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Hydrodynamic flow of expanding Bose-Einstein condensates

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We study expansion of quasi-one-dimensional (1D) Bose-Einstein condensate (BEC) after switching off the confining harmonic potential. Exact solution of dynamical equations is obtained in the framework of the hydrodynamic approximation and it is compared with the direct numerical simulation of the full problem, showing excellent agreement at realistic values of physical parameters. We analyze the maximum of the current density and estimate the velocity of expansion. The results of the 1D analysis provides also qualitative understanding of some properties of BEC expansion observed in experiments.

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Many current experiments with Bose-Einstein condensates (BEC) include regime of a free expansion of the gas which initially was (magnetically or optically) trapped by a confining potential [1,2]. The respective phase of the BEC evolution is of practical interest since much information about coherent matter waves is experimentally obtained at this stage (say, by absorption imaging), and also because of the importance of a free BEC flow in the context of recently proposed new devices, for example, an atomic laser [3]. At ultralow temperatures, a trapped BEC is well described by the three-dimensional (3D) Gross-Pitaevskii (GP) equation which in the absence of a trap potential is known as a nonlinear Schrödinger (NLS) equation. In a number of specific cases it can be reduced to a quasi-one-dimensional one. More specifically, this happens when one can neglect (due to some reasons) interaction between transverse modes. In the absence of a trap the problem thus reduces to an integrable model, 1D NLS equation, which is a very well-studied fundamental model of the nonlinear physics.

The problem of theoretical description of evolution of BEC confined by harmonic potential with varying parameters has been addressed in several papers [4]. In the experiment [5] the realization of BEC expansion in quasi-1D waveguide has been reported and again excellent agreement has been found with theoretical predictions at long enough values of time of evolution. These results have been confirmed by numerical solution of corresponding dynamical equations for the quasi-1D case [4,6]. In the present paper, we give a complete analytical treatment of this problem. We consider only BECs with positive scattering lengths. In terms of the NLS equation, the situation we are dealing with corresponds to the evolution of an initially localized pulse with the initial profile corresponding to the ground state of a BEC confined in a parabolic potential, i.e., to the defocusing NLS equation with zero boundary conditions at infinity. As it is known, such a problem does not have soliton solutions, and independent of the number of particles the condensate will spread out and its density will tend to zero with time. The formal analytical description of the respective solution at $t\rightarrow\infty$ was obtained rather long ago [7]. However, for practical purposes of analysis of experimental data it is desirable to have a description of the condensate evolution during the initial stages of time, too. It turns out that such evolution also admits a rather complete analytical description in a number of cases, and, in particular, when the initial density distribution is smooth enough, or in terms of BEC in the limit of a large number of atoms which leads to the so-called Thomas-Fermi (TF) approximation. Then the hydrodynamic approach allows one to describe analytically evolution of the density and velocity fields of the condensate. The evolution of BEC at initial stages appears to be quite rich. For example, as it has been recently shown in Ref. [8], it may display wave breaking for some specific initial distributions of the BEC density.

In the present Brief Report we describe the free expansion of a BEC initially confined in a harmonic potential. On the basis of the hydrodynamic approach we find the time dependence of the density and velocity distributions and calculate such characteristics of the gas flow as the value and space and time coordinates of the maximum of current, velocity of the "edge" points of the condensate, as well as asymptotic density and velocity distributions. Also, we show that simple analytical estimates of the 1D problem allow one to give qualitative understanding of the phenomena observed in experiments with effectively 2D BEC.

Let us start with the 3D GP equation for the order parameter $\psi\equiv\psi(\mathbf{r},t)$,

$$i\hbar(\partial\psi/\partial t) = -(\hbar^2/2m)\Delta\psi + V_{trap}(\mathbf{r})\psi + g_0|\psi|^2\psi, \quad (1)$$

