Optical interferometry at the Heisenberg limit with twin Fock states and parity measurements

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Holland and Burnett [Phys. Rev. Lett. **71**, 1355 (1993)] have argued that twin Fock states of equal photon number N injected at both input ports of a Mach-Zehnder interferometer lead to phase measurements with accuracies approaching the Heisenberg limit $\Delta\varphi_{HL}=1/(2N)$. However, the method of phase detection suggested by those authors, obtaining the difference of the photocurrents at the output ports of the interferometer, is not sensitive to the phase difference between the two interferometer paths; in fact, the photocurrent vanishes. In this paper we show that the use of parity measurements on just one of the output modes not only is sensitive to the phase difference but that the sensitivity approaches the Heisenberg limit for large N.

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Coherent radiation fields with Poisson-distributed numbers of photons, as commonly found in the output of phase-stabilized lasers, provide interferometric phase sensitivities bound by the familiar standard quantum limit (or shot noise limit) of $\Delta\varphi_{SQL}=1/\sqrt{\bar{N}}$, where $\bar{N}$ is the average number of photons counted in a chosen time interval. Quantum-state synthesis, which holds the promise for advantageous technological progress in areas of information science, such as computing and cryptography, also offers the opportunity to surpass the standard quantum limit of interferometric phase sensitivity and to reach the so-called Heisenberg limit $\Delta\varphi_{HL}=1/\bar{N}$, the ultimate level of sensitivity allowed by quantum mechanics [1]. This goal requires joint consideration of quantum-state generation and detection methods. This paper exemplifies such an assessment for interferometry with twin Fock states, together with a parity detection method on one of the output beams.

Some years ago, Holland and Burnett [2] studied the uncertainties in optical phase measurements obtained in a Mach-Zehnder interferometer (MZI) under the assumption of twin Fock states $|N\rangle_a|N\rangle_b$ at the inputs of the first beam as shown in Fig. 1. The goal is to measure φ, the phase difference between the two paths of the MZI. By studying the phase-difference distribution for the states *inside* the MZI just prior to the second beam splitter BS2, these authors concluded that the uncertainty in the measurement of the phase difference approaches the Heisenberg limit $\Delta\varphi_{HL}=1/(2N)$ relevant for a total of $2N$ photons passing through the interferometer. The input state $|N\rangle_a|N\rangle_b$ itself does not realize the Heisenberg limit exactly, but, according to Holland and Burnett [2], approaches the limit asymptotically as N becomes large. On the one hand, it was shown by Bollinger *et al.* [3] that if the field state just after BS1 is somehow a maximally entangled state (MES), i.e., if it is of the form

$$|2N::0\rangle_{a,b}^{\Phi_N}\equiv\frac{1}{\sqrt{2}}(|2N\rangle_a|0\rangle_b+e^{i\Phi_N}|0\rangle_a|2N\rangle_b), \tag{1}$$

where Φ_N is a phase that may depend on N, then the phase uncertainty is *exactly* at the Heisenberg limit $\Delta\varphi_{HL}=1/(2N)$. Obtaining such a state inside the MZI is not easy and, in fact, cannot be done with an ordinary beam splitter alone. Schemes for generating such states using nonlinear devices and linear devices in conjunction with conditional measurements are now known and under development [4]. On the other hand, referring to Fig. 1, with the input twin Fock states $|N\rangle_a|N\rangle_b$, the state inside the interferometer just after BS1 is given by the expansion [5]

$$|\psi_{2N}\rangle=\sum_{k=0}^{N}(-1)^{N-k}\left[\binom{2k}{k}\binom{2N-2k}{N-k}\left(\frac{1}{2}\right)^{2N}\right]^{1/2}\times|2k\rangle_a|2N-2k\rangle_b, \tag{2}$$

which we refer to as the arcsine (AS) state. We assumed that the 50:50 beam splitter BS1 of Fig. 1 is described by the transformation [5,6] $\hat{U}_{BS1}=\exp[\pi(\hat{a}^\dagger\hat{b}-\hat{b}^\dagger\hat{a})/4]$. The two-photon $(N=1)$ version of this state,

