Capacity of a channel assisted by two-mode squeezed states

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(Received 29 November 2002; published 13 August 2003)

The main purpose of this paper is to derive the capacity of attenuated or amplified noisy quantum channel assisted by two-mode squeezed states. In our previous works, formulas for capacity and reliability function have been obtained for continuous variable communication channel in the case where the classical information is conveyed by unentangled quantum Gaussian states. This paper investigates how two-mode squeezed states can enhance classical communication over quantum Gaussian channel.

DOI: 10.1103/PhysRevA.68.022303 PACS number(s): 03.67.−a, 89.70.+c

I. INTRODUCTION

Classical communication over ideal quantum channel assisted by entangled states, called quantum dense coding, was first proposed by Bennett and Wiesner [1]. The dense coding scheme has been generalized by several researchers. Bowen obtained the capacity of ideal qubit channel assisted by shared states that are not maximally entangled, showing that the optimal encoding is again given by the Pauli matrices [2]. Ban and independently Braunstein and Kimble investigated quantum dense coding with continuous variables [3,4]. They considered transmission through the ideal channel with assistance of two-mode squeezed states, employing encoding based on phase-space displacement operators and separable (unentangled) measurement with homodyne detection.

Bennett, Shor, Smolin, and Thapliyal [5,6] extended the dense coding to the general noisy quantum channel Φ, which is given by a completely positive, trace preserving map on trace class operators, and established important formula for the entanglement-assisted classical capacity

$$C_{ea}(\Phi)=\max_{\rho_A} I(\rho_A;\Phi), \tag{1}$$

where $I(\rho_A;\Phi)$ is the quantum mutual information,

$$I(\rho_A;\Phi)=H(\rho_A)+H(\Phi[\rho_A])-H(\rho_A;\Phi), \tag{2}$$

with the entropy exchange $H(\rho_A;\Phi)$, and ρ_A is varying over all density operators on the channel input. Calculation of this quantity for concrete channels is a nontrivial and important problem. Bennett *et al.* computed it for the amplitude damping channel [6], and Holevo and Werner for the attenuated or amplified noisy channel [7].

Our interest is devoted to a more practical case of a noisy channel assisted by special bipartite entangled states. In particular, we compute the capacity C_{sq} of attenuated or amplified channel assisted by two-mode squeezed states, and compare it with the entanglement-assisted capacity C_{ea}. The two-mode squeezed state, which was used in the experiments on quantum teleportation by Braunstein, Kimble, and Milburn [8,9], is a fundamental entanglement resource in continuous variable system. In our setting, we assume that the classical information is encoded by applying unitary displacement operators on Alice's (transmitter) part of the two-mode squeezed state and Bob (receiver) employs entangled measurements. In Sec. II we recall definitions of the entanglement-assisted capacity C_{ea} and the unassisted classical capacity C_{cl}. In Sec. III, we compute the capacity C_{sq} of attenuated or amplified noisy channel assisted by two-mode squeezed states. In Sec. IV, we estimate the ability of two-mode squeezed state to enhance classical communication, comparing C_{sq} with C_{ea} and C_{cl}. We show, in particular, that C_{sq} approaches the entanglement-assisted capacity C_{ea} for sufficiently large input energy and that two-mode squeezed states enhance classical communication well for nearly ideal channels.

II. DEFINITIONS OF CAPACITIES

Let us recall the protocol for the classical information transmission through a quantum channel Φ from Alice's Hilbert space $\mathcal{H}_A$ to Bob's space $\mathcal{H}_B$ with assistance of shared entanglement. Suppose that Alice and Bob may share unlimitedly entangled states $\rho_{AB}=|\psi\rangle_{AB}\langle\psi|$ to enhance classical communication. Following Refs. [6,10] we consider the entanglement-assisted communication for channels with constrained inputs. Alice encodes a continuous classical signal x from a finite-dimensional Euclidean space by using the map $\mathcal{E}_A^x$ to get a state $\rho_{AB}^x=(\mathcal{E}_A^x\otimes I_B)[\rho_{AB}]$, and sends it to Bob through the quantum channel Φ, yielding the state $\tilde{\rho}_{AB}^x=(\Phi\otimes I_B)(\mathcal{E}_A^x\otimes I_B)[\rho_{AB}]$. Moreover, the energy constraint on the codeword states $\rho_{AB}^{x_1}\otimes\cdots\otimes\rho_{AB}^{x_m}$ is imposed as