where we use the standard notation $g_0=4\pi\hbar^2 a_s/m$, a_s being the s-wave scattering length, which is considered positive and m being the atomic mass; $V_{trap}(\mathbf{r})$ is a trap potential. Considering the case of a two-dimensional drop let BEC [9], we take $V_{trap}=(m/2)\omega^2x^2$, where ω is the harmonic-oscillator frequency, and in the transverse direction (i.e., in the direction orthogonal to the x axis) the size of the condensate is supposed large enough to be considered infinite in the first approximation. In the longitudinal direction the size of the condensate is of the order of magnitude $a=(\hbar/m\omega)^{1/2}$. It is convenient to introduce some typical reference frequency ω_0 so that the trap frequency is measured in units of ω_0, $\omega=\nu\omega_0$, ν being the dimensionless trap frequency. To make the dynamical equation dimensionless, we introduce

$$\psi(\mathbf{r},t)=(2\sqrt{2}\pi a_0^2 a_s)^{-1/2}\exp\left(i\mathbf{k}_\perp\cdot\mathbf{r}_\perp - i\frac{\hbar k_\perp^2}{2m}t\right)\Psi(x,t), \tag{2}$$

where $\mathbf{r}_\perp=(y,z)$ and $a_0^2=\hbar/m\omega_0=a^2\nu$, and make a change of independent variables $x=2^{-1/4}a_0x'$ and $t=2^{1/2}t'/\omega_0$. This results in the canonical form of the NLS equation with a parabolic potential,

$$i(\partial\Psi/\partial t)+(\partial^2\Psi/\partial x^2)-2|\Psi|^2\Psi=\tfrac{1}{2}\nu^2x^2\Psi, \tag{3}$$

where the primes were suppressed. The dimensionless BEC wave function $\Psi(x,t)$ is normalized according to

$$\int_{-\infty}^{\infty}|\Psi(x,t)|^2dx=\frac{4\pi}{2^{1/4}}\frac{\mathcal{N}a_sa_0}{S}, \tag{4}$$

where $\mathcal{N}$ is the total number of particles and S is an effective area of the transverse cross section of the condensate.

A stationary solution $\Psi(x,t)=\Psi(x)\exp(-i\mu t)$, corresponding to the ground state of BEC, is given by $\Psi(x)$ satisfying the equation

$$(d^2\Psi/dx^2)+\mu\Psi-2|\Psi|^2\Psi=\tfrac{1}{2}\nu^2x^2\Psi, \tag{5}$$

subject to the zero boundary condition at $|x|\to\infty$ and having no other zeros. The eigenvalue μ (chemical potential) is determined by normalization (4). In dimensionless units the longitudinal size of the condensate is of the order of magnitude $\mu^{1/2}/\nu$ and if $\mu\gg\nu$, the considerable part of the condensate can be described by the TF approximation in which the term with the second space derivative in Eq. (5) can be neglected almost everywhere, so that

$$\rho_0(x)\equiv|\Psi_{TF}(x)|^2=\tfrac{1}{2}(\mu-\tfrac{1}{2}\nu^2x^2), \tag{6}$$

with normalization

$$\int_{-\sqrt{2\mu}/\nu}^{\sqrt{2\mu}/\nu}|\Psi_{TF}(x)|^2dx=\frac{(2\mu)^{3/2}}{3\nu}. \tag{7}$$

The TF approximation (6) fails at the tails of the density distribution where the density decays exponentially as $|x|\to\infty$ instead of vanishing at finite distance $|x|=\sqrt{2\mu}/\nu$ (called TF radius) according to Eq. (6). Since only a small part of the condensate's mass is concentrated in these tails (for example, for $\mu=2$ and $\nu=0.5$, it is less than 1.5%), in the limit $\mu\gg\nu$ distribution (6) is assumed to be a good approximation of the initial density distribution of BEC before switching off the external potential.

By equating Eqs. (4) and (7) we determine the value of the dimensionless chemical potential μ in terms of experimentally measurable parameters,

$$\mu=[2^{1/4}3\pi\nu(\mathcal{N}a_sa_0/S)]^{2/3}. \tag{8}$$

Then the condition $\mu\gg\nu$ yields the criterion $\mathcal{N}a_sa/S\gg1$ of applicability of the TF approximation in physical units. In what follows we deal mainly with two parameters ν and μ, which completely define the initial distribution of the condensate density.

