Photon polarization and Wigner's little group

Paweł Caban* and Jakub Rembieliński†
Department of Theoretical Physics, University of Łódź, Pomorska 149/153, 90-236 Łódź, Poland
(Received 16 May 2003; published 10 October 2003)

To discuss one-photon polarization states we find the explicit form of the Wigner's little group element in the massless case for arbitrary Lorentz transformation. As is well known, when analyzing the transformation properties of the physical states, only the value of the phase factor is relevant. We show that this phase factor depends only on the direction of the momentum $\mathbf{k}/|\mathbf{k}|$ and does not depend on the frequency k^0. Finally, we use this observation to discuss the transformation properties of the linearly polarized photons and the corresponding reduced density matrix. We find that they transform properly under Lorentz group.

DOI: 10.1103/PhysRevA.68.042107 PACS number(s): 03.65.Fd, 03.65.Ud, 03.30.+p

I. INTRODUCTION

In recent years a lot of interest has been devoted to the study of the quantum entanglement and Einstein-Podolsky-Rosen correlation function under the Lorentz transformations for massive particles [1–7]. In recent papers [8,9] also the massless particle case was discussed. One of the key ingredients of these papers is the calculation of the explicit form of the little group element for massless particle in some special cases to analyze transformation properties of entangled states and reduced density matrix.

In this paper we derive the explicit form of the Wigner's little group element in the massless case for an arbitrary Lorentz transformation and discuss the transformation properties of the linearly polarized photons and the corresponding reduced density matrix obtained by tracing out kinematical degrees of freedom. As is well known in the Hilbert space of massless particles the one-particle momentum eigenvectors under Lorentz transformation Λ are multiplied by a phase factor depending on Λ and the particle four-momentum k^μ. We show that this phase factor depends only on Λ and the direction of the momentum $\mathbf{k}/|\mathbf{k}|$ but does not depend on the frequency k^0. In contrast to other papers [8] this observation enables us to give the description of the transformation rules of the linearly polarized photons which are not necessarily monochromatic.

II. WIGNER'S LITTLE GROUP FOR MASSLESS PARTICLES

As is well known, the pure quantum states are identified with rays in the Hilbert space. For this reason, on the quantum level, we should use ray representations of the classical symmetry groups. In our case of the proper ortochronous Poincaré group $P^\uparrow_+$, which is the semidirect product of the proper ortochronous Lorentz group $L^\uparrow_+$ and the translations group T^4, its ray representations (so called double-valued representations) are faithful representations of the universal covering of $P^\uparrow_+$, i.e., the semidirect product of $\mathrm{SL}(2,\mathbb{C})$ and T^4. Moreover, the faithful representations of $P^\uparrow_+$ are homomorphic representations of its universal covering group.

We use the canonical homomorphism between the group $\mathrm{SL}(2,\mathbb{C})$ (universal covering of the proper ortochronous Lorentz group $L^\uparrow_+$) and the Lorentz group $L^\uparrow_+ \sim \mathrm{SO}(1,3)_0$ [10]. This homomorphism is defined as follows: With every four-vector k^μ we associate a two-dimensional Hermitian matrix k such that

$$\mathsf{k} = k^\mu \sigma_\mu, \tag{1}$$

where $\sigma_i, i=1,2,3$ are the standard Pauli matrices and $\sigma_0 = I$. In the space of two-dimensional Hermitian matrices (1), the Lorentz group action is given by

$$\mathsf{k}' = A\mathsf{k}A^\dagger, \tag{2}$$

where A denotes element of the $\mathrm{SL}(2,\mathbb{C})$ group corresponding to the Lorentz transformation $\Lambda(A)$ which converts the four-vector k to k' (i.e., $k'^\mu = \Lambda^\mu_\nu k^\nu$) and $\mathsf{k}' = k'^\mu \sigma_\mu$. The kernel of this homomorphism is isomorphic to $\mathbb{Z}_2$ [the center of the $\mathrm{SL}(2,\mathbb{C})$].