$$f(x_1)+\cdots+f(x_m)\leq mN_{tr}, \tag{3}$$

where f is the energy function defined as

$$f(x)=\mathrm{Tr}\rho_{AB}^x a_A^\dagger a_A. \tag{4}$$

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Then the one-shot classical capacity is given by the formula [10,11]

$$C_{ea}^{(1)}(\mathcal{E}_A^x,\rho_{AB},\Phi)=\max_{\pi\in\mathcal{P}_1}\left\{H\left(\int\tilde{\rho}_{AB}^x\pi(dx)\right)-\int H(\tilde{\rho}_{AB}^x)\pi(dx)\right\}, \tag{5}$$

where $\mathcal{P}_1$ is the set of *a priori* probability distribution satisfying the energy constraint,

$$\int f(x)\pi(dx)\leqslant N_{tr}. \tag{6}$$

Optimizing the capacity $C_{ea}^{(1)}(\mathcal{E}_A^x,\rho_{AB},\Phi)$ with respect to $\mathcal{E}_A^x$ and ρ_{AB}, we obtain one-shot entanglement-assisted capacity $C_{ea}^{(1)}(\Phi)$. In the same way considering n uses of the channel, we can define the n-shot capacity $C_{ea}^{(n)}(\Phi)$. The full entanglement-assisted capacity is then defined as

$$C_{ea}(\Phi)=\lim_{n\to\infty}\frac{1}{n}C_{ea}^{(n)}(\Phi). \tag{7}$$

Then it can be proved [10] that formula (1) holds, where ρ_A varies over all density operators satisfying the constraint

$$\mathrm{Tr}\rho_A a_A^\dagger a_A\leqslant N_{tr}. \tag{8}$$

The attenuator is described by the transformation

$$\tilde{a}_A=ka_A+\sqrt{1-k^2}a_E,\quad k<1, \tag{9}$$

and the amplifier

$$\tilde{a}_A=ka_A+\sqrt{k^2-1}a_E^\dagger,\quad k>1, \tag{10}$$

in the Heisenberg picture. Here a_A is the annihilation operator of the Alice's mode, and a_E is that of another mode in the Hilbert space $\mathcal{H}_E$ of "environment," which is initially in the vacuum state. We assume also the additive classical complex thermal noise with zero mean and variance $\hbar N_c$. The capacity $C_{sq}(\Phi_{k,N_c})$ of attenuated or amplified noisy channel Φ_{k,N_c} assisted by two-mode squeezed states is defined to be capacity (5) with the encoding maps $\mathcal{E}_A^x$ acting as $\mathcal{E}_A^x[\rho_A]=D(x)\rho_A D(x)^\dagger$, where $D(x)$ is the displacement operator, ρ_{AB} is the two-mode squeezed state, and $\Phi=\Phi_{k,N_c}$.

In the following sections we compute the capacity $C_{sq}(\Phi_{k,N_c})$ and compare it with the entanglement-assisted capacity $C_{ea}(\Phi_{k,N_c})$ and the unassisted capacity $C_{cl}(\Phi_{k,N_c})$ for the attenuated noisy channel Φ_{k,N_c}. Let us summarize results about $C_{cl}(\Phi_{k,N_c})$ and $C_{ea}(\Phi_{k,N_c})$. The unassisted capacity $C_{cl}(\Phi_{k,N_c})$ is conjectured [7] to be achieved by the coherent states with the Gaussian probability density, resulting in

$$C_{cl}(\Phi_{k,N_c})=g(k^2N_{tr}+N_c+m(k))-g(N_c+m(k)), \tag{11}$$

TABLE I. Comparison of capacities $C_{ea}^{(1)}(\mathcal{E}_A^x,\rho_{AB},\Phi)$, $C_{ea}^{(1)}(\Phi)$, $C_{ea}(\Phi)$, and $C_{cl}(\Phi)$. Here n and m are numbers used to describe codeword state as $\rho^{x_1}\otimes\cdots\otimes\rho^{x_m}$ with $\rho^{x_j}\in\mathcal{H}_A^{\otimes n}\otimes\mathcal{H}_B^{\otimes n}$ (Only in the case of $C_{cl}(\Phi)$, where $\rho^{x_j}\in\mathcal{H}_A$, the value of n makes no sense).

	Encoding	Entangled State	n	Entangled Measurement	m
$C_{ea}^{(1)}(\mathcal{E}_A^x,\rho_{AB},\Phi)$ C_{sq}	Fixed	Fixed	1	Optimized	∞
$C_{ea}^{(1)}(\Phi)$	Optimized	Optimized	1	Optimized	∞
$C_{ea}(\Phi)$	Optimized	Optimized	∞	Optimized	∞
$C_{cl}(\Phi)$	Optimized			Optimized	∞

where

$$g(x)=(x+1)\log(x+1)-x\log x, \tag{12}$$

and

$$m(k)=\max\{0,k^2-1\}. \tag{13}$$

The entanglement-assisted capacity $C_{ea}(\Phi_{k,N_c})$ was computed in Ref. [7] as follows:

$$C_{ea}(\Phi_{k,N_c})=g(N_{tr})+g(k^2N_{tr}+N_c+m(k))-\mathcal{I}(k,N_c,N_{tr}), \tag{14}$$

where

$$\mathcal{I}(k,N_c,N_{tr})=g(\xi_+-1/2)+g(\xi_--1/2), \tag{15}$$

is the entropy exchange and

$$\xi_\pm=\tfrac{1}{2}\{\pm[(k^2-1)N_{tr}+N_c+m(k)]+\sqrt{[(k^2+1)N_{tr}+N_c+m(k)+1]^2-4k^2N_{tr}(N_{tr}+1)}\}.$$

In Table I we summarize the capacities introduced above. From the definitions the following inequalities between these capacities hold:

$$C_{cl}(\Phi_{k,N_c})\leqslant C_{sq}(\Phi_{k,N_c})\leqslant C_{ea}^{(1)}(\Phi_{k,N_c})\leqslant C_{ea}(\Phi_{k,N_c}). \tag{16}$$

III. CAPACITY OF ATTENUATED OR AMPLIFIED NOISY CHANNEL ASSISTED BY TWO-MODE SQUEEZED STATES

The purpose of this section is to compute the capacity $C_{sq}(\Phi_{k,N_c})$. Consider a two-dimensional real vector $x=[x^q,x^p]$ as a classical signal. Then Alice encodes the classical signal x by applying a displacement unitary operator

$$D(x)=\exp\frac{i}{\hbar}(x^pq_A-x^qp_A) \tag{17}$$

to her part of shared two-mode squeezed state $|\psi_{sq}\rangle_{AB}$, yielding the quantum state $|\psi_{sq}(x)\rangle_{AB}=[D(x)\otimes I_B]|\psi_{sq}\rangle_{AB}$. Bob obtains the state $\tilde{\rho}_{sq}^x=(\Phi_{k,N_c}\otimes I_B)$

$\times[|\psi_{sq}(x)\rangle_{AB}\langle\psi_{sq}(x)|]$ by receiving Alice's part of $|\psi_{sq}(x)\rangle_{AB}$ through the attenuated or amplified noisy channel Φ_{k,N_c}.

We explain the two-mode squeezed state and attenuation or amplification for it in Secs. III A and III B, respectively, and compute the capacity C_{sq} in Sec. III C. Our computation is based on the general formula of von Neumann entropy for a general Gaussian state [12]

$$H(\rho)=\frac{1}{2}\mathrm{Sp}g\left(\mathrm{abs}(\Delta_s^{-1}\alpha)-\frac{I_{2s}}{2}\right), \tag{18}$$

where Sp denotes trace of matrices as distinct from trace of operators Tr, and Δ_s, α are correspondingly the commutation and correlation matrices of the canonical quadrature observables

$$q_j=\sqrt{\frac{\hbar}{2\omega_j}}(a_j+a_j^\dagger),\quad p_j=i\sqrt{\frac{\hbar\omega_j}{2}}(a_j^\dagger-a_j),$$
$$j=1,\ldots,s, \tag{19}$$

and $\mathrm{abs}(\cdot)$ is defined as follows: for a diagonalizable matrix $M=T\mathrm{diag}(t_j)T^{-1}$ with a non-singular matrix T and $t_j\in\mathbf{C}$, we put $\mathrm{abs}M=T\mathrm{diag}(|t_j|)T^{-1}$. In the following calculation we use formula (18) with $s=2$, where we put $\omega_1=\omega_2=1$ for simplicity.

A. Two-mode squeezed state

Let a_A and a_B be the annihilation operators for the systems of Alice and Bob, respectively. Ignoring an unimportant phase factor, we can represent the two-mode squeezed state as

$$|\psi_{sq}\rangle_{AB}=S(r)|0\rangle_{AB}. \tag{20}$$

Here $|0\rangle_{AB}$ is the two-mode vacuum state $|0\rangle_A\otimes|0\rangle_B$ and

$$S(r)=\exp[-r(a_A^\dagger a_B^\dagger-a_Aa_B)] \tag{21}$$

is the squeezing operator, which transforms annihilation operators a_A and a_B according to the relations

$$a_A'=S(r)^\dagger a_AS(r)=a_A\cosh r-a_B^\dagger\sinh r, \tag{22}$$
$$a_B'=S(r)^\dagger a_BS(r)=-a_A^\dagger\sinh r+a_B\cosh r.$$

In order to apply formula (18), we must obtain the correlation matrix α of the two-mode squeezed state. Introducing vector representations $\mathcal{R}=[q_A,p_A;q_B,p_B]^T$, we define the unitary operators for four-dimensional real vectors z,

$$V(z)=\exp[i\mathcal{R}^Tz], \tag{23}$$

by which the characteristic function is defined (see Refs. [12,13]). Using the vector representations $\mathcal{R}$ and $\mathcal{R}'=[q_A',p_A';q_B',p_B']^T$ with $q_j'=S(r)^\dagger q_jS(r)$ and $p_j'=S(r)^\dagger p_jS(r)$ $(j=A,B)$, we can rewrite Eq. (22) in a real setting as

$$\mathcal{R}'=\mathcal{L}\mathcal{R}, \tag{24}$$

where

$$\mathcal{L}=\begin{bmatrix}\cosh rI_2 & \sinh rJ_2\\ \sinh rJ_2 & \cosh rI_2\end{bmatrix}, \tag{25}$$

with the 2×2 diagonal matrices

$$I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad J_2=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}. \tag{26}$$

It follows from Eq. (24) that

$$S(r)^\dagger V(z)S(r)=\exp[i(\mathcal{R}')^Tz]=\exp[i\mathcal{R}^T(\mathcal{L}^Tz)]=V(\mathcal{L}^Tz). \tag{27}$$

Thus, the characteristic function of $|\psi_{sq}\rangle_{AB}$ is given by

$$\mathrm{Tr}|\psi_{sq}\rangle_{AB}\langle\psi_{sq}|V(z)=\mathrm{Tr}|0\rangle_{AB}\langle0|S(r)^\dagger V(z)S(r)$$
$$=\exp\left[-\frac{1}{2}\frac{\hbar}{2}z^T\mathcal{L}\mathcal{L}^Tz\right], \tag{28}$$

which means the correlation matrix of two-mode squeezed state is

$$\alpha=\frac{\hbar}{2}\mathcal{L}\mathcal{L}^T=\frac{\hbar}{2}\begin{bmatrix}\cosh 2rI_2 & \sinh 2rJ_2\\ \sinh 2rJ_2 & \cosh 2rI_2\end{bmatrix}. \tag{29}$$

B. Attenuation and amplification for Gaussian states

Bob gets states $(\Phi_{k,N_c}\otimes I_B)[|\psi_{sq}(x)\rangle_{AB}\langle\psi_{sq}(x)|]$ when Alice's part of encoded two-mode squeezed states are transmitted through the attenuated or amplified noisy channel Φ_{k,N_c}. These are Gaussian states with mean vectors kx and the same correlation matrix $\tilde{\alpha}$.

In this subsection, we obtain the correlation matrix $\tilde{\alpha}$. As a first step, let us consider an attenuated channel without thermal noise, $\Phi_{k,0}$ $(k<1)$. From Eqs. (22) and (9), we deduce that the transformation $|0\rangle_{AB}\langle0|\to(\Phi_{k,N_c}\otimes I_B)[|\psi_{sq}(x)\rangle_{AB}\langle\psi_{sq}(x)|]$ is described in the Heisenberg picture by the relation

$$\tilde{a}_A=ka_A\cosh r-ka_B^\dagger\sinh r+\sqrt{1-k^2}a_E, \tag{30}$$
$$\tilde{a}_B=-a_A^\dagger\sinh r+a_B\cosh r.$$

Put $\tilde{q}_j=\sqrt{\hbar/2}(\tilde{a}_j+\tilde{a}_j^\dagger)$, $\tilde{p}_j=i\sqrt{\hbar/2}(\tilde{a}_j^\dagger-\tilde{a}_j)$, $q_j=\sqrt{\hbar/2}(a_j+a_j^\dagger)$, $p_j=i\sqrt{\hbar/2}(a_j^\dagger-a_j)$ for $j=A,B,E$ and introduce vector representations $\tilde{\mathcal{R}}=[\tilde{q}_A,\tilde{p}_A,\tilde{q}_B,\tilde{p}_B]^T$, $\mathcal{R}_0=[q_A,p_A,q_B,p_B,q_E,p_E]^T$. Then Eq. (30) can be rewritten in a real setting as

$$\tilde{\mathcal{R}}=\mathcal{M}\mathcal{R}_0, \tag{31}$$

where

$$\mathcal{M}=\begin{bmatrix}k\cosh rI_2 & k\sinh rJ_2 & \sqrt{1-k^2}I_2\\ \sinh rJ_2 & \cosh rI_2 & 0\end{bmatrix}. \tag{32}$$

Like in Sec. III A we can find that the correlation matrix in this case is

$$\frac{\hbar}{2}\mathcal{M}\mathcal{M}^T=\frac{\hbar}{2}\begin{bmatrix}(k^2\cosh 2r+|1-k^2|)I_2 & k\sinh 2rJ_2\\ k\sinh 2rJ_2 & \cosh 2rI_2\end{bmatrix}. \tag{33}$$

It is easy to see that the correlation matrix in the case of the amplification ($k>1$) is also given by Eq. (33). Moreover, in the Gaussian case, we can separately deal with the effect of attenuation or amplification and that of thermal noise [14]. Hence, considering the effect of thermal noise with zero mean and variance $\hbar N_c$, we obtain the correlation matrix of the state $(\Phi_{k,N_c}\otimes I_B)[|\psi_{sq}(x)\rangle_{AB}\langle\psi_{sq}(x)|]$ in the form

$$\tilde{\alpha}=\frac{\hbar}{2}\mathcal{M}\mathcal{M}^T+\hbar\begin{bmatrix}N_cI_2 & 0\\ 0 & 0\end{bmatrix}, \tag{34}$$

for any value of k.

C. Computation of capacity assisted by two-mode squeezed states

Here we compute the capacity $C_{sq}(\Phi_{k,N_c})$. Energy function (4) for the two-mode squeezed state $|\psi_{sq}(x)\rangle_{AB}\langle\psi_{sq}(x)|$ is

$$f(x)=\frac{x^Tx}{2\hbar}+N_{sq}, \tag{35}$$

with a squeezing energy $N_{sq}=(\cosh 2r-1)/2$. First, let us mention that we can restrict an *a priori* probability distribution to Gaussian one without loss of generality. To show the validity of this restriction, we first point out that the second term in the right-hand side of Eq. (5) can be ignored because it takes the same value for any *a priori* distribution. Thus, it suffices to show that the first term is maximized by an *a priori* Gaussian distribution. For arbitrary distribution $\hat{\pi}\in\mathcal{P}_1$, there exists such Gaussian distribution $\tilde{\pi}$ that $\tilde{\rho}_{AB}=\int\tilde{\rho}(x)\tilde{\pi}(dx)$ has the same first and second moment as $\hat{\rho}_{AB}=\int\tilde{\rho}(x)\hat{\pi}(dx)$. Then it is known [12] that $H(\tilde{\rho}_{AB})\geqslant H(\hat{\rho}_{AB})$ holds. This means that an optimal *a priori* distribution is given by Gaussian one. In the following, we denote the correlation matrix of an *a priori* Gaussian distribution $\tilde{\pi}$ by

$$\tilde{\beta}=\begin{bmatrix}\beta & O\\ O & O\end{bmatrix}, \tag{36}$$

where O is the 2×2 zero matrix and β is a 2×2 real symmetric matrix

$$\hbar\begin{bmatrix}\beta^{qq} & \beta^{qp}\\ \beta^{qp} & \beta^{pp}\end{bmatrix}, \tag{37}$$

satisfying the positivity condition

$$\beta^{qq}\beta^{pp}-(\beta^{qp})^2\geqslant 0. \tag{38}$$

Then correlation matrix of Gaussian state $\tilde{\rