial vector

$$b_{lm}=a_{l0}(|a_{l0}|^2)^{-1/2}\delta_{m0}. \tag{75}$$

We thus have

$$b_{lm}=\delta_{m0}\,(a_{l0}/|a_{l0}|), \tag{76}$$

$$|B\rangle=\sum_{l=0}^{n-1}\sqrt{2l+1}(a_{l0}/|a_{l0}|)|l0\rangle. \tag{77}$$

In order to determine $\langle\cos\omega_z\rangle$ in Eq. (50), we note that

$$\langle A|U(\alpha\beta\gamma)|B\rangle=\sum_{l=0}^{n-1}\sqrt{2l+1}\,a_{l0}^*b_{l0}\langle l0|\mathcal{D}^{(l)}(\alpha\beta\gamma)|l0\rangle,$$

$$=\sum_{l=0}^{n-1}\sqrt{2l+1}|a_{l0}|d_{00}^{(l)}(\beta). \tag{78}$$

We insert this expression into Eq. (50). The result, obtained by using Eqs. (19)–(21) of Ref. [3], is

$$\langle\cos\omega_z\rangle=\sum_{kl}A_{lk}|a_{l0}a_{k0}|, \tag{79}$$

where A_{lk} is a real symmetric matrix whose nonvanishing elements are

$$A_{l,l-1}=A_{l-1,l}=l/\sqrt{4l^2-1}, \tag{80}$$

and $a_{l\,0}=C_{-jj0}^{jj\,l}$. The results are summarized in Fig. 1, in which the mean-square error is plotted versus n. The $|K,\mathbf{z}\rangle$ state gives fidelity better than the circular state $|ll\rangle$, but for $n>3$ its fidelity is substantially less than optimal [3,4] and goes asymptotically to $1/(4n-2)$. This raises the question whether it is possible to build a "natural" POVM by setting Bob's vector to $|B\rangle=\sqrt{N}|K,\mathbf{z}\rangle$, so that POVM elements are

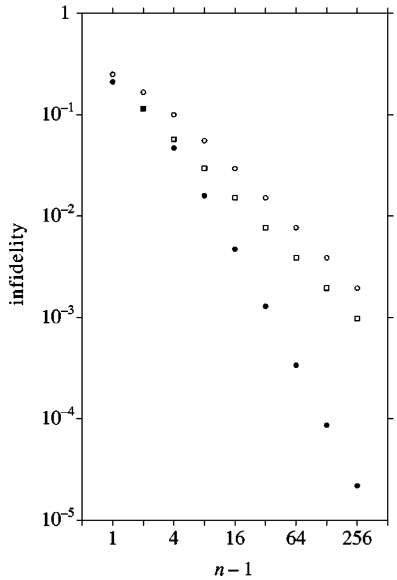

FIG. 1. Mean-square error as a function of n for the transmission of a single axis using the circular state (open circles), the extreme Stark state (squares), and the optimal state (closed circles).

$$N|K,\mathbf{u}_{\theta\phi}\rangle\langle K,\mathbf{u}_{\theta\phi}|, \tag{81}$$

where $|K,\mathbf{u}_{\theta\phi}\rangle$ was defined in Eq. (67) and N is a normalization factor. Unfortunately, $|K,\mathbf{z}\rangle$ contains a superposition of all values of l, as can be seen from Eq. (62). Thus $|K,\mathbf{z}\rangle$ does not belong to one irreducible subspace of the representation of the SO(3) rotation group. As a result, the operator

$$B=\int d_{\theta\phi}|K,\mathbf{u}_{\theta\phi}\rangle\langle K,\mathbf{u}_{\theta\phi}| \tag{82}$$

is not proportional to the identity, but is a block-diagonal matrix with different blocks for each irreducible representation of the rotation group. Moreover, the resulting POVM includes an element which corresponds to the absence of any answer, thus reducing fidelity. A natural POVM which uses the SO(4) group will be discussed in Sec. IV.

The direction of the minor axis of a classical nondegenerate ellipse is that of $\mathbf{L}\times\mathbf{K}$. A quantum ellipse also has this property. Taking Alice's state as a quantum ellipse with eccentricity $0<e<1$, with both $\mathbf{k}$ and ℓ lying in the xy plane so that $\mathbf{w}=\mathbf{z}$, the resulting fidelity can be compared with the cases where $\mathbf{k}$ or ℓ points along the z axis and the eccentricity of the ellipse is 0 or 1, respectively. The fidelity for transmission using the semiminor axis reaches a maximum at eccentricity of about $e=0.7$ (a different eccentricity for each value of n). A comparison of the mean-square error for using the three options is given in Table I.