Now, let us focus on the case of massless particles. An explicit matrix representation (1) of the null (light-cone) four-vector k can be written as

$$\mathsf{k} = k^0 \begin{pmatrix} 1+n^3 & n_- \\ n_+ & 1-n^3 \end{pmatrix}, \tag{3}$$

where $n_\pm = n^1 \pm i n^2$, $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$, $k^0 = |\mathbf{k}|$ and $\det\mathsf{k} = k^\mu k_\mu = 0$. In this case we choose the standard four-vector as $\tilde{k} = (1,0,0,1)$. In the matrix representation (3) the following matrix is associated with $\tilde{k}$:

$$\tilde{\mathsf{k}} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

Now, let us find the stability group of $\tilde{\mathsf{k}}$, i.e., $A_0 \in \mathrm{SL}(2,\mathbb{C})$ which leaves $\tilde{\mathsf{k}}$ invariant. All such A_0 form a subgroup of the $\mathrm{SL}(2,\mathbb{C})$ group, i.e.,

$$(\text{stability group}) = \{A_0 \in \mathrm{SL}(2,\mathbb{C}) : \tilde{\mathsf{k}} = A_0 \tilde{\mathsf{k}} A_0^\dagger\}. \tag{5}$$

*Electronic address: P.Caban@merlin.fic.uni.lodz.pl
†Electronic address: J.Rembielinski@merlin.fic.uni.lodz.pl

As is well known [10], the stability group of the four-vector $\tilde{k}$ is isomorphic to the $E(2)$ group of rigid motions of Euclidean plane. We can easily find the most general A_0 by solving the equation $\tilde{\mathsf{k}}=A_0\tilde{\mathsf{k}}A_0^\dagger$. We get

$$A_0=\begin{pmatrix} e^{i(\psi/2)} & z \\ 0 & e^{-i(\psi/2)} \end{pmatrix}, \tag{6}$$

where z is an arbitrary complex number. Since the SL(2,$\mathbb{C}$) is the two-fold covering of the Lorentz group, we restricted the values of ψ to the interval $\langle 0,2\pi)$. Our purpose is to find the Wigner's little group element $W(\Lambda,k)$ corresponding to k and the Lorentz transformation Λ, namely,

$$W(\Lambda,k)=L^{-1}_{\Lambda k}\Lambda L_k, \tag{7}$$

where $L_k\in L^\uparrow_+$ is determined uniquely by the following conditions:

$$k=L_k\tilde{k}, \quad L_{\tilde{k}}=I. \tag{8}$$

In order to find the corresponding element $S(\Lambda,k)$ in SL(2,$\mathbb{C}$) such that $W(\Lambda,k)=\Lambda(S(\Lambda,k))$, i.e.,

$$S(\Lambda,k)=A^{-1}_{\Lambda k}AA_k, \tag{9}$$

where $\Lambda(A_k)=L_k$, we have to first calculate the matrix A_k. We can do it by solving the matrix equation

$$\mathsf{k}=A_k\tilde{\mathsf{k}}A_k^\dagger. \tag{10}$$

After simple calculation we get

$$A_k=U_{\mathbf{n}}B(k^0), \tag{11}$$

where

$$U_{\mathbf{n}}=\frac{1}{\sqrt{2(1+n^3)}}\begin{pmatrix} 1+n^3 & -n_- \\ n_+ & 1+n^3 \end{pmatrix} \tag{12}$$

represents rotation $R_{\mathbf{n}}$ which converts the spatial vector $(0,0,1)$ to $\mathbf{n}$, while

$$B(k^0)=\begin{pmatrix} \sqrt{k^0} & 0 \\ 0 & \dfrac{1}{\sqrt{k^0}} \end{pmatrix} \tag{13}$$

represents boost along the z axis which converts $\tilde{k}$ to $k^0\tilde{k}$. Therefore,

$$A_k=\frac{1}{\sqrt{2k^0(1+n^3)}}\begin{pmatrix} k^0(1+n^3) & -n_- \\ k^0n_+ & 1+n^3 \end{pmatrix}. \tag{14}$$