$$|\psi_2\rangle=\frac{1}{\sqrt{2}}(|2\rangle_a|0\rangle_b-|0\rangle_a|2\rangle_b)=|2::0\rangle_{a,b}^{\pi}, \tag{3}$$

a MES, has long been available in the laboratory [7]. The four-photon version $(N=2)$ given by

$$\begin{aligned}|\psi_4\rangle&=\sqrt{\frac{3}{8}}(|4\rangle_a|0\rangle_b+|0\rangle_a|4\rangle_b)-\frac{1}{2}|2\rangle_a|2\rangle_b\\&=\sqrt{\frac{3}{4}}|4::0\rangle_{a,b}^{0}-\frac{1}{2}|2\rangle_a|2\rangle_b\end{aligned} \tag{4}$$

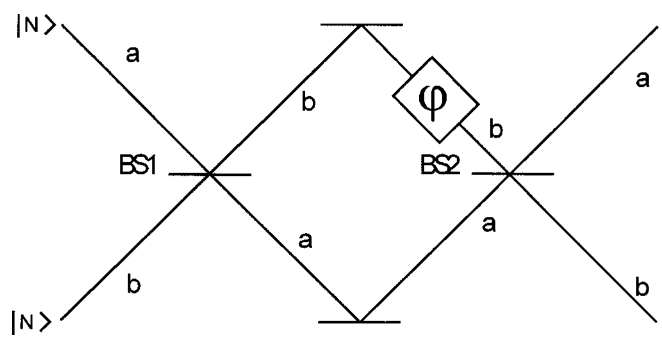

FIG. 1. Schematic of the Mach-Zehnder interferometer for the detection of the phase difference φ when the twin Fock states $|N\rangle_a|N\rangle_b$ are injected into the first beam splitter.

was detected in recent experiments by Ou *et al.* [8]. With the phase shift operator in the b mode (the clockwise path as in Fig. 1) represented by $\hat{U}(\varphi)=\exp(i\varphi\hat{b}^{\dagger}\hat{b})$, the state just prior to the second beam splitter BS2 is

$$|\psi_{2N}(\varphi)\rangle=\sum_{k=0}^{N}(-1)^{N-k}e^{i\varphi(2N-2k)}\left[\binom{2k}{k}\binom{2N-2k}{N-k}\times\left(\frac{1}{2}\right)^{2N}\right]^{1/2}|2k\rangle_a|2N-2k\rangle_b. \tag{5}$$

Evidently, there are strong correlations between the photon number states of the two modes. Because of this, the only nonzero elements of the joint photon number probability distribution are the joint probabilities for finding $2k$ photons in mode a and $2N-2k$ in mode b given by

$$\begin{aligned}P_{AS}(2k,2N-2k)&=|{}_a\langle 2k|_b\langle 2N-2k|\psi_{2N}\rangle|^2\\&=\binom{2k}{k}\binom{2N-2k}{N-k}\left(\frac{1}{2}\right)^{2N},\quad k\in[0,N],\end{aligned} \tag{6}$$

forming a distribution known in probability theory as the fixed-multiplicative discrete arcsine law of order N [9]; hence the name for our states. This should be contrasted with the distribution for the MES of Eq. (1), which takes the form $P_{MES}(2k,2N-2k)=(\delta_{k,0}+\delta_{k,N})/2$.

We now illustrate the phase properties of these states. In the most general case, if $\hat{\rho}(\varphi)$ is the density operator of the field inside the MZI just prior to BS2, then the phase distribution is defined by [10]

$$\mathcal{P}(\theta_a,\theta_b|\varphi)=\langle\theta_a|\langle\theta_b|\hat{\rho}(\varphi)|\theta_a\rangle|\theta_b\rangle/(2\pi)^2, \tag{7}$$

where the $|\theta_i\rangle=\sum_n e^{in\theta_i}|n\rangle_i$ $(i=a,b)$ are the phase states. In terms of the number basis we have

$$\mathcal{P}(\theta_{\rm a},\theta_b|\varphi)=\frac{1}{(2\pi)^2}\sum_{n,n',m,m'=0}^{\infty}{}_a\langle n'|_b\langle m'|\hat{\rho}|n\rangle_a|m\rangle_b\times e^{i\theta_a(n-n')}e^{i\theta_b(m-m')}, \tag{8}$$

which amounts to a discrete Fourier transform of the density operator's elements. For the sake of comparison, suppose the state just before the second beam splitter is the MES $|2N::0\rangle_{a,b}^{2N\varphi}$, where the phase shift φ has been taken into account. The phase-difference distribution is obtained by integration over the sum of phases,

$$\begin{aligned}\mathcal{D}_{MES}(\theta|\varphi)&\equiv\frac{1}{2}\int_0^{2\pi}\mathcal{P}_{MES}\left(\frac{\theta_s-\theta}{2},\frac{\theta_s+\theta}{2}\middle|\varphi\right)d\theta_s\\&=\frac{1}{2\pi}\cos^2[N(\theta+\varphi)],\end{aligned} \tag{9}$$

where $\theta,\theta_s=\theta_b\mp\theta_a$ are the phase difference and sum, respectively. In Fig. 2 we plot (a) the elements of the joint number distribution $P_{MES}(2k,2N-2k)$ and (b) the phase-difference distribution $\mathcal{D}_{MES}(\theta|0)$ for $2N=20$. The latter of course oscillates with "frequency" N and thus the peak-to-trough spacings are on the order of $1/2N$ (the Heisenberg limit). For the arcsine states we similarly obtain

$$\mathcal{D}_{AS}(\theta+\pi/2|\varphi)\equiv\frac{1}{2\pi}g^*(\theta-\varphi+\pi/2)g(\theta-\varphi+\pi/2),$$

$$g(u)=e^{iN(u+\pi)}\sum_{n=-N/2}^{N/2}\left[\binom{N+2n}{N/2+n}\binom{N-2n}{N/2-n}\left(\frac{1}{2}\right)^{2N}\right]^{1/2}e^{i2nu}. \tag{10}$$

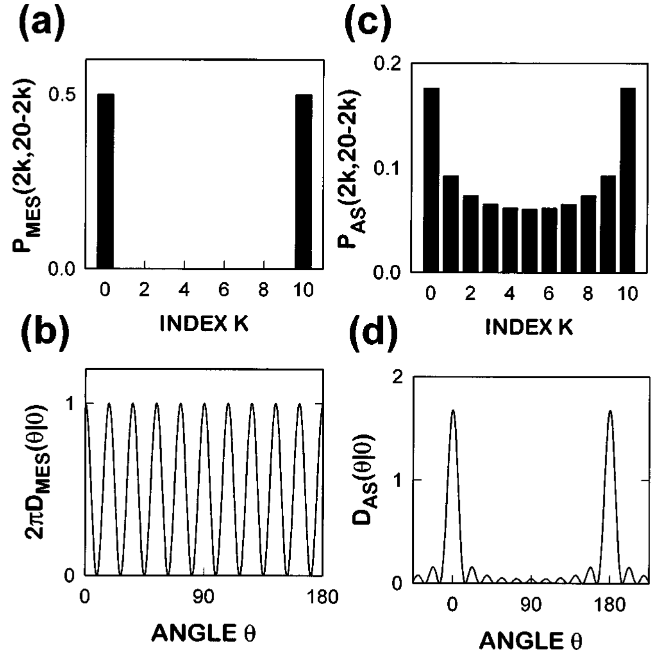

FIG. 2. (a) Photon number distribution and (b) phase-difference distribution for the maximally entangled states for $2N=20$; (c) and (d) the same but for the arcsine state.