TABLE I. Eccentricities e and mean-square errors η for transmission of a single direction using $\mathbf{z}=\mathbf{w}$, $\mathbf{z}=\ell$, or $\mathbf{z}=\mathbf{k}$, for $n=5$ or 10.

n	$\mathbf{z}=\mathbf{w}$	$\mathbf{z}=\ell$	$\mathbf{z}=\mathbf{k}$
5	$e=0.6963$	$e=0$	$e=1$
	$\eta=0.193967$	$\eta=0.1$	$\eta=0.0573645$
10	$e=0.701261$	$e=0$	$e=1$
	$\eta=0.0861934$	$\eta=0.05$	$\eta=0.0264067$

Comparison between elliptic wave functions and optimal wave functions

We shall now compare the extreme Stark state with Alice's optimal vector for the transmission of one axis as calculated in Ref. [3]. They are both eigenstates of L_z with $m=0$, and since Eq. (79) holds, we will present them in the notation $(|a_{00}|,|a_{10}|,|a_{20}|,\ldots,|a_{n-1,0}|)$, where $a_{l\,0}=C^{jj\,l}_{-jj0}$ as before. For $n=3$ we have

$$|K,\mathbf{z}\rangle=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{6}}\right), \tag{83}$$

while Alice's optimal state is

$$|A_{\rm opt}\rangle=\left(\sqrt{\frac{5}{3\sqrt{2}}},\frac{1}{\sqrt{2}},\sqrt{\frac{2}{3}}\right). \tag{84}$$

Thus for $n=3$ the overlap between the extreme Stark state and the optimal state is

$$|\langle K,\mathbf{z}|A_{\rm opt}\rangle|^2=0.993\,491. \tag{85}$$

Both states give almost the same fidelity for transmission of one axis. For higher values of n, they become more and more different. For $n=10$ the overlap is

$$|\langle K,\mathbf{z}|A_{\rm opt}\rangle|^2=0.764\,06. \tag{86}$$

The various components are given in Table II. We see that the extreme Stark state has coefficients peaked at lower values of l than the optimal state.

IV. TRANSMISSION OF TWO AXES

Alice now wants to transmit a Cartesian frame by indicating the directions of two axes, the third one being inferred from them. Which elliptic state is optimal? Obviously, states with $e=0$ and $e=1$ will not do in this case, since they define only one direction. We have to find the optimal eccentricity. Let

TABLE II. Coefficients $|a_{l0}|$ for Alice's optimal state and for the extreme Stark state when $n=10$.

l	0	1	2	3	4	5	6	7	8	9
$\vert K,\mathbf{z}\rangle$	0.3162	0.4954	0.5222	0.4534	0.3365	0.2148	0.1167	0.0526	0.0186	0.0045
Optimal	0.1825	0.3079	0.3767	0.4098	0.4130	0.3894	0.3422	0.2751	0.1923	0.0989

$$\Delta K_\perp = \sqrt{(\Delta K_\ell)^2 + (\Delta K_w)^2}, \tag{87}$$

$$\Delta L_\perp = \sqrt{(\Delta L_k)^2 + (\Delta L_w)^2}, \tag{88}$$

where

$$\Delta K_\ell = \sqrt{\langle K_\ell^2 \rangle - \langle K_\ell \rangle^2} = \sqrt{\langle K_\ell^2 \rangle}, \tag{89}$$

owing to Eq. (40). We define similar expressions for the other components. When we want to transmit **k**, namely, the direction of the classical LRL vector, then a smaller $\Delta K_\perp / \langle K_k \rangle$ improves the fidelity. A similar argument holds for the transmission of $\boldsymbol{\ell}$. Thus when transmitting two axes, a heuristic guideline is to look for states that satisfy

$$\frac{\Delta K_\perp}{\langle K_k \rangle} \approx \frac{\Delta L_\perp}{\langle L_l \rangle}. \tag{90}$$

A straightforward calculation [14] gives

$$\Delta K_w = \Delta L_w = \sqrt{\tfrac{1}{2}(n-1)}, \tag{91}$$

$$\Delta K_\ell = \sqrt{\tfrac{1}{2}(n-1)} \sin\zeta, \tag{92}$$

$$\Delta L_k = \sqrt{\tfrac{1}{2}(n-1)} \cos\zeta. \tag{93}$$

Together with Eqs. (38) and (39), this gives an equation for the eccentricity,

$$\frac{\sqrt{1+\sin^2\zeta}}{\sqrt{2(n-1)}\sin\zeta} \approx \frac{\sqrt{1+\cos^2\zeta}}{\sqrt{2(n-1)}\cos\zeta}. \tag{94}$$

Therefore we expect that the optimal eccentricity is approximately

$$e = \sin\zeta = \cos\zeta = 1/\sqrt{2}. \tag{95}$$

More accurate numerical results are given below.

