Optimal conclusive teleportation of quantum states

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(Received 3 April 2003; published 21 August 2003)

Quantum teleportation of qudits is revisited. In particular, we analyze the case where the quantum channel corresponds to a nonmaximally entangled state and show that the success of the protocol is directly related to the problem of distinguishing nonorthogonal quantum states. The teleportation channel can be seen as a coherent superposition of two channels, one of them being a maximally entangled state, thus leading to perfect teleportation, and the other, corresponding to a nonmaximally entangled state living in a subspace of the d-dimensional Hilbert space. The second channel leads to a teleported state with reduced fidelity. We calculate the average fidelity of the process and show its optimality.

DOI: 10.1103/PhysRevA.68.022310 PACS number(s): 03.67.Hk

I. INTRODUCTION

Entanglement is a fundamental property of quantum mechanical systems [1]. It is one of the most interesting and puzzling ideas associated with composite systems [2]. The postulates of quantum mechanics state that the state space of a composite physical system made up of two (or more) distinct physical systems is a tensor product of the state spaces of the component systems. A consequence of this structure is that there are states in the composite state space for which the correlations between the component systems cannot be accounted for classically. By classical we mean here that only local operations and classical communications (LOCC) are considered.

Although there is still not a complete theory on entanglement, it is considered a fundamental resource of nature whose importance is comparable to energy, information, and entropy [3], among others. Recently, the field of entanglement has become an intense research area due to its key role in many applications of quantum information processing [4]. An important example of this is teleportation of quantum states, where a maximally entangled state shared by two parties is used as a channel to transmit an unknown state using LOCCs. Teleportation protocols can also be used to transmit quantum operations [5,6] and to implement protocols for quantum cryptography [7].

In this paper we study the teleportation of a quantum state belonging to a d-dimensional Hilbert space (qudit). Our protocol considers the use of a nonmaximally entangled pure state of two qudits as quantum channel. We relate quantum teleportation to the problem of quantum states discrimination and show that the success of this scheme has a fundamental limit determined by how accurately a set of nonorthogonal linearly independent quantum states can be unambiguously distinguished. Thereby, we generalize to arbitrary dimensions the result by Mor and Horodecki [8] concerning conclusive quantum teleportation of qubits. We show that the generalized protocol for conclusive quantum teleportation can be interpreted in terms of the coherent superposition of two quantum channels. One of them allows for perfect teleportation. The other channel corresponds to a superposition of $d-1$ product states and leads to the failure of the process. This interpretation of conclusive quantum teleportation allows us to calculate easily the average fidelity over the entire Hilbert space. Finally, we demonstrate the optimality of the average fidelity.

This paper is organized as follows: In Sec. II we review the standard teleportation protocol considering both maximally and nonmaximally entangled states as quantum channel. In this section we also relate quantum teleportation to quantum state discrimination. In Sec. III we discuss in detail the quantum state discrimination protocol used in this paper. The results of this section allows us to calculate in Sec. IV the average fidelity of conclusive state teleportation and demonstrate its optimality.

II. QUANTUM STATE TELEPORTATION

In the process of teleporting a quantum state two parties, sender and receiver, share a maximally entangled two-qudit pure state. The sender has a third qudit in the state to be teleported. The sender carries out a generalized Bell measurement on his two particles and communicates the outcome of the measurement to the receiver. Conditional on the measurement result the receiver applies an unitary transformation on his particle. Thereafter, receiver's particle is in the state to be teleported.