Note that according to Eq. (8) $A_{\tilde{k}}=I$. Now, an arbitrary Lorentz transformation $\Lambda(A)$ is represented in SL(2,$\mathbb{C}$) by the corresponding complex unimodular matrix

$$A=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta-\beta\gamma=1. \tag{15}$$

To calculate $A_{\Lambda k}$ we simply use the formulas (2) and (3) to find k' and then identify $k'=\Lambda k$. We find

$$k'^0=\frac{1}{2}k^0a, \tag{16}$$

$$n'^3=\frac{2b}{a}-1, \tag{17}$$

$$n'_+=\frac{2c}{a}, \tag{18}$$

$$n'_-=n'^*_+, \tag{19}$$

where

$$a=(|\alpha|^2+|\gamma|^2)(1+n^3)+(|\beta|^2+|\delta|^2)(1-n^3)+(\alpha\beta^*+\gamma\delta^*)n_-+(\alpha^*\beta+\gamma^*\delta)n_+, \tag{20}$$

$$b=|\alpha|^2(1+n^3)+|\beta|^2(1-n^3)+\alpha\beta^*n_-+\alpha^*\beta n_+, \tag{21}$$

$$c=\alpha^*\gamma(1+n^3)+\beta^*\delta(1-n^3)+\beta^*\gamma n_-+\alpha^*\delta n_+, \tag{22}$$

and $\mathbf{n}'=\mathbf{k}'/|\mathbf{k}'|$. Therefore we can find the explicit form of $S(\Lambda,k)$ by means of Eqs. (9) and (14). We have to calculate only the elements $S(\Lambda,k)_{11}$ and $S(\Lambda,k)_{12}$, since the general little group element (6) depends only on the phase factor $e^{i(\psi/2)}$ and complex number z. A straightforward calculation yields finally the following formulas:

$$e^{i(\psi(\Lambda,k)/2)}=\frac{[\alpha(1+n^3)+\beta n_+]b+[\gamma(1+n^3)+\delta n_+]c^*}{a\sqrt{b(1+n^3)}} \tag{23}$$

$$z(\Lambda,k)=\frac{[-\alpha n_-+\beta(1+n^3)]b+[-\gamma n_-+\delta(1+n^3)]c^*}{k^0a\sqrt{b(1+n^3)}}, \tag{24}$$

where a, b, and c are given by Eqs. (20)–(22).

The unitary irreducible representations of the Poincaré group are induced from the unitary irreducible representations of the little group of the four-momentum k^μ [i.e., the $E(2)$ group in the case of the massless particles] [11,10]. Now, we have two classes of the unitary irreducible representations of $E(2)$: the faithful infinite dimensional representations and the one-dimensional homomorphic representations of $E(2)$, isomorphic to its compact subgroup SO(2)$\subset E(2)$. Because there is no evidence for existence of massless particles with a continuous intrinsic degrees of freedom the physical choice is the last one [11]. Thus by means of the induction procedure [10] the four-momentum eigenstates transform according to the formula

$$U(\Lambda)|k,\lambda\rangle=e^{i\lambda\psi(\Lambda,k)}|\Lambda k,\lambda\rangle. \tag{25}$$

In the above formula, $U(\Lambda)$ denotes unitary operator representing Λ in the unitary representation of the Poincaré group

while the helicity λ fixes irreducible unitary representation of the Poincaré group induced from SO(2), λ takes integer and half-integer values only [11,10]. We use invariant normalization of the four-momentum eigenstates $|k,\lambda\rangle$, i.e., $\langle p,\sigma|k,\lambda\rangle = 2k^0\delta_{\sigma\lambda}\delta(\mathbf{k}-\mathbf{p})$. Thus, when analyzing the transformation properties of physical states only the value of the phase $\psi(\Lambda,k)$ is relevant [Eq. (23)]. So it is very important to stress that *the value of the phase ψ depends only on Λ and* $\mathbf{n}$ *and does not depend on the frequency* k^0:

$$\psi(\Lambda,k)=\psi\left(\Lambda,\frac{\mathbf{k}}{|\mathbf{k}|}\right)=\psi(\Lambda,\mathbf{n}). \tag{26}$$