We now evaluate the fidelity for the transmission of two axes. Alice uses an elliptic state with $\mathbf{k} = \mathbf{x}$ and $\boldsymbol{\ell} = \mathbf{y}$ (the unit vectors in the x and y directions, respectively). The eccentricity $e = \sin\zeta$ has to be optimized. Recall that ζ is defined to be half the angle between $\mathbf{u}_1$ and $\mathbf{u}_2$. The definitions of **k** and $\boldsymbol{\ell}$ are given in Eq. (29). Thus in order to meet the above requirements we set in $|A\rangle = |\mathbf{u}_{\theta_1\phi_1}, \mathbf{u}_{\theta_2\phi_2}\rangle$ the parameters $\theta_1 = \theta_2 = \pi/2$, and

$$\phi_1 = \tfrac{1}{2}\pi - \zeta, \quad \phi_2 = \tfrac{3}{2}\pi - \zeta. \tag{96}$$

Fidelities now must be defined for each one of the axes. Note that $\cos\omega_k$ (for the kth axis) is given by the corresponding diagonal element of the orthogonal (classical) rotation matrix. For the transmission of the x and y axes, we thus need [11]

$$\langle \cos\omega_x + \cos\omega_y \rangle = \langle (1+\cos\beta)(\cos(\alpha+\gamma)) \rangle. \tag{97}$$

We expand $|A\rangle$ and $|B\rangle$ as in Eqs. (41) and (45). Bob's optimal fiducial vector is still given by Eq. (47), and $\langle \cos\omega_x + \cos\omega_y \rangle$ is calculated using Eqs. (23)–(26) of Ref. [7]. The mean-square error per axis is plotted in Fig. 2 for $n = 5, 10$, and 20. The error is minimal at $e \approx 0.708$ for $n = 5$, at $e \approx 0.704$ for $n = 10$, and $e \approx 0.674$ for $n = 20$.

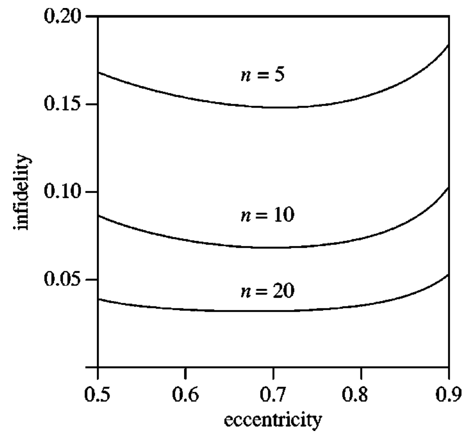

FIG. 2. Mean-square error (per axis) as a function of eccentricity for $n = 5$, 10, and 20.

Note that the shape of the curve flattens with increasing n, so that the minimum is hard to find numerically. The intuitive explanation is that in the limit of large n, as if Alice were to transmit a "classical atom," i.e., a classical two-body Kepler system, then the direction of the classical angular momentum and LRL vectors could be found irrespective of the eccentricity. Therefore, the transmission accuracy would be the same for any eccentricity that is not close to zero or one.

The deviation of the optimum from $e = 1/\sqrt{2}$ was expected, since transmission of the **k** direction ($e = 1$) achieved higher fidelity than the transmission of the ℓ direction ($e = 0$). Thus the ellipse with optimal eccentricity for transmission of two axes is biased to give $\Delta L_\perp / L < \Delta K_\perp / K$ in order to compensate the difference and make the contribution to the error from the **k** direction about equal to that from the $\boldsymbol{\ell}$ direction.

Elliptic states give results very close to the optimal ones. The mean-square error for transmission of two axes by elliptic states with optimal eccentricity is compared to the optimal results [7] in Table III.