After switching off the potential, the condensate evolves according to Eq. (3) with $\nu=0$. If evolution does not lead to wave breaking of the pulse, then we still can neglect the dispersion effects and use the hydrodynamic approximation for description of this evolution. As we shall see, this is the case of initial distribution (6), so that the hydrodynamic approximation is valid even at asymptotically large values of time considered in Ref. [7]. Therefore we pass from the NLS equation (3) to its hydrodynamic representation. We represent $\Psi(x,t)$ in the form

$$\Psi(x,t)=\sqrt{\rho(x,t)}\exp\left(i\int^x v(x',t)dx'\right), \tag{9}$$

so that substitution of Eq. (9) into Eq. (3) with $\nu=0$ yields

$$\tfrac{1}{2}\rho_t+(\rho v)_x=0,\quad \tfrac{1}{2}v_t+vv_x+\rho_x=0, \tag{10}$$

which is subject to the initial conditions

$$\rho(x,0)=\rho_0(x),\quad v(x,0)=0, \tag{11}$$

where $\rho_0(x)$ is taken to be the initial distribution (6). In Eq. (10) we have neglected the higher space derivatives of the density $\rho(x,t)$ that correspond to the quantum-pressure contribution.

A problem similar to Eqs. (10) and (11) was studied in nonlinear optics long ago [10] for the opposite sign of the "pressure" ρ_x in Eq. (10) which corresponds to evolution of an optical beam in a focusing Kerr medium. More complex initial conditions were considered in Refs. [11,12], and in the recent paper [8] the same nonlinear geometrical optics method has been applied to the investigation of wave breaking phenomena in BEC. Here we shall apply the method of Ref. [10] to the problem of BEC expansion.

We look for a solution of Eqs. (10) and (11) in the form

$$\rho(x,t)=\frac{\mu}{2f(t)}(1-\nu^2x^2/2\mu f(t)^2),\quad v(x,t)=x\phi(t), \tag{12}$$

where according to Eq. (11) the functions $f(t)$ and $\phi(t)$ must satisfy the initial conditions

$$f(0)=1,\quad \phi(0)=0. \tag{13}$$

Substitution of Eq. (12) into Eq. (10) gives the relationship between $f(t)$ and $\phi(t)$,

$$\phi(t)=f'(t)/2f(t), \tag{14}$$

as well as the differential equation for $f(t)$,

$$f^2f''=2\nu^2. \tag{15}$$

This equation can be readily solved with the initial conditions $f(0)=1$, $f'(0)=2f(0)\phi(0)=0$ [see Eqs. (13) and (14)] to give

$$4\nu t = 2\sqrt{f(f-1)} + \ln[2f - 1 + 2\sqrt{f(f-1)}]. \quad (16)$$

This formula determines implicitly f as a function of t and, hence, the distribution of density $\rho(x,t)$ according to Eq. (12). Then the function $\phi(t)$ is determined by Eq. (14), so that distribution of velocities $v(x,t) = x\phi(t)$ is given by

$$v(x,t) = [x\nu/f(t)]\sqrt{1 - 1/f(t)}. \quad (17)$$

Thus, Eqs. (12), (14), and (16) give the complete analytic solution of the posed problem in the TF approximation.

One of the effects which accompanies the expansion of the condensate is a nonmonotonic behavior of the current density

$$J(x,t) = \rho(x,t)v(x,t). \quad (18)$$

Equating its derivatives with respect to x and t to zero yields with the use of Eqs. (10), (12), and (17) its maximum value as well as corresponding values of x and t:

$$J_m(x_m, t_m) = \frac{(2\mu)^{3/2}}{27},$$

$$x_m = \sqrt{3\mu/2\nu^2}, \quad t_m = [\sqrt{3} + \ln(2+\sqrt{3})]4\nu, \quad (19)$$

which with the use of Eq. (8) can be expressed in terms of experimentally measurable parameters