Note also that momenta of massless particles which are parallel in one inertial frame are parallel for every inertial observer, i.e.,

$$\frac{\mathbf{k}}{|\mathbf{k}|}=\frac{\mathbf{p}}{|\mathbf{p}|}\Rightarrow\frac{\mathbf{k}'}{|\mathbf{k}'|}=\frac{\mathbf{p}'}{|\mathbf{p}'|}, \tag{27}$$

where $k'=\Lambda k$, $p'=\Lambda p$. Indeed, for massless particles, $\mathbf{k}$ and $\mathbf{p}$ are parallel iff the corresponding four-momenta are Lorentz orthogonal, i.e., $k_\mu p^\mu=0$. Since $k_\mu p^\mu$ is a Lorentz invariant then this holds in all inertial frames. Equation (27) can be also verified explicitly by using Eqs. (17)–(19). The above property holds good only in the massless case.

Now, using Eq. (23) we can immediately obtain the value of $e^{i\psi(\Lambda,k)}$ in a number of special cases considered elsewhere.

Rotations. In the case $\Lambda=R$, we have $R=\Lambda(U)$ where $U\in \mathrm{SU}(2)\subset \mathrm{SL}(2,\mathbb{C})$, thus we put

$$\delta=\alpha^*,\quad \gamma=-\beta^*,\quad |\alpha|^2+|\beta|^2=1 \tag{28}$$

and from Eq. (23), we get the following simple formula:

$$e^{i\psi(R,k)}=\frac{\alpha(1+n^3)+\beta n_+}{\alpha^*(1+n^3)+\beta^* n_-}. \tag{29}$$

For the given rotation R the explicit form of α and β can be expressed by, e.g., Euler angles (see, for example, Ref. [12]). Also note that from Eq. (24), we get

$$z(R,k)=0. \tag{30}$$

Now let us consider the special case of the rotation $R(\chi\mathbf{n})$ around the direction $\mathbf{n}$. The matrix $U_{R(\chi\mathbf{n})}\in \mathrm{SU}(2)$ representing $R(\chi\mathbf{n})$ can be written in the form [13]

$$U_{R(\chi\mathbf{n})}=e^{(i/2)\chi n^j\sigma_j}=\cos\frac{\chi}{2}+in^j\sigma_j\sin\frac{\chi}{2}. \tag{31}$$

Thus,

$$U_{R(\chi\mathbf{n})}=\begin{pmatrix}\cos\frac{\chi}{2}+in^3\sin\frac{\chi}{2} & in_-\sin\frac{\chi}{2}\\ in_+\sin\frac{\chi}{2} & \cos\frac{\chi}{2}-in^3\sin\frac{\chi}{2}\end{pmatrix}. \tag{32}$$

Thus inserting the corresponding values of α and β to Eq. (29), we find in this case

$$e^{i\psi[R(\chi\mathbf{n}),\mathbf{n}]}=e^{i\chi}. \tag{33}$$

As the next example, we consider the rotation $R(\chi\hat{\mathbf{z}})$ around the z axis. In this case (see Ref. [13])

$$U_{R(\chi\hat{\mathbf{z}})}=\begin{pmatrix}e^{i(\chi/2)} & 0\\ 0 & e^{-i(\chi/2)}\end{pmatrix}, \tag{34}$$

therefore from Eq. (29), we get the same formula as previously,

$$e^{i\psi[R(\chi\hat{\mathbf{z}}),\mathbf{n}]}=e^{i\chi}. \tag{35}$$

Boosts. Pure Lorentz boost $\Lambda(\mathbf{v})$ in an arbitrary direction $\mathbf{e}=\mathbf{v}/|\mathbf{v}|$ can be represented by the following SL(2,$\mathbb{C}$) matrix [10]:

$$A(\mathbf{v})=e^{(1/2)\xi e^j\sigma_j}=\begin{pmatrix}\cosh\frac{\xi}{2}+e^3\sinh\frac{\xi}{2} & e_-\sinh\frac{\xi}{2}\\ e_+\sinh\frac{\xi}{2} & \cosh\frac{\xi}{2}-e^3\sinh\frac{\xi}{2}\end{pmatrix}, \tag{36}$$

where the parameter ξ is connected with the velocity of the boosted frame by the relation

$$\tanh\xi=-v \tag{37}$$

and $e_\pm=e^1\pm ie^2$ (we use natural units with the light velocity equal to 1). Inserting the corresponding values of α and β into Eqs. (20)–(22), we arrive at the relations of the form

$$a=2(\cosh\xi+\mathbf{e}\cdot\mathbf{n}\sinh\xi), \tag{38}$$

$$b=\cosh\xi+n^3+(e^3+\mathbf{e}\cdot\mathbf{n})\sinh\xi+e^3\mathbf{e}\cdot\mathbf{n}(\cosh\xi-1), \tag{39}$$

$$c=n_++[\sinh\xi+\mathbf{e}\cdot\mathbf{n}(\cosh\xi-1)]e_+. \tag{40}$$

Now, the corresponding little group element can be obtained from Eqs. (23)–(24). We consider here two special cases: boost $A(v\mathbf{n})$ along the $\mathbf{n}$ direction and boost $\Lambda(v\hat{\mathbf{z}})$ along the z direction. In the first case, we have $\mathbf{e}=\mathbf{n}$ and from Eqs. (38)–(40) and Eqs. (23)–(24), we find

$$e^{i\psi(\Lambda(v\mathbf{n}),k)}=1,\quad z(\Lambda(v\mathbf{n}),k)=0. \tag{41}$$

In the second case, we have

$$\alpha=\frac{1}{\delta}=\sqrt[4]{\frac{1-v}{1+v}},\quad \beta=\gamma=0. \tag{42}$$

Inserting the above values into Eqs. (23) and (24), we get

$$\psi(\Lambda(v\hat{\mathbf{z}}),k)=0,\quad z(\Lambda(v\hat{\mathbf{z}}),k)=\frac{n_-}{k^0\left(\frac{1}{v}-n^3\right)}. \tag{43}$$

III. TRANSFORMATION LAW FOR LINEARLY POLARIZED LIGHT

Now, we apply the above results to discuss some transformation properties of the polarized states and reduced density matrix for photons.

Let us consider first the classical electromagnetic field. As is well known the monochromatic plane electromagnetic wave is in general polarized elliptically. In the special case of the linear polarization we can deal also with the plane wave which is not necessarily monochromatic. In this case the electromagnetic field tensor can be written as

$$F^{\mu\nu}(x)=\mathcal{F}^{\mu\nu}f(ct-\mathbf{n}\cdot\mathbf{x}) \tag{44}$$

for the lineary polarized wave propagating in the **n** direction, where the tensor $\mathcal{F}^{\mu\nu}$ is x^{μ} independent. It is evident that the above formula is covariant under Lorentz transformations. It means that Lorentz transformations preserve linear polarization of an arbitrary plane wave (not necessarily monochromatic). We show that it is also the case on the quantum level.

As is well known (see, e.g., Refs. [11–14]) the one-photon representation space is spanned by the vectors $\{|k,1\rangle,|k,-1\rangle\}$ because the parity operator changes the sign of the helicity. Let us consider first the monochromatic linearly polarized plane wave. The photon state corresponding to such a wave is of the form [14]

$$|k,\phi\rangle\equiv|(k^0,|\mathbf{k}|\mathbf{n}),\phi\rangle=\frac{1}{\sqrt{2}}\sum_{\substack{\lambda=-1\\ \lambda\neq 0}}^{1}e^{i\lambda\phi}|k,\lambda\rangle, \tag{45}$$

where $k^0=|\mathbf{k}|$ and the momentum independent angle ϕ determines the direction of the polarization in the plane perpendicular to the direction of the propagation **n**. The general linearly polarized state corresponding to the wave (44) has the form

$$|g,\phi,\mathbf{n}\rangle=\frac{1}{\sqrt{2}}\sum_{\lambda}e^{i\lambda\phi}\int_0^{\infty}d|\mathbf{k}|g(|\mathbf{k}|)|(k^0,|\mathbf{k}|\mathbf{n}),\lambda\rangle, \tag{46}$$

where **n** is fixed. The state (46) is a tensor product of momentum direction and polarization states in each Lorentz frame. Let us note that states belonging to the proper Hilbert space (wave packets) cannot be exactly linearly polarized states. However, linearly polarized states (46) can be approximated (as tempered distributions) with an arbitrary accuracy by sequences of wave packets. It is interesting to point out a parallelism between classical and quantum description of ideal linearly polarized states. Namely, on the classical level they have infinite total electromagnetic energy while on the quantum level they lie out of the proper Hilbert space of the wave packets, i.e., they are distributions.

Now, we show that *for every inertial observer the linearly polarized state* (46) remains linearly polarized. Indeed, taking into account Eqs. (23)–(26), we find

$$\begin{aligned}U(\Lambda)|g,\phi,\mathbf{n}\rangle&=\frac{1}{\sqrt{2}}\sum_{\lambda}e^{i\lambda(\phi+\psi(\Lambda,\mathbf{n}))}\\&\quad\times\int_0^{\infty}d|\mathbf{k}|g'(|\mathbf{k}|)|(k^0,|\mathbf{k}|\mathbf{n}'),\lambda\rangle\\&=|g',\phi+\psi,\mathbf{n}'\rangle,\end{aligned} \tag{47}$$

where

$$g'(|\mathbf{k}|)=\frac{2}{a}g\left(\frac{2|\mathbf{k}|}{a}\right), \tag{48}$$

a is given by Eq. (20), and by virtue of Eq. (27) the direction $\mathbf{n}'$ is fixed, too. Therefore, the state we received is again linearly polarized.

Now, we discuss the transformation of the reduced density matrix for linearly polarized plane wave. In general for the reduced density matrix describing the helicity properties of the state

$$|f\rangle=\sum_{\lambda}\int d\mu(k)f_{\lambda}(k)|k,\lambda\rangle \tag{49}$$

we obtain the following formula:

$$\hat{\rho}_{\sigma\lambda}=\frac{\int d\mu(k)f_{\sigma}(k)f_{\lambda}^{*}(k)}{\sum_{\lambda}\int d\mu(k)|f_{\lambda}(k)|^2}, \tag{50}$$

where we have used the Lorentz invariant measure

$$d\mu(k)=\theta(k^0)\delta(k^2)d^4k\equiv\frac{d^3\mathbf{k}}{2|\mathbf{k}|}. \tag{51}$$

It should be noted that in general the state $|f\rangle$ can be a tempered distribution (it does not necessarily belong to the Hilbert space but rather to the Gel'fand triple), for example, four-momentum eigenstates. In such a situation the formula (50) should be understood as a result of a proper regularization procedure. Applying the above considerations to the density matrix describing the state $|g,\phi,\mathbf{n}\rangle$, we get the following reduced density matrix:

$$\rho_{\lambda\sigma}(g,\phi,\mathbf{n})=\tfrac{1}{2}e^{i(\lambda-\sigma)\phi}, \tag{52}$$

i.e.,

$$\rho(g,\phi,\mathbf{n})=\frac{1}{2}\begin{pmatrix}1 & e^{2i\phi}\\ e^{-2i\phi} & 1\end{pmatrix}, \tag{53}$$

which in fact represents a reduced pure state because $\rho^2=\rho$. The above density matrix transforms properly under Lorentz transformations, namely,

$$\rho' = \begin{pmatrix} e^{i\psi(\Lambda,\mathbf{n})} & 0 \\ 0 & e^{-i\psi(\Lambda,\mathbf{n})} \end{pmatrix} \rho(g,\phi,\mathbf{n}) \begin{pmatrix} e^{-i\psi(\Lambda,\mathbf{n})} & 0 \\ 0 & e^{i\psi(\Lambda,\mathbf{n})} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & e^{2i(\phi+\psi)} \\ e^{-2i(\phi+\psi)} & 1 \end{pmatrix} = \rho(g',\phi+\psi,\mathbf{n}'). \qquad (54)$$

We stress on the fact that the linearly polarized state admits a covariant reduced density matrix description in terms of helicity degrees of freedom is related to the property of the Lorentz transformation and that it does not generate entanglement between momentum direction and helicity.

Finally, let us note that the von Neumann entropy corresponding to the density matrix (53) is equal to zero. Evidently it is Lorentz-invariant in view of Eq. (54).

Our discussion can be easily recast in terms of polarization vectors defined according to Ref. [11], for different approach see also Ref. [15].

IV. CONCLUSIONS

We have found in this paper the explicit form of the Wigner's little group element in the massless case for arbitrary Lorentz transformation. Using this result we have shown that the light wave which is linearly polarized (but not necessarily monochromatic) for one inertial observer remains linearly polarized also for an arbitrary inertial observer. We have also shown that the reduced density matrix describing linearly polarized photon, obtained by tracing out kinematical degrees of freedom, transforms properly under Lorentz group action. Moreover the corresponding von Neumann entropy is a Lorentz scalar.

ACKNOWLEDGMENTS

This work was supported by University of Łódź and the Laboratory of Physical Bases of Processing of Information.

[1] A. Peres, P.F. Scudo, and D.R. Terno, Phys. Rev. Lett. **88**, 230402 (2002).
[2] M. Czachor, Phys. Rev. A **55**, 72 (1997).
[3] M. Czachor, in *Proceedings of SPIE*, edited by S.P. Holating and A.R. Pirich (SPIE, Bellingham, WA, 1997), Vol. 3076, p. 141.
[4] J. Rembieliński and K.A. Smoliński, Phys. Rev. A **66**, 052114 (2002); e-print quant-ph/0204155.
[5] P.A. Alsing and G.J. Milburn, Quantum Inf. Comput. **2**, 487 (2002).
[6] H. Terashima and M. Ueda, Int. J. Quantum Inf. **1**, 93 (2003).
[7] R.M. Gingrich and C. Adami, Phys. Rev. Lett. **89**, 270402 (2002).
[8] N.H. Lindner, A. Peres, and D.R. Terno, J. Phys. A: Math. Gen. **36**, L449 (2003).
[9] A.J. Bergou, M. Gingrich, and C. Adami, Phys. Rev. A **68** (to be published), e-print quant-ph/0302095.
[10] A. O. Barut and R. Rączka, *Theory of Group Representations and Applications* (PWN, Warszawa, 1977).
[11] S. Weinberg, in *Lectures on Particles and Field Theory*, edited by S. Deser and K. W. Ford, Lectures Delivered at Brandeis Summer Institute in Theoretical Physics Vol. II (Prentice-Hall, Inc., Englewood Cliffs, NJ, 1964), p. 405.
[12] G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers* (McGraw-Hill Book Company, New York, 1961).
[13] F. Gürsey, in *Relativity, Groups, and Topology*, 1963 Les Houches Lectures, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1963), p. 91.
[14] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1996), Vol. I.
[15] A. Peres and D. R. Terno, J. Mod. Opt. **50**, 1165 (2003); eprint quant-ph/0208128.