V. POVM FOR SO(4)

We now construct a POVM based on the SO(4) group and use it in order to transmit two axes. This POVM is naturally built with the SO(4) coherent states which are, as we have seen, direct products of two SO(3) coherent states. We shall use for each one of the SO(3) subspaces the notation

TABLE III. Mean-square error η for transmission of two axes by an elliptic state with optimal eccentricity, and by the optimal method [7] for $n = 5$,10, and 20.

n	Elliptic	Optimal
5	$\eta = 0.14765$	$\eta = 0.14465$
10	$\eta = 0.06822$	$\eta = 0.06793$
20	$\eta = 0.03190$	$\eta = 0.03088$

$$|\psi\theta\phi\rangle_{\mathbf{u}}=\sqrt{2j+1}U(\psi\theta\phi)|\mathbf{u}\rangle, \tag{98}$$

and

$$dE(\psi\theta\phi)=d_{\psi\theta\phi}|\psi\theta\phi\rangle_{\mathbf{u}}\langle\psi\theta\phi|_{\mathbf{u}}, \tag{99}$$

where $\mathbf{u}$ labels the direction to be transmitted, and $d_{\psi\theta\phi}=\sin\theta d\theta d\phi d\psi/8\pi^2$ as in Eq. (44). By applying Schur's lemma to each of the SO(3) subspaces, we have

$$\int\int dE_1(\psi_1\theta_1\phi_1)\otimes dE_2(\psi_2\theta_2\phi_2)=\mathbb{1}_1\otimes\mathbb{1}_2=\mathbb{1}. \tag{100}$$

We are now ready to discuss the transmission of Alice's x and y axes by means of an elliptic state. We take

$$|A\rangle=|\mathbf{xy}\rangle=|\mathbf{x}\rangle\otimes|\mathbf{y}\rangle. \tag{101}$$

This equation was written in Alice's notation. We also define a fiducial vector for Bob,

$$|B\rangle=(2j+1)|\mathbf{x}\rangle\otimes|\mathbf{y}\rangle, \tag{102}$$

written in Bob's notations. Thus the POVM element is constructed from the vector

$$|\psi_1\theta_1\phi_1\rangle_{\mathbf{x}}\otimes|\psi_2\theta_2\phi_2\rangle_{\mathbf{y}}=(2j+1)(U_1\otimes U_2)|B\rangle. \tag{103}$$

The result of Bob's measurement consists of two sets of Euler angles, $\psi_1\theta_1\phi_1$ and $\psi_2\theta_2\phi_2$. The first one gives Bob's estimate of the active rotation needed to bring his x axis to Alice's x axis. Likewise, the second set gives Bob's estimate of the active rotation needed to bring his y axis to Alice's y axis. The detection probability of these sets of angles is

$$dP(\psi_1\cdots\phi_2)=d_{\psi_1\theta_1\phi_1}d_{\psi_2\theta_2\phi_2}|\langle A|U_1\otimes U_2|B\rangle|^2. \tag{104}$$

Recall that Eq. (101) was written in Alice's notations, while Eq. (102) was in Bob's notations. To compute the result explicitly, we need a uniform system of notations. For this we introduce, as in Ref. [7], the Euler angles $\xi\eta\zeta$ that rotate Bob's xyz axes into Alice's axes. (The Euler angle ζ should not be confused with the eccentricity parameter introduced in Sec. II.) The unitary operator $U(\xi\eta\zeta)$ represents an active transformation of Bob's state vectors to the corresponding state vectors of Alice's system. Therefore, $U(\xi\eta\zeta)$ is also the passive transformation from Alice's notations to Bob's notations. Written in Bob's notations, Alice's vector $|A\rangle$ becomes $U(\xi\eta\zeta)|A\rangle$ so that in Eq. (104), $\langle A|$ becomes $\langle A|U(\xi\eta\zeta)^\dagger$. Owing to the commutation relations (7) and (8),

$$U(\xi\eta\zeta)=e^{-iL_z\xi}e^{-iL_y\eta}e^{-iL_z\zeta}, \tag{105}$$

$$=U_1(\xi\eta\zeta)\otimes U_2(\xi\eta\zeta), \tag{106}$$

where again U_1 and U_2 are defined as in Eq. (73). We thus have

$$U(\xi\eta\zeta)|\mathbf{u}_1\mathbf{u}_2\rangle=U_1(\xi\eta\zeta)|\mathbf{u}_1\rangle\otimes U_2(\xi\eta\zeta)|\mathbf{u}_2\rangle. \tag{107}$$

Let us therefore define

$$U_1(\alpha_1\beta_1\gamma_1)=U_1^\dagger(\xi\eta\zeta)U_1(\psi_1\theta_1\phi_1) \tag{108}$$

and

$$U_2(\alpha_2\beta_2\gamma_2)=U_2^\dagger(\xi\eta\zeta)U_2(\psi_2\theta_2\phi_2). \tag{109}$$

We shall henceforth use the left-hand sides of Eqs. (108) and (109) as the new definitions of the symbols U_1 and U_2. As before, the Euler angles $\alpha_1\beta_1\gamma_1$ have the effect of rotating Bob's x axis into his estimate of Alice's x axis and then rotating back the result by the true rotation from Alice's to Bob's frame. The action of the Euler angles $\alpha_2\beta_2\gamma_2$ is similar for the y axis. Thus the Euler angles $\alpha_i\beta_i\gamma_i$ indicate Bob's measurement error, and the probability of that error is

$$dP(\alpha_1\cdots\gamma_2)=d_{\alpha_1\beta_1\gamma_1}d_{\alpha_2\beta_2\gamma_2}|\langle A|U_1\otimes U_2|B\rangle|^2. \tag{110}$$

Note the similarity with Eq. (104). The difference is that Eq. (104) referred to the probability of *detection* of a particular set of Euler angles, while Eq. (110) gives the probability of *error* in that detection.

The transmission mean-square error *per axis* is, as in Eq. (57),

$$R=\tfrac{1}{4}(1-\cos\omega_x)+\tfrac{1}{4}(1-\cos\omega_y), \tag{111}$$

where ω_x and ω_y are the angles between the true and estimated directions of the and y axes, respectively. Since Bob infers the direction of the x axis from the angles $\psi_1\theta_1\phi_1$, the value of $\cos\omega_x$ depends only on $\alpha_1\beta_1\gamma_1$. Likewise, the value of $\cos\omega_y$ depends only on the angles $\alpha_2\beta_2\gamma_2$. We have

$$\langle\cos\omega_x\rangle=\int d_{\alpha_1\beta_1\gamma_1}|\langle\mathbf{x}|U_1|\mathbf{x}\rangle|^2\cos\omega_x, \tag{112}$$

where we have used Eq. (110) and Schur's lemma for the second set of angles, namely,

$$(2j+1)\int d_{\alpha_2\beta_2\gamma_2}U_2|\mathbf{y}\rangle\langle\mathbf{y}|U_2^\dagger=\mathbb{1}_2. \tag{113}$$

The evaluation of Eq. (112) is identical to the one performed in Eq. (55), with n replaced by $\frac{1}{2}(n+1)$ everywhere, and we get

$$\langle\cos\omega_x\rangle=(n-1)/(n+1). \tag{114}$$

Likewise,

$$\langle\cos\omega_y\rangle=(n-1)/(n+1). \tag{115}$$

Thus the infidelity (mean-square error) per axis is

$$\tfrac{1}{4}(1-\langle\cos\omega_x\rangle)+\tfrac{1}{4}(1-\langle\cos\omega_y\rangle)=1/(n+1). \tag{116}$$

In Ref. [7] it was found that the *optimal* POVM (not restricted to elliptic states) for transmission of two axes using a nonrelativistic hydrogen atom is of the form given by Eq.

(44). It was shown that using this POVM, the infidelity per axis for Alice's optimal signal approaches $1/(3n)$ asymptotically. Using SO(4) instead of SO(3) as in Ref. [7], we obtain an infidelity per axis that is exactly $1/(n+1)$ for all values of n. As shown in the Appendix, an adjustment procedure to obtain orthogonal axes will further decrease the mean-square error by a factor which, for large values of n, tends to 3/4.

The SO(4) POVM also enables the transmission of two directions which are not orthogonal, by means of a particle in an elliptic state. To transmit the directions of two general unit vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, Alice's prepares the elliptic state

$$|A\rangle=|\mathbf{v}_1\mathbf{v}_2\rangle=|\mathbf{v}_1\rangle\otimes|\mathbf{v}_2\rangle, \tag{117}$$

(in her notations) while Bob's vector is (in his notations)

$$|B\rangle=(2j+1)|\mathbf{v}_1\rangle\otimes|\mathbf{v}_2\rangle. \tag{118}$$

As before, the infidelity for each direction is $1/(n+1)$. It should be noted that transmission of two nonorthogonal directions with one hydrogen atom is not possible with the SO(3) POVM.

VI. SUMMARY AND CONCLUDING REMARKS

We have shown how elliptic Rydberg states can transfer information on the orientation of one direction, or more generally that of a Cartesian frame. For increasing values of n, the fidelity obtained for a single direction falls rapidly below the optimal ones. However, for a Cartesian frame the results are very close to the optimal ones. Note that we have assumed that Alice and Bob have the same chirality. If their chiralities are opposite, then when angular momenta are used for the transmission the direction inferred by Bob should be reversed (because directions are polar vectors while angular momentum is an axial vector). However, the LRL vector is a polar vector, thus even if Bob and Alice have opposite chiralities, the direction inferred by Bob is correct. We have also shown how elliptic Rydberg states can be prepared to encode two arbitrary directions, when the measurement is based on the SO(4) rotation group.

ACKNOWLEDGMENTS

Work by A.P. was supported by the Gerard Swope Fund and the Fund for Promotion of Research. Work by N.H.L. was supported by a grant from the Technion Graduate School.

APPENDIX: REDUCTION OF ERRORS BY ORTHOGONALIZATION

As we have seen in Sec. V, Bob's estimates of Alice's x and y axes may not be exactly orthogonal. The probability for the estimate of the x axis to have an angular error ω_x, as can be seen from Eq. (64), is

$$\rho(\omega_x)\propto\cos^{2n-2}(\omega_x/2), \tag{A1}$$

and likewise for the y axis. Thus for large values of n, the error probability distribution will be highly peaked. We now calculate the gain in fidelity achieved if Bob performs a simple orthogonalization of his two estimates $\hat{\mathbf{r}}_x$ and $\hat{\mathbf{r}}_y$, by rotating the two vectors in their plane by the same angle, so that they become orthogonal.

Let us define two pairs of spherical angles that give the position of the estimated directions with respect to the (unknown) true axes. These positions are given by

$$\hat{\mathbf{r}}_x=(\sin\theta_1\cos\phi_1,\sin\theta_1\sin\phi_1,\cos\theta_1) \tag{A2}$$

and

$$\hat{\mathbf{r}}_y=(\sin\theta_2\cos\phi_2,\sin\theta_2\sin\phi_2,\cos\theta_2). \tag{A3}$$

The probability distributions will be denoted by $\rho_i(\theta_i,\phi_i)$. In the limit of large n, the deviation angles ω_x and ω_y are small. Hence the distribution is centered as

$$\rho_x=\rho(\theta_1-\tfrac{1}{2}\pi,\phi_1), \tag{A4}$$

$$\rho_y=\rho(\theta_2-\tfrac{1}{2}\pi,\phi_2-\tfrac{1}{2}\pi), \tag{A5}$$

where $\rho(\xi,\mu)$ is peaked around $(0,0)$. Here we used the fact that the SO(4) POVM gives probabilities of error for each axis, which are identical and independent. Define new variables

$$\tilde{\theta}_i=\theta_i-\tfrac{1}{2}\pi, \tag{A6}$$

$$\tilde{\phi}_2=\phi_2-\tfrac{1}{2}\pi. \tag{A7}$$

The deviation angles are given by $\cos\omega_x=\hat{\mathbf{r}}_x\cdot\hat{\mathbf{x}}$ and $\cos\omega_y=\hat{\mathbf{r}}_y\cdot\hat{\mathbf{y}}$, namely,

$$\cos\omega_x=\sin\theta_1\cos\phi_1\approx1-\tfrac{1}{2}\tilde{\theta}_1^2-\tfrac{1}{2}\phi_1^2 \tag{A8}$$

and

$$\cos\omega_y=\sin\theta_2\sin\phi_2\approx1-\tfrac{1}{2}\tilde{\theta}_2^2-\tfrac{1}{2}\tilde{\phi}_2^2. \tag{A9}$$

Let g denote the infidelity per axis before the adjustment. The infidelities for both axes are equal, thus

$$g\equiv\tfrac{1}{2}(1-\langle\cos\omega_x\rangle), \tag{A10}$$

$$\approx\tfrac{1}{4}\int(\tilde{\theta}_1^2+\phi_1^2)d\rho_i\equiv\tfrac{1}{4}\langle\tilde{\theta}_1^2+\phi_1^2\rangle, \tag{A11}$$

where

$$d\rho_i=\rho(\tilde{\theta}_i,\phi_i)\sin\tilde{\theta}_i d\tilde{\theta}_i d\phi_i \tag{A12}$$

fulfills

$$\int d\rho_i=1. \tag{A13}$$

Equivalently, we can write the infidelity in terms of $\tilde{\theta}_2$ and $\tilde{\phi}_2$ as

$$g\approx\tfrac{1}{4}\int(\tilde{\theta}_2^2+\tilde{\phi}_2^2)d\rho_i\equiv\tfrac{1}{4}\langle\tilde{\theta}_2^2+\tilde{\phi}_2^2\rangle. \qquad \text{(A14)}$$

In *first* order we have, by combining Eqs. (A2) and (A3) with the definitions (A6) and (A7),

$$\hat{\mathbf{r}}_x\approx(1,\phi_1,-\tilde{\theta}_1),\quad \hat{\mathbf{r}}_y\approx(-\tilde{\phi}_2,1,-\tilde{\theta}_2), \qquad \text{(A15)}$$

and the angle Ω between them is given by

$$\cos\Omega=\hat{\mathbf{r}}_x\cdot\hat{\mathbf{r}}_y\approx\phi_1-\tilde{\phi}_2. \qquad \text{(A16)}$$

The bisector of $\hat{\mathbf{r}}_x$ and $\hat{\mathbf{r}}_y$ is given by the *unit* vector $\hat{\mathbf{b}}=(\hat{\mathbf{r}}_1+\hat{\mathbf{r}}_2)/|\hat{\mathbf{r}}_1+\hat{\mathbf{r}}_2|$. Using Eq. (A15) and keeping only first-order terms, we have

$$\hat{\mathbf{b}}\approx[1-\tfrac{1}{2}(\phi_1+\tilde{\phi}_2),1+\tfrac{1}{2}(\phi_1+\tilde{\phi}_2),-\tilde{\theta}_1-\tilde{\theta}_2]/\sqrt{2}, \qquad \text{(A17)}$$

where we used

$$|\hat{\mathbf{r}}_1+\hat{\mathbf{r}}_2|\approx\sqrt{2}(1+\tfrac{1}{2}\phi_1-\tfrac{1}{2}\tilde{\phi}_2). \qquad \text{(A18)}$$

We can also express the bisector $\hat{\mathbf{b}}$ in terms of its spherical angles which we shall denote by (τ,φ). Since the errors are small, we have $\varphi\approx\frac{1}{4}\pi$, and it is convenient to define

$$\tilde{\varphi}=\varphi-\tfrac{1}{4}\pi. \qquad \text{(A19)}$$

Comparison of the two expressions for $\hat{\mathbf{b}}$ gives

$$\xi=\tfrac{1}{2}\pi+\sqrt{\tfrac{1}{2}}(\tilde{\theta}_1+\tilde{\theta}_2),\quad \tilde{\varphi}=\tfrac{1}{2}(\phi_1+\tilde{\phi}_2). \qquad \text{(A20)}$$

In first order, as Eq. (A16) shows, the orthogonalization consists in changing the angles ϕ_i irrespective of θ_i, without changing the θ_i themselves. Hence, in first order, the procedure defines

$$\phi_1'=\varphi-\tfrac{1}{4}\pi,\quad \phi_2'=\varphi+\tfrac{1}{4}\pi, \qquad \text{(A21)}$$

i.e.,

$$\phi_1'=\tilde{\phi}_2'=\tilde{\varphi}=\tfrac{1}{2}(\phi_1+\tilde{\phi}_2), \qquad \text{(A22)}$$

where again $\tilde{\phi}_2'=\phi_2'-\frac{1}{2}\pi$. The change in $\tilde{\theta}_i$ is of higher order, $\tilde{\theta}_i'=\tilde{\theta}_i+O(\theta^2,\phi^2)$. The new infidelity per axis, g^{new}, is

$$g^{\text{new}}=\tfrac{1}{4}\langle\phi_1'^2+\tilde{\theta}_1^2\rangle=\tfrac{1}{4}\langle\tilde{\phi}_2'^2+\tilde{\theta}_2^2\rangle. \qquad \text{(A23)}$$

Returning to Eqs. (A11) and (A14), consider the integrals over ϕ_i. Define

$$\rho_\phi(\phi)\equiv\int\rho(\theta,\phi)\sin\theta d\theta. \qquad \text{(A24)}$$

Keeping in mind that the distributions for ϕ_1 and $\tilde{\phi}_2$ are identical, the ϕ part of the infidelity per axis *before* the adjustment is

$$g_\phi=\tfrac{1}{4}\langle\phi_1^2\rangle=\tfrac{1}{4}\langle\tilde{\phi}_2^2\rangle, \qquad \text{(A25)}$$

$$=\int\phi_1^2\rho_\phi(\phi_1)\rho_\phi(\tilde{\phi}_2)d\phi_1 d\tilde{\phi}_2/4, \qquad \text{(A26)}$$

$$=\int\tilde{\phi}_2^2\rho_\phi(\tilde{\phi}_1)\rho_\phi(\tilde{\phi}_2)d\phi_1 d\tilde{\phi}_2/4. \qquad \text{(A27)}$$

The ϕ parts of the infidelities *after* the adjustment, denoted by g_ϕ^{new}, are

$$g_\phi^{\text{new}}=\tfrac{1}{4}\langle\phi_1'^2\rangle=\tfrac{1}{4}\langle\tilde{\phi}_2'^2\rangle=\tfrac{1}{16}\langle\phi_1^2+2\phi_1\tilde{\phi}_2+\tilde{\phi}_2^2\rangle. \qquad \text{(A28)}$$

The functions $\rho_\phi(\phi_1)$ and $\rho_\phi(\tilde{\phi}_2)$ are even, because the probability distribution ρ depends only on the angles ω_x or ω_y, which are independent of the signs of ϕ_1 and $\tilde{\phi}_2$. Thus

$$\langle\phi_1\tilde{\phi}_2\rangle=\int\phi_1\tilde{\phi}_2\rho_\phi(\phi_1)\,\rho_\phi(\tilde{\phi}_2)\,d\phi_1 d\tilde{\phi}_2=0. \qquad \text{(A29)}$$

With the help of Eq. (A25) we obtain

$$g_\phi^{\text{new}}=\tfrac{1}{16}\langle\phi_1^2+\tilde{\phi}_2^2\rangle=\tfrac{1}{8}\langle\phi_1^2\rangle. \qquad \text{(A30)}$$

Thus the ϕ parts of the infidelity are halved,

$$g_\phi^{\text{new}}=\tfrac{1}{2}g_\phi. \qquad \text{(A31)}$$

As already stated, the angles $\tilde{\theta}_i$ are unchanged in first order. Since the probability function $\rho_x(\phi_1,\tilde{\theta}_1)$ depends only on the angle $\omega_x=\hat{\mathbf{r}}_x\cdot\hat{\mathbf{x}}$, it is symmetric with respect to rotations around the x axis. A similar argument holds for the y axis. Thus

$$\langle\tilde{\theta}_1^2\rangle=\langle\phi_1^2\rangle=\langle\tilde{\theta}_2^2\rangle=\langle\phi_2^2\rangle, \qquad \text{(A32)}$$

and we have finally

$$g^{\text{new}}=\tfrac{3}{4}g. \qquad \text{(A33)}$$

[1] S. Massar and S. Popescu, Phys. Rev. Lett. **74**, 1259 (1995).
[2] N. Gisin and S. Popescu, Phys. Rev. Lett. **83**, 432 (1999).
[3] A. Peres and P.F. Scudo, Phys. Rev. Lett. **86**, 4160 (2001).
[4] E. Bagan, M. Baig, A. Brey, R. Muñoz-Tapia, and R. Tarrach, Phys. Rev. A **63**, 052309 (2001).
[5] E. Bagan, M. Baig, and R. Muñoz-Tapia, Phys. Rev. Lett. **87**, 257903 (2001).
[6] A. Peres and P. Scudo, J. Mod. Opt. **49**, 1235 (2002).
[7] A. Peres and P.F. Scudo, Phys. Rev. Lett. **87**, 167901 (2001).
[8] P. Rozmej, M. Turek, R. Arvieu, and I.S. Averbukh, J. Phys. A **35**, 7803 (2002).
[9] E.S. Chang, Phys. Rev. A **31**, 495 (1985).
[10] J. Yeazell and C.R. Stroud, Jr., Phys. Rev. Lett. **60**, 1494 (1988).
[11] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1980).
[12] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1977).
[13] M. Bander and C. Itzykson, Rev. Mod. Phys. **38**, 330 (1966).
[14] J.C. Gay, D. Delande, and A. Brommier, Phys. Rev. A **39**, 6587 (1989).
[15] A.R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1957).
[16] M. Tinkham, *Group Theory and Quantum Mechanics* (McGraw-Hill, New York, 1964), p. 121.
[17] E. Wigner, *Group Theory* (Academic Press, New York, 1959), p. 76.
[18] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, 1993).
[19] B.G. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974).