The teleportation of a quantum state $|\psi\rangle$ of a d-dimensional Hilbert space, spanned by the basis $\{|n\rangle\}$ with $n=0,\ldots,d-1$, can be shortly described by the following identity:

$$|\Psi_{0,0}\rangle_{12}\otimes|\psi\rangle_3=\frac{1}{d}\sum_{l,k=0}^{d-1} Z_1^{d-l}X_1^k|\psi\rangle_1\otimes|\Psi_{l,k}\rangle_{23}, \qquad (1)$$

where particles 2 and 3 belong to the sender and particle 1 belong to the receiver. The states $|\Psi_{l,k}\rangle$ with $l,k=0,\ldots,d-1$ are a generalization of the Bell basis to the case of two d-dimensional quantum systems,

$$|\Psi_{n,m}\rangle=\frac{1}{\sqrt{d}}\sum_{j} e^{2\pi ijn/d}|j\rangle\otimes|j\oplus m\rangle, \tag{2}$$

where $j\oplus m$ denotes the sum $j+m$ modulus d. In this case the maximally entangled state $|\Psi_{0,0}\rangle$ has been chosen as quantum channel. The unitary operators X and Z are defined by

$$X=\sum_{n=0}^{d-1}|n+1\rangle\langle n|, \quad Z=\sum_{n=0}^{d-1}\exp\left(\frac{2\pi i}{d}n\right)|n\rangle\langle n|. \tag{3}$$

Instead of a direct Bell measurement it is possible to apply a generalized control-not gate (GXOR_{23}) [9] in order to map the states $|\Psi_{l,k}\rangle_{23}$ onto the unentangled states $F_2|l\rangle_2\otimes|k\rangle_3$, where F denotes the discrete Fourier transform, i.e.,

$$F|l\rangle=\frac{1}{\sqrt{d}}\sum_{k=0}^{d-1} e^{i2\pi lk/d}|k\rangle. \tag{4}$$

The generalized control-not gate is defined by $\mathrm{GXOR}_{ab}|i\rangle_a|j\rangle_b=|i\rangle_a|i\ominus j\rangle$, where $i\ominus j$ stands for the difference $i-j$ modulo d; thereby, Eq. (1) becomes

$$\mathrm{GXOR}_{23}|\Psi_{0,0}\rangle_{12}\otimes|\psi\rangle_3=\frac{1}{d}\sum_{l,k=0}^{d-1} Z_1^{d-l}X_1^k|\psi\rangle_1\otimes F_2|l\rangle_2\otimes|k\rangle_3. \tag{5}$$

The protocol for quantum state teleportation can be straightforwardly read out from this equation. In the case of a nonmaximally entangled pure quantum state $|\Psi\rangle_{12}$ as quantum channel, defined by

$$|\Psi\rangle_{12}=\sum_{m=0}^{d-1} A_m|m\rangle_1\otimes|m\rangle_2, \tag{6}$$

where $\{|m\rangle_i\}$ with $i=1,2$ are orthonormal basis defined by the Schmidt decomposition and the coefficients A_m are real and satisfy the normalization condition, the previous identity [Eq. (5)] is replaced by

$$\mathrm{GXOR}_{23}|\Psi\rangle_{12}\otimes|\psi\rangle_3=\frac{1}{d}\sum_{l,k=0}^{d-1} Z_1^{d-l}X_1^k|\psi\rangle_1\otimes|\nu_l\rangle_2\otimes|k\rangle_3, \tag{7}$$

where the states $|\nu_l\rangle_2$ are given by

$$|\nu_l\rangle_2=Z^l\sum_{k=0}^{d-1} A_k|k\rangle_2 \quad \text{with} \quad l=0,\ldots,d-1. \tag{8}$$

The previous identity [Eq. (7)] resembles Eq. (5) where now the states $F|l\rangle_2$ have been replaced by the states $|\nu_l\rangle$ of Eq. (8). Let us now recall that in the teleportation of states it is necessary to measure the state of particle 2. Conditional on the outcome of this measurement a unitary operator is applied on particle 1. These operators are in a one-to-one relation with the outcomes of the measurements. Therefore, it is necessary to distinguish the possible states of particle 2 perfectly. However, it is clear from the definition of the states $|\nu_l\rangle_2$ [Eq. (8)] that, in general, these states are nonorthogonal. In fact, the inner product between any two of these states is given by $\langle\nu_n|\nu_m\rangle=\Sigma_{k=0}^{d-1}\exp[(2\pi i/d)k(n-m)]|A_k|^2$. Only in the case of a maximally entangled state as quantum channel, that is, $A_k=1/\sqrt{d}\ \forall k=0,\ldots,d-1$, the overlap vanishes and the states $|\nu_l\rangle_2$ are simply the states $F_2|l\rangle_2$, which are mutually orthogonal. Thus, in the case of a nonmaximally entangled state $|\psi\rangle_{12}$ the success of the teleportation protocol is limited by our capability of distinguishing among the set $\{|\nu_l\rangle_2\}$ of d nonorthogonal quantum states.

This problem has been previously studied by Mor and Horodecky [8] in the case of two-dimensional quantum systems. They proposed the use of unambiguous state discrimination in combination with the usual quantum teleportation protocol. In this way, it is possible to distinguish perfectly among the two states $|\nu_0\rangle_2$ and $|\nu_1\rangle_2$ with some probability. For those events in which the discrimination is successful the teleported state has fidelity 1. However, if the discrimination fails the postmeasurement states might still allow one to teleport thought with reduced fidelity. The need for state discrimination measurements when teleporting a quantum state has also been considered in Ref. [24,25].

In the following section we review briefly the problem of quantum state discrimination. We show the optimal conclusive state discrimination protocol for the states $|\nu_l\rangle_2$ [Eq. (8)] and obtain the postmeasurement states.

III. QUANTUM STATE DISCRIMINATION

The problem of quantum state discrimination has deserved considerable attention. An overview of the main strategies has been given by Chefles [10]. In the later, generalized measurements are used to construct an error-free strategy for discriminating among a finite number of nonorthogonal states with given *a priori* probabilities. The scheme can occasionally lead to inconclusive results. This idea first proposed by Ivanovic [11] has been studied by Dieks [12] and Peres [13] for two nonorthogonal states generated with equal *a priori* probabilities. The result was later generalized by Jaeger and Shimony [14] for arbitrary *a priori* probabilities. The qudit case was then considered by Chefles [15] and Peres and Terno [16] where the former showed that results for the qudit case simply generalize form those of qubits when the states are linearly independent. The linearly dependent case can be considered only when copies of the state are available [17]. Here we consider a method developed by Sun *et al.* [18], which allows the construction of the optimal conclusive state discrimination scheme for a given set of linearly dependent states.

According to the quantum operations formalism [19] the most general transformation of a quantum system can be represented by a completely positive, trace preserving map; thereby, it is possible to perform a predetermined nonunitary transformation with some probability. Furthermore, in general, it is possible to known whether the desired transformation has been successfully implemented or not.

In particular, it is possible to change probabilistically the inner product between pure states of a quantum system. This is the essence of unambiguous state discrimination among elements of a set $\Omega=\{|\nu_l\rangle\}$ of nonorthogonal states. The state $|\nu_l\rangle$ is mapped probabilistically onto the state $|e_l\rangle$ that belongs to a set of orthogonal states. This mapping has certain probability of failure. In this case, the system is mapped onto a state $|\phi_l\rangle$ that does not allow a conclusive identification of the initial state. Thus, the failure probability of the mapping is identified with the total probability of obtaining an inconclusive identification of the states. A major problem consists in finding the optimal mapping, that is, the mapping with the smallest inconclusive probability.

Necessary and sufficient conditions for the existence of a conclusive discrimination scheme have been found by Chefles [17], namely, the states in Ω must be linearly independent (LI). Thus, the state $\{|\phi_l\rangle\}$ must be linearly independent. Otherwise, it would be possible to use another mapping that allows the discrimination among these states. Based on this observation, Sun *et al.* [18] have developed a method to find the optimal conclusive discrimination scheme and proposed a physical implementation in terms of optical multiports. In their approach conclusive state discrimination is described in terms of a unitary operator U and projective measurements. The states $\{|\nu_l\rangle\}$ generated with *a priori* probabilities $\{\eta_l\}$ are considered to belong to a Hilbert space $\mathcal{K}$ that can be decomposed as a direct sum of two subspaces, i.e., $\mathcal{K}=\mathcal{U}\oplus\mathcal{A}$. These subspaces are spanned by the basis states $\{|u_l\rangle\}$ and $\{|a_l\rangle\}$, respectively. The action of the unitary transformation is such that

$$U|\nu_l\rangle=\sqrt{p_l}|u_l\rangle+|\phi_l\rangle, \tag{9}$$

where the set of not necessarily normalized, linearly dependent states $\{|\phi_l\rangle\}$ is in $\mathcal{A}$, and p_l denotes the probability of discriminating successfully the state $|\nu_l\rangle$. The unitary transformation U [Eq. (9)] is followed by a measurement that projects the state of the particle onto one of the basis states of $\{|u_l\rangle\}$ or $\{|a_l\rangle\}$. The states $|\nu_l\rangle$ and $|u_l\rangle$ are in one-to-one correspondence. This and the orthogonality of the states $|u_l\rangle$ allow one to discriminate among the states of the set $\{|\nu_l\rangle\}$. However, in general, each basis state in $\mathcal{A}$ has a component in all the states $\{|\phi_l\rangle\}$ and thus it is not possible to assign them a particular state $|\nu_l\rangle$.

The optimal average probability of succes $S=\Sigma_{l=0}^{d-1}\eta_l p_l$ can be found under the constraint $\det(Q)=0$ where Q is the positive semidefinite matrix whose matrical elements are given by

$$Q_{k,l}=\langle\phi_k|\phi_l\rangle=\langle\nu_k|\nu_l\rangle-p_k\delta_{k,l}\,. \tag{10}$$

The states $|\phi_l\rangle$ can be defined as $|\phi_l\rangle=A|a_l\rangle$ with $Q=A^\dagger A$. Thereby, it is possible to find the form of U in Eq. (9).

It turns out that the states $|\nu_l\rangle$ are not necessarily well suited for conclusive discrimination. In fact, these states can be linearly dependent. The states $|\nu_l\rangle$ are LI under the condition

$$\sum_{l=0}^{d-1}C_l|\nu_l\rangle_2=0 \quad \text{iff} \quad C_l=0 \;\; \forall l=0,\dots,d-1, \tag{11}$$

or equivalently,

$$\sum_{n=0}^{d-1}\langle c|F|n\rangle A_n|n\rangle=0$$

with

$$|c\rangle=\sum_{k=0}^{d-1}C_k^*|k\rangle. \tag{12}$$

In the case that all the amplitudes A_n are nonzero, all the coefficients $\langle c|F|n\rangle$ must be null. Since $F|n\rangle$ form a basis, the only solution to Eq. (12) is $|c\rangle=0$. Therefore, all the coefficients C_l must vanish. Thus, if all the amplitudes A_m are different from 0, the states $|\nu_l\rangle$ are LI. Otherwise, when a subset of the amplitudes A_m are 0, Eq. (12) can be satisfied by taking $|c\rangle=\Sigma_{\{m\}}a_mF|m\rangle$ for any $a_m\neq 0$. Thus, in this case the states $|\nu_l\rangle$ are LD.

In the case that all the amplitudes A_m are different from 0, all the states $|\nu_l\rangle$ are different. Thus, the only source of error is the scheme of discrimination, itself. For example, in the process of conclusive state discrimination, there is a probability for the failure in the discrimination process. This event leads to a failure in the teleportation of states because it is not possible to decide which unitary operator must be applied in order to recover the state to be teleported. When only one of the amplitudes is different from 0, the states $|\nu_l\rangle$ are all equal. Henceforth, it is impossible to discriminate among them at all and both two processes fail completely.

Generally, when $1<n<d$ of the d amplitudes A_m are different from 0, then the d states $|\nu_l\rangle_2$ are LD. In this case it has been shown [17] that a conclusive state discrimination protocol can be formulated when copies for each state $|\nu_l\rangle_2$ are available. This adds an extra source of error to the teleportation of states and unitaries. In fact, the no cloning theorem [20] states that it is not possible to copy perfectly an unknown quantum state due to the linear character of quantum mechanics. Thus, besides the success probability of the discrimination protocol itself the success probability of a probabilistic cloning machine must be considered.

Let us now calculate the optimal average failure probability $F=1-S$ for the states $\{|\nu_l\rangle\}$ under the condition $A_m\neq 0$ $\forall m=0,\dots,d-1$. The calculations can be greatly simplified if the matrix Q is Fourier transformed. In fact, the matrix Q becomes

$$FQF^\dagger=\tilde{Q}=\sum_{m=0}^{d-1}\left[\left(\sum_{k=0}^{d-1}\frac{f_k}{d}\right)-1+dA_m^2\right]|m\rangle\langle m| + \sum_{m\neq n}^{d-1}\left(\sum_{k=0}^{d-1}\frac{f_k}{d}\epsilon^{k(n-m)}\right)|m\rangle\langle n|, \tag{13}$$

where f_k denotes the failure probability asociated with the state $|\nu_k\rangle$. For d arbitrary, the determinant of $\tilde{Q}$ has the generic form

$$\det(\tilde{Q}) = C_0(A_0^2, \ldots, A_{d-1}^2) + C_1(A_0^2, \ldots, A_{d-1}^2) \times \sum_{k=0}^{d-1} f_k + \Pi_{k=0}^{d-1} f_k . \quad (14)$$

Applying the method of Lagrange multipliers we obtain

$$\frac{\partial}{\partial f_i}[F + \lambda \det(\tilde{Q})] = \frac{1}{d} + \lambda C_1(A_0^2, \ldots, A_{d-1}^2) + \lambda \Pi_{k\neq i}^{d-1} f_k , \quad (15)$$

where λ is the Lagrange multiplier and we have made use of the fact that the states $\{|\nu_l\rangle\}$ are generated with the same probability, i.e., $\eta_l = 1/d \ \forall l = 0, \ldots, d-1$. The derivatives [Eq. (15)] are invariant under permutations of f_i's. In particular, they can be obtained from the derivative with respect to f_0 by suitably permuting f_i's. Thereby, the condition

$$\frac{\partial}{\partial f_i}[F + \lambda \det(\tilde{Q})] = 0 \quad (16)$$

implies that $f_0 = f_1 = \ldots = f_{d-1} = f$. Thus, the failure probabilities are all equal. Under this condition, matrix $\tilde{Q}$ has the simpler expression

$$\tilde{Q} = \sum_{m=0}^{d-1} (f - 1 + dA_m^2)|m\rangle\langle m|, \quad (17)$$

which turns out to be diagonal. This simplifies the analysis considerably. In fact, the determinant of $\tilde{Q}$ is now given by

$$\mathrm{Det}(\tilde{Q}) = \mathrm{Det}(Q) = \Pi_{k=0}^{d-1}(f - 1 + dA_k^2). \quad (18)$$

Thereby, the condition $\mathrm{Det}(Q)=0$ implies $f = 1 - dA_k^2$ for some k. The condition $f \leqslant 1$ must also hold. This condition can be satisfied if $A_k^2 \leqslant 1/d$. This rules out the choice of $f = 1 - dA_{d-1}^2$ where A_{d-1} is the the largest amplitude. The matrix Q (and $\tilde{Q}$) must also be positive semidefinite; this can be guaranteed if all the principal minors of $\tilde{Q}$ are non-negative, that is,

$$\Pi_{k=0}^{n}(f - 1 + dA_k^2) \geqslant 0 \quad \forall \quad n = 0, \ldots, d-1. \quad (19)$$

This condition implies that

$$f \geqslant 1 - dA_k^2 \quad \forall k = 0, \ldots, d-1 \quad (20)$$

and can be satisfied iff $f = 1 - d \min\{A_k^2\}_{k=0,\ldots,d-1}$. Thus, the optimal average failure probability F_{min} is given by

$$F_{min} = 1 - S_{max} = 1 - dA_{min}^2 , \quad (21)$$

where A_{min} is the smallest coefficient in the state [Eq. (6)]. The set of states $\{|\phi_l\rangle\}$ can be readily found. Noting that $Q = F^\dagger \tilde{Q} F = F^\dagger \tilde{A}^\dagger \tilde{A} F = F^\dagger \tilde{A}^\dagger F F^\dagger \tilde{A} F$, we obtain

$$|\phi_l\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} A_k^l |a_k\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \left[\sum_{m=0}^{d-1} \exp\left(\frac{2\pi i}{d} m(l-k)\right) \sqrt{A_m^2 - A_{min}^2} \right] |a_k\rangle. \quad (22)$$

IV. FIDELITY OF CONCLUSIVE STATE TELEPORTATION

With the results of the preceding section we can now state precisely the protocol for conclusive quantum state teleportation. Our starting point is the identity in Eq. (5) and the definition of states $|\nu_l\rangle$ of Eq. (8). The standard teleportation protocol consists in measuring the states of particles 2 and 3 and communicating the outcomes (l,k) of these measurements to the receiver. This projects particle 1 to the state $Z^{d-l}X^k|\psi\rangle$ from which the state $|\psi\rangle$ to be teleported can be obtained via applying the operator $X^{d-k}Z^l$. However, the states $|\nu_l\rangle$ cannot be distinguished with certainty, affecting the overall performance of the process, in this stage enters optimal conclusive quantum state discrimination. Before measuring particles 2 and 3, the unitary transformation U in Eq. (9) is applied onto particle 2. This leads to the join state

$$U_2 \,\mathrm{GXOR}_{23}|\Psi\rangle_{12} \otimes |\psi\rangle_3 = \frac{1}{d} \sum_{l,k=0}^{d-1} Z_1^{d-l} X_1^k |\psi\rangle_1 \otimes (\sqrt{S_{max}}|e_l\rangle_2 + |\phi_l\rangle_2) \otimes |k\rangle_3 . \quad (23)$$

Measurements on particles 2 and 3 project the particle one to the state

$$|\Psi_{l,k}^{\mathcal{U}}\rangle = \frac{1}{d} Z^{d-l} X^k |\psi\rangle \quad (24)$$

with probability S_{max} and to the state

$$|\Psi_{s,k}^{\mathcal{A}}\rangle = \frac{1}{d\sqrt{d}} \left(\sum_{l=0}^{d-1} A_s^l Z^{d-l} X^k \right) |\psi\rangle \quad (25)$$

with probabilities $1 - S_{max}$. The state $|\Psi_{l,k}^{\mathcal{U}}\rangle$ [Eq. (24)] is associated with the conclusive events in the discrimination of states and clearly leads to the perfect teleportation of the state $|\psi\rangle$. However, the state $|\Psi_{s,k}^{\mathcal{A}}\rangle$ [Eq. (25)] leads to a failure of the process. Nevertheless, this state has some fidelity with respect to the state $|\psi\rangle$ to be teleported. The average fidelity F of teleportation is given by

$$F = dA_{min}^2 + \sum_{s,k=0}^{d-1} \int d\psi |\langle \psi | \Psi_{s,k}\rangle|^2, \quad (26)$$

where the states $|\Psi\rangle_{s,k}$ are the states of particle 1 after the teleportation protocol has been carried out for the particular pair of outcomes (s,k) and the integral is performed over the entire Hilbert space. In the case that the teleportation protocol is interrupted after the measurement of particles 2 and 3, that is, $|\Psi\rangle_{s,k}=|\Psi\rangle_{s,k}^{\mathcal{A}}$, we obtain for the average fidelity

$$F=dA_{min}^2+\sum_{n,k=0}^{d-1}(A_{n+k}^2-A_{min}^2)\int d\psi|\langle n+k|\phi\rangle|^2|\langle n|\phi\rangle|^2, \tag{27}$$

where the states $\{|n\rangle\}$ for $n=0,\dots,d-1$ for a basis for the Hilbert space of particle one. The integral entering in Eq. (27) is equal to $(\delta_{n,k}+1)/d(d+1)$. Thereby, the average fidelity of teleportation becomes

$$F_0=\frac{1}{d}+(d-1)A_{min}^2. \tag{28}$$

The fidelity can be increased by using the information available about the outcomes of the measurements carried out on particles 2 and 3. The state of Eq. (25) can be cast in the form

$$|\Psi_{s,k}^{\mathcal{A}}\rangle=\frac{1}{\sqrt{d}}\left[\sum_{l=0}^{d-1}A_s^l\exp\left(-\frac{2\pi i}{d}lk\right)Z^{d-l}\right]|\psi\rangle \tag{29}$$

by appliying onto particle 1 the operator X^{d-k} conditional on the outcome k of the measurement of particle 3. This state leads to the following fidelity:

$$F_1=\frac{2+d(d-1)A_{min}^2}{d+1}, \tag{30}$$

which is clearly larger than F_0. A further increase in the average fidelity can be achieved by observing that the distribution $(A_s^l)^2$ has its maximum at $s=l$. This suggests to complete the protocol for unambiguous state teleportation by applying the operator Z^s onto particle 1 conditional on the outcome s of the measurement carried out on particle 3. In this case the average fidelity becomes

$$F_2=\frac{1}{d+1}[2+d(d-1)A_{min}^2]+\frac{1}{d+1}\sum_{n\neq r}\sqrt{A_n^2-A_{min}^2}\sqrt{A_r^2-A_{min}^2}. \tag{31}$$

Clearly, $F_0\leqslant F_1\leqslant F_2$ for all $A_{min}\in[0,1/\sqrt{d}]$. In what follows we will show this result to be optimal. This can be done by noting that the states $|\phi_l\rangle$ can be cast in the form

$$\tilde{F}|\phi_l\rangle=\tilde{Z}^l\sum_{n=0}^{d-1}\sqrt{A_n^2-A_{min}^2}|a_n\rangle, \tag{32}$$

where the operators $\tilde{F}$ and $\tilde{Z}$ act now on the subspace $\mathcal{A}$. Thereby, the transformation U is replaced by $\tilde{F}U$. This resembles the definition of the states $|\nu_l\rangle$ and suggests that the states $F|\phi_l\rangle$ originates in a quantum channel $|ch\rangle_{12}$ of the form

$$|ch\rangle_{12}=\sum_{n=0}^{d-1}\sqrt{\frac{A_n^2-A_{min}^2}{1-dA_{min}^2}}|a_n\rangle_1\otimes|a_n\rangle_2. \tag{33}$$

Thus, conclusive state teleportation can be described as starting with a coherent superposition of two quantum channels, i.e.,

$$|\psi\rangle_{12}=\sqrt{dA_{min}^2}\sum_{n=0}^{d-1}\frac{1}{\sqrt{d}}|e_n\rangle_1\otimes|e_n\rangle_2+\sqrt{1-dA_{min}^2}\sum_{n=0}^{d-1}\sqrt{\frac{A_n^2-A_{min}^2}{1-dA_{min}^2}}|a_n\rangle_1\otimes|a_n\rangle_2. \tag{34}$$

The first term at the right-hand side (rhs) of Eq. (34) corresponds to a perfectly entangled state in the subspace $\mathcal{H}$. This part of the channel is responsible for the events in which teleportation success occurs with unity fidelity. The second term at the rhs of Eq. (34) describes the ambiguous events that lead to a failure of the teleportation. This term corresponds to a nonmaximally entangled state in the subspace $\mathcal{A}$ and is formed by the superposition of only $d-1$ states. The protocol for conclusive quantum state teleportation is easily obtained from

$$GXOR2{,}3|\psi\rangle_{12}\otimes|\psi\rangle_3=\frac{1}{d}\sum_{l,k=0}^{d-1}Z_1^{d-l}X_1^k|\psi\rangle_1\otimes(\sqrt{dA_{min}^2}|e_l\rangle_2+\sqrt{1-dA_{min}^2}|\nu_l\rangle_2)\otimes|k\rangle_3, \tag{35}$$

where

$$|\nu_l\rangle=\tilde{Z}^l\sum_{n=0}^{d-1}\sqrt{\frac{A_n^2-A_{min}^2}{1-dA_{min}^2}}|a_n\rangle. \tag{36}$$

Now we can recall Banaszek's result [21] concerning the maximal average fidelity of teleportation

$$F_B\leqslant\frac{1}{d+1}\left[1-\left(\sum_{k=0}^{d-1}t_k\right)^2\right] \tag{37}$$

through a quantum channel $|ch\rangle_{12}=\Sigma_{k=0}^{d-1}t_k|k\rangle_1\otimes|k\rangle_2$. Inserting the coefficients of the quantum channel [Eq. (34)] into the previous definition [Eq. (37)], the fidelity for this channel is given by F_2 [Eq. (31)]. Therefore, the protocol for unambiguous state teleportation achieves the maximal possible fidelity.

V. CONCLUSIONS

The problem of teleporting the state of a d-dimensional quantum system through a quantum channel corresponding to a nonmaximally entangled state is directly related to that of distinguishing between a set of d nonorthogonal states. In the teleportation process, a generalized control-not gate is used to map the entangled state of the sender into an unentangled state. The sender then measures the state, which is, in general, nonorthogonal, and transmits the outcome to the receiver. The receiver then proceeds to use this information to choose among a set of transformations that must be applied to his state in order to recover the teleported state. The state's fidelity depends on the discrimination scheme used by the sender to distinguish its state. The optimal conclusive discrimination protocol proposed by Sun *et al.* [18] is at the center of the teleportation procedure presented here.

The optimal conclusive discrimination scheme is based on the idea of mapping the non-orthogonal state onto a set of orthogonal states in a probabilistic fashion. When the map is successful, the state can then be distinguished with certainty, thus leading to a perfect teleportation. The failure probability of the mapping procedure is responsible for a reduction in the fidelity of the teleported state. We conclude that this can be visualized in the following way: the nonmaximally entangled quantum channel is a coherent superposition of two channels, one allowing for perfect teleportation because the channel corresponds to a maximally entangled state related to a successful map in the discrimination procedure and a nonmaximally entangled channel living in a subspace of the Hilbert space. The truncated channel is generated by the failure probability of the map and teleportation through this channel leads to a state with reduced fidelity. Linear independence of the set of non-orthogonal states is a crucial factor in the scheme. The success of the procedure depends strongly on the number of states that are linearly independent. If no states are dependent, then the only source of error is the discrimination scheme itself. The dependent states cause further errors in the scheme because these states cannot be distinguished. Obviously, when all the states are linearly dependent, the scheme fails completely. In the protocol proposed here, the average fidelity when all the nonorthogonal states are linearly independent is optimal, i.e., it achieves the maximal average fidelity possible for a teleportation procedure. Therefore, we know with certainty that any further local operation would only decrease its performance [22–24].

We are currently investigating other possible applications of the quantum channel (34) including the teleportation of unitary evolutions, which allow for the remote implementation of quantum gates. We are also interested in extending the scheme to continuous variables.

ACKNOWLEDGMENTS

We would like to thank Carlton M. Caves, Ivan H. Deutsch, and the rest of the Information Physics group at the University of New Mexico for their hospitality. This work was supported under the auspices of the Office of Naval Research, Grant Nos. N00014-00-1-0575 and N00014-00-1-0578. A.D. acknowledges support from Fondecyt under Grant No. 1030671. L.R. thanks Fondecyt for support under Grant Nos. 1010010, 1030671, and UdeC P.I. 96011016-12. I.F.-G. would like to thank Consejo Nacional de Ciencia y Tecnologia (Mexico), Grant No. 115569/135963, for financial support.

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