$$J_m(x_m, t_m) = 1.17\nu(\mathcal{N}a_s a_0/S),$$

$$x_m = 2.74\nu^{-2/3}(\mathcal{N}a_s a_0/S)^{1/3},$$

$$t_m = [\sqrt{3} + \ln(2+\sqrt{3})]/4\nu \approx 0.76\nu^{-1}. \quad (20)$$

The maximum of the density flow is proportional to the number of particles $\mathcal{N}$, although the time coordinate of the current maximum does not depend on the density but only on the condensate aspect ratio ν. Let us consider the typical experiments on the condensate with $\mathcal{N} = 10^5$ atoms of ^{87}Rb with scattering length $a_s \approx 5$ nm [13]. Taking the length scale of the condensate of the order of $a_0 \approx 1$ μm (which corresponds to the frequency $\omega_0 \sim 5\times10^3$ Hz) and transversal radius ≈ 10 μm one can obtain for $\nu = 0.2$ the maximum of the current density $J_m \approx 100$ atoms μm^{-2} ms^{-1} in the coordinate $x_m \approx 11.5$ μm (initial TF radius was ≈ 13.3 μm) and at the time $t_m \approx 1.1$ ms.

In asymptotic limit of large t, when $f(t) \gg f(0)$, we have

$$f(t) \cong 2\nu t, \quad \phi(t) \cong 1/2t. \quad (21)$$

Hence, solution (12) takes the form

$$\rho(x,t) \cong \frac{\mu}{4\nu t}(1 - x^2/8\mu t^2), \quad v(x,t) \cong x/2t. \quad (22)$$

These formulas describe a hydrodynamic flow "by inertia" when the density becomes so small that the pressure does not accelerate the gas anymore [factor 1/2 in the second formula (22) corresponds to definition of the "time" variable in Eqs. (10)].

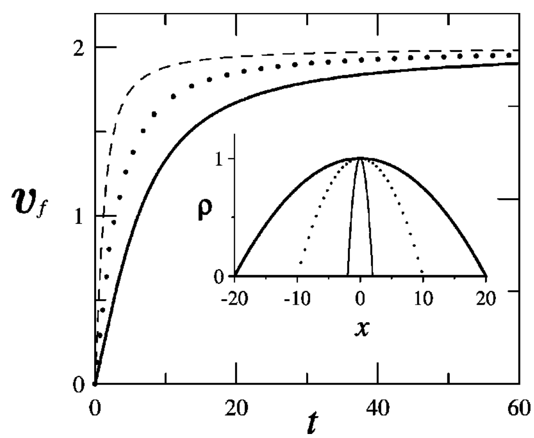

FIG. 1. Time dependence of the front velocity $v_f(t)$ for $\mu = 2$ and for different ν (solid line $\nu = 0.1$, dotted line $\nu = 0.2$, and dashed line $\nu = 0.5$). In the inset the corresponding initial distributions of the condensate density $\rho_0(x)$ are depicted.

From Eq. (22) we can easily find asymptotic distribution of particles on their velocities:

$$W(v)dv = \frac{\mu}{2\nu}\left(1 - \frac{v^2}{2\mu}\right)dv, \quad (23)$$

which gives the number of atoms with velocities in the interval $(v, v+dv)$. Also, in the framework of the TF approximation, Eq. (12), and with the use of Eq. (15) one obtains the velocity of the condensate front,

$$v_f(t) = \sqrt{\frac{\mu}{2}}\frac{1}{\nu}\frac{df}{dt} = \left[2\mu\left(1 - \frac{1}{f}\right)\right]^{1/2}, \quad (24)$$

which at $t \to \infty$ asymptotically goes to $v_f(\infty) \cong \sqrt{2\mu}$. The corresponding plots are shown in Fig. 1.

To verify the above findings, we have carried out numerical simulations of the condensate dynamics governed by Eq. (3) after switching off the trap potential, i.e., with $\nu = 0$. Initial profile of the condensate was taken as a numerical solution of Eq. (5).

In agreement with analytical predictions we found that during expansion the current density $J(x,t) = \rho(x,t)v(x,t)$ has a maximum (see Fig. 2). The space and time coordinates of this maximum which were calculated with the use of the TF approximation and by direct numerical calculations practically coincide (see Fig. 3). The discrepancy between the analytical and numerical calculations increases with increase

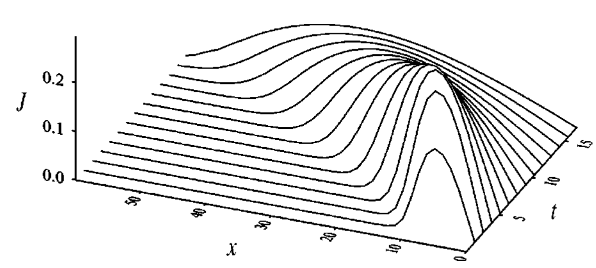

FIG. 2. Spatiotemporal behavior of the density flow $J(x,t)$ for $\mu = 2$ and $\nu = 0.2$.