The fixed phase translation by $\pi/2$ results from the specific choice of beam splitter type for BS1; the distribution $\mathcal{D}_{AS}(\theta|\varphi)$ therefore represents generic compensation to the phase origin. In Fig. 2 we also plot (c) the elements of the joint photon number distribution $P_{AS}(2k,2N-2k)$ and (d) the phase-difference distribution $\mathcal{D}_{AS}(\theta|\varphi)$, again for $2N=20$ and for $\varphi=0$. For $P_{AS}(2k,2N-2k)$ we see the characteristic "bathtub" shape of an arcsine distribution. Although it would seem, from the pictured joint distributions, that the arcsine states represent a poor candidate for precision interferometry, they in fact provide an excellent approximation to the Heisenberg limit in selected ranges of the phase difference. This is evident from the corresponding phase-difference distributions of Figs. 2(b) and 2(d). The MES produces a simple harmonic dependence $\cos^2(N\theta)$ for a total of $2N$ photons, while the arcsine states approximate to $\sin^2(N\theta+\theta)/\sin^2\theta$ for large N, equivalent to a uniform distribution of photon pairs. A familiar analogy is two-slit versus $(N+1)$-slit classical interference from classical optics [11]. For the MES, the peak-to-trough distance along the horizontal axis is $\sim 1/2N$, approximately the Heisenberg limit of resolution. For the arcsine states, interference cancels out all the oscillations except for a set of two spikes separated by a phase difference of π. Their widths, however, are still $\sim 1/2N$ for large N. Evidently the distribution of photons for certain specific non-MES states, such as the arcsine, can still attain Heisenberg-limited sensitivity, e.g., MES performance, over specific ranges of phase difference. This restriction is similar to the situation that occurs, for example, in optical squeezing, where field quadrature fluctuations can be reduced within certain useful measurement windows.

Holland and Burnett [1] assumed that the measurement of the phase difference in the two arms of the MZI could be carried out in the usual way by subtracting the currents of the photodetectors placed at the output ports of the second beam splitter BS2, essentially measuring the operator $\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}$ at the output. But for states of the type of Eq. (3) or Eq. (2) inside a MZI, this difference will vanish, thus yielding no information on the relative path lengths. This is a result of the symmetry between the two modes of the state inside the MZI. Alternatively, as Hillery, Zou, and Buzek [12] pointed out, the phase-difference distribution just prior to BS2 consists of two narrow peaks, as shown in Fig. 2(d), and it is this double peaked structure that accounts for the vanishing of the difference in the output fields at BS2. In any case, the photon number difference operator is not a useful measure of the phase difference φ for input twin Fock states.

How then can we measure φ and at the same time attain sensitivity at the Heisenberg limit? Kim *et al.* [13] and Han [14] considered the square of the difference of the outputs $(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b})^2$ but this measure, although sensitive to the phase shift φ, does not have the desired accuracy. Bollinger *et al.* [3], in connection with spectroscopy using MES of a system of N trapped ions whose optical analog has similar phase distribution properties, showed that a measurement of the parity of just one of the output field modes is not only sensitive to the phase shift but does, in fact, yield accuracies in phase measurements at the Heisenberg limit. Indeed, we previously discussed the use of this measure for optical interferometry with MES [4(a),(b),(c),(g)]. As the expectation of the number difference operator vanishes in both the MES of Eq. (1) and the arcsine states of Eq. (2), it is reasonable to consider the use of parity measurements in the case of the input twin Fock states $|N\rangle_a|N\rangle_b$.

We consider a detector that is placed at one of the output beams, for instance, the b mode. We write the parity operator for this mode as $\hat{O}=(-1)^{\hat{b}^\dagger\hat{b}}=\exp(i\pi\hat{b}^\dagger\hat{b})$. With the operator representation for the beam splitter BS2 [5,6] taken as $\hat{U}_{BS2}=\exp[-i\pi(\hat{a}^\dagger\hat{b}+\hat{a}\hat{b}^\dagger)/4]$, the expectation value of the parity operator is

$$\langle\hat{O}\rangle_N=\langle\psi_{2N}(\varphi)|\hat{U}_{BS2}^\dagger\hat{O}\,\hat{U}_{BS2}|\psi_{2N}(\varphi)\rangle$$
$$=\sum_{k=0}^{N} e^{i2\varphi(N-2k)}\binom{2k}{k}\binom{2N-2k}{N-k}\left(\frac{1}{2}\right)^{2N}. \qquad (11)$$

The imaginary part of this function sums identically to zero as it is the product of an even times an odd function of k. The real part is identically a Legendre polynomial [15]:

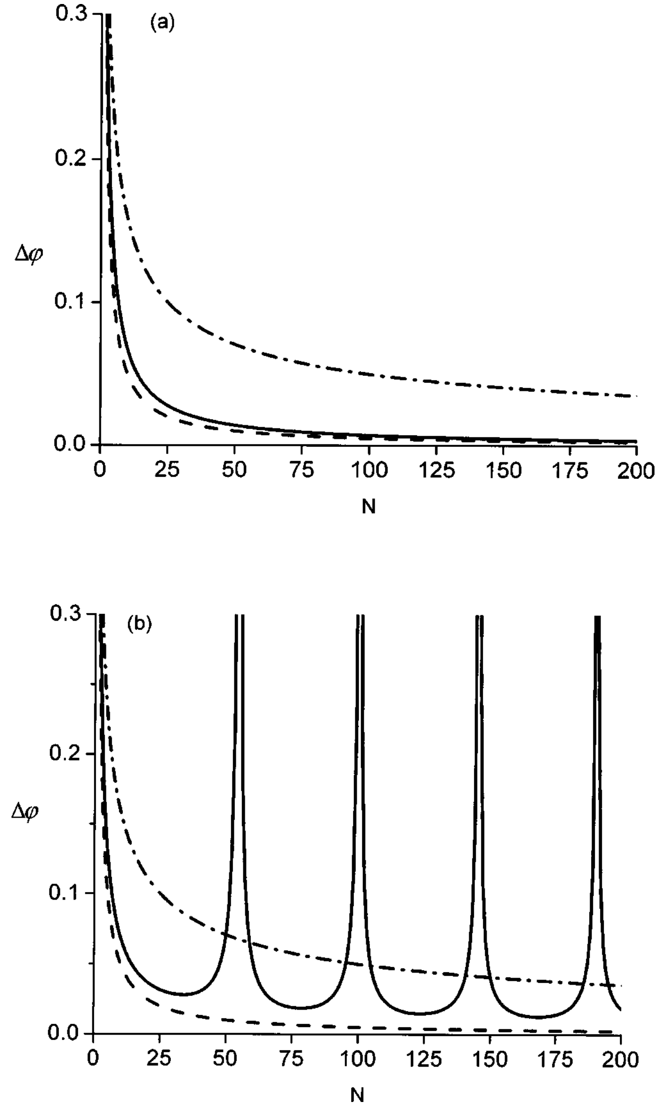

FIG. 3. $\Delta\varphi$ versus N (solid line) along with $\Delta\varphi_{SQL}$ (dot-dashed line) and $\Delta\varphi_{HL}$ (dashed line) for (a) $\varphi=0$, (b) $\varphi=\pi/90$.

$$\langle\hat{O}\rangle_N=P_N[\cos(2\varphi)]. \qquad (12)$$

The phase uncertainty determined from the error propagation calculus is given by

$$\Delta\varphi=\frac{\Delta O}{\left|\frac{\partial\langle\hat{O}\rangle}{\partial\varphi}\right|}, \qquad (13)$$

where $\Delta O=\sqrt{\langle\hat{O}^2\rangle-\langle\hat{O}\rangle^2}=\sqrt{1-\langle\hat{O}\rangle^2}$ as $\hat{O}^2=\hat{I}$. For $N=1$ we have $\langle\hat{O}\rangle_1=\cos 2\varphi$ so that $\Delta\varphi=1/2$, the Heisenberg limit for a total of $2N=2$ photons. For $N=2$ (the photomixing of two photons with two photons) we have $\langle\hat{O}\rangle_2=(3\cos^2 2\varphi-1)/2=1/4+3/4\cos 4\varphi$, from which we obtain, in the limit $\varphi\to 0$, $\Delta\varphi=1/\sqrt{12}=0.2886$, which is just above the Heisenberg limit of $\Delta\varphi_{HL}=1/4=0.25$, and still considerably below the standard quantum limit of $\Delta\varphi_{SQL} = 1/\sqrt{4} = 0.5$ for a total of $2N=4$ photons passing through the interferometer.

Assuming now that $\varphi=0$, we plot in Fig. 3(a) the phase uncertainty $\Delta\varphi$ obtained from our states as a function of N along with $\Delta\varphi_{HL}=1/(2N)$ (dashed line) and $\Delta\varphi_{SQL} = 1/\sqrt{2N}$ (dash-dotted line). We notice that the phase uncertainty for the parity measurement very rapidly approaches the Heisenberg limit for increasing photon number $2N$ and is always much less than the standard quantum limit. In Fig. 3(b) we plot the phase uncertainty for $\varphi=\pi/90 = 0.0349$ rad from the origin. Evidently, for certain photon numbers, the phase uncertainty blows up due to the periodic nature of $\langle \hat{O} \rangle$, but then there are other photon numbers where the uncertainty is still below the standard quantum limit. Therefore the twin Fock states $|N\rangle_a|N\rangle_b$ may still be of use for interferometry for certain N even when the phase difference $\varphi \neq 0$. Of course, for very weak phase shifts, as are expected from gravity waves, and starting from a balanced interferometer where $\varphi=0$, we still can expect high-resolution phase measurements over a wide range of input photon numbers.

Some comments are in order with respect to the measurement of parity. Operationally, one possible way to perform such measurements is to count the number of photons and raise -1 to that power. This type of measurement has previously been described by Banaszek and Wódkiewicz [16] in a proposed experimental scheme for testing nonlocality in phase space. Of course, photon number detectors capable of measuring photon numbers at single photon resolution are required. Presently, such detectors are not available although there have been recent developments toward that goal [17]. On the other hand, it is well known [18] that for some quantum state $|\psi\rangle$ the Wigner function $W(\alpha)$ can be written as

$$W(\alpha) = \frac{2}{\pi}\sum_{n=0}^{\infty} (-1)^n \langle \psi|\hat{D}(\alpha)|n\rangle\langle n|\hat{D}^{\dagger}(\alpha)|\psi\rangle, \quad (14)$$

where $\hat{D}(\alpha)$ is the displacement operator, from which it follows that the expectation value of the parity operator $\hat{O}$ is, apart for a numerical factor, the Wigner function evaluated at the origin: $\langle \hat{O} \rangle = (\pi/2)\, W(0)$. Banaszek *et al.* [19] have discussed the direct measurement of the Wigner function by photon counting while Banaszek and Wódkiewicz [20] have discussed the effects of detector efficiency on this procedure. The authors showed that, while the correction for nonunit quantum efficiency introduces significant errors for large displacements in phase space, the Wigner function is very well reconstructed near the origin, the region of interest for parity detection. Of course, it is quite possible that other methods not requiring direct photon counting, such as homodyning [21], might be used to reconstruct the Wigner function. Finally, other methods, insensitive to photon number, but where a nonlinear device coupling two modes acts like a parity dependent switch [22], have been proposed.

To conclude, we note that optical parity detection is a suitable method to achieve Heisenberg-limited interferometry when twin Fock states are presented at the inputs of the system. The method circumvents the lack of signal phase sensitivity in the traditional balanced homodyne detection if applied to this system. Finally, we remark that the phase measurement scheme proposed here for twin Fock states of photonic fields also has applications for phase resolution between two Bose-Einstein condensates where the number of atoms in each condensate is initially identical. Indeed, Dunningham and Burnett [23] have recently adapted the scheme of Ref. [2] for precisely that situation; the difficulties of their scheme, as discussed above, still apply but may be circumvented by the methods described in the present article, suitably adapted.

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[1] W. Heitler, *The Quantum Theory of Radiation*, 3rd ed. (Oxford University Press, London, 1954).

[2] M. J. Holland and K. Burnett, Phys. Rev. Lett. **71**, 1355 (1993).

[3] J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A **54**, R4649 (1996). The discussion in this paper is in the context of spectroscopy using MES of the internal states of a system of N trap ions.

[4] (a) C. C. Gerry, Phys. Rev. A **61**, 043811 (2000); (b) C. C. Gerry and R. A. Campos, *ibid.* **64**, 063814 (2001); (c) C. C. Gerry and A. Benmoussa, *ibid.* **65**, 033822 (2002); (d) P. Kok, H. Lee, and J. P. Dowling, *ibid.* **65**, 052104 (2002); (e) H. Lee, P. Kok, N. J. Cerf, and J. P. Dowling, *ibid.* **65**, 030101 (2002); (f) J. Fiurášek, *ibid.* **65**, 053818 (2002); (g) C. C. Gerry, A. Benmoussa, and R. A. Campos, *ibid.* **66**, 013804 (2002).

[5] R. A. Campos, B. E. A. Saleh, and M. C. Teich, Phys. Rev. A **40**, 1371 (1989).

[6] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).

[7] C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987).

[8] Z. Y. Ou, J.-K. Rhee, and L. J. Wang, Phys. Rev. Lett. **83**, 959 (1999). Both $|\psi_2\rangle$ and $|\psi_4\rangle$ are generated from the $|1\rangle_a|1\rangle_b$ and $|2\rangle_a|2\rangle_b$ components, respectively, of a two-mode squeezed vacuum state $|\xi\rangle = (1-|\xi|^2)^{1/2}\Sigma_{N=0}^{\infty}\xi^N|N\rangle_a|N\rangle_b$, which in turn is generated by a down-conversion process. Different experimental detection schemes are required to project out $|\psi_2\rangle$ and $|\psi_4\rangle$.

[9] W. Feller, *An Introduction to Probability Theory and Its Applications*, 3rd ed. (Wiley, New York, 1968), Vol. I.

[10] For a general discussion of phase distributions with several examples, see G. S. Agarwal, S. Chaturvedi, K. Tara, and V. Srinivasan, Phys. Rev. A **45**, 4904 (1992).

[11] M. V. Klein, *Optics* (Wiley, New York, 1970), Sec. 7.3, p. 335.

[12] M. Hillery, M. Zou, and V. Bužek, Quantum Semiclassic. Opt. **8**, 1041 (1996).
[13] T. Kim, O. Pfister, M. J. Holland, J. Noh, and J. L. Hall, Phys. Rev. A **57**, 4004 (1998).
[14] B. Han, Ph.D. thesis, Hunter College, City University of New York, 1998.
[15] *Handbook of Mathematical Functions*, Natl. Bur. Stand. Appl. Math. Ser. No. 55, edited by M. Abramowitz and I. A. Stegun (U.S. GPO, Washington, D.C., 1964), p. 776.
[16] K. Banaszek and K. Wódkiewicz, Phys. Rev. Lett. **82**, 2009 (1999).
[17] See J. Řeháček *et al.*, Phys. Rev. A **67**, 061801(R) (2003), and references therein.
[18] A. Royer, Phys. Rev. A **15**, 449 (1977).
[19] K. Banaszek, C. Radzewicz, K. Wódkiewicz, and J. S. Krasinski, Phys. Rev. A **60**, 674 (1999).
[20] K. Banaszek and K. Wódkiewicz, J. Mod. Opt. **44**, 2441 (1997).
[21] See U. Leonhardt, *Measuring the Quantum State of Light* (Cambridge University Press, Cambridge, England, 1997).
[22] See B. Yurke and D. Stoler, Phys. Rev. Lett. **79**, 4941 (1997); C. C. Gerry, A. Benmoussa, and R. A. Campos, Phys. Rev. A **66**, 013804 (2002).
[23] J. A. Dunningham and K. Burnett, Phys. Rev. A **61**, 065601 (2000); J. A. Dunningham, K. Burnett, and S. M. Barnett, Phys. Rev. Lett. **89**, 150401 (2002).