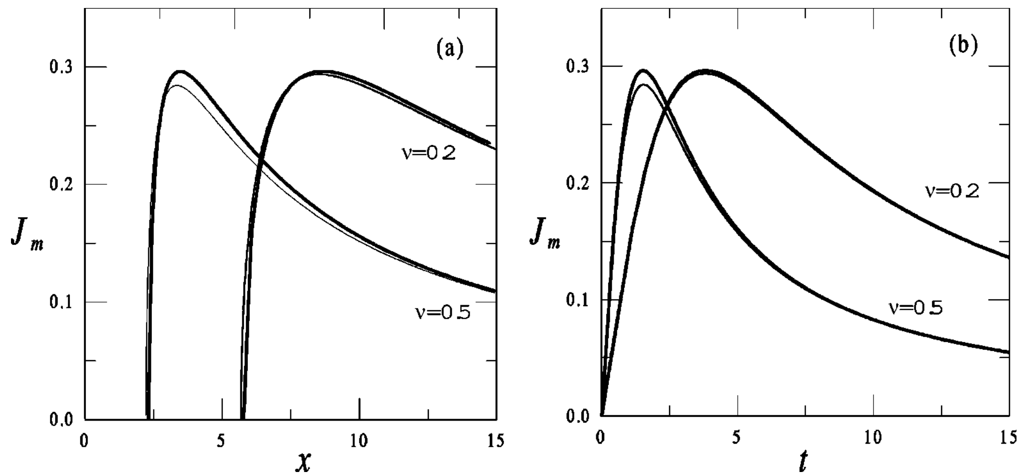

FIG. 3. Dependence of the maximum of the density flow with coordinate (a) and time (b). Here solid thick lines correspond to the analytical solution and thin lines show the results of numerical calculations.

of the parameter ν (see Fig. 3) which can be explained by the loss of the accuracy of the TF approximation in accordance with the criterion $\mu \gg \nu$.

Let us estimate the time when we can consider an expansion of the condensate cloud only in the axial direction. In the particular case of the pancake geometry this can be done by a comparison of the respective kinetic parts of the initial 3D GP equation. This gives us a criterium $T_0/T_\perp \approx a^2/a_\perp^2 \approx 0.01 \ll 1$, where T_0 and $T_\perp$ are characteristic times of the processes in the axial and the radial direction of the condensate, respectively. Considering time scale $t \ll T_\perp$ one can neglect the kinetics of the condensate in the radial direction and thus consider it to be unchanged in this direction. In order to estimate T_0 and $T_\perp$ we notice that velocity has an order of magnitude $v \sim \sqrt{\mu}$ [see Eq. (24)] and thus $T_0 \simeq a_\perp/\sqrt{\mu}$, $a_\perp$ being the transverse radius. In the case of numerical calculations reported in Figs. 1 and 2 one estimates $T_0 \sim 0.4$ ms (1 dimensional unit) and $T_\perp \sim 40$ ms (100 dimensional units) and thus the asymptotic limit described by Eq. (21) is indeed achieved.

To conclude, it is interesting to make a comparison of our simplified 1D model with some experiments where the expansion of the condensate cloud was observed. To follow the real experimental setup we take $\mu = \text{const}$ which corresponds to constant density ρ_0 at the condensate center [see Eq. (6)] and look for the front velocities $v_f(t)$ with different ν which correspond to different sizes (number of particles $\mathcal{N}$) of the condensate in the transversal directions as was shown in Fig. 1. For larger values of ν (more narrow profile of the condensate) the front velocity v_f is higher than for the condensate with smaller ν. This result of the 1D model is in qualitative agreement with the experimental observations where faster expansion of the condensate cloud was observed in the direction with smaller transverse size [1,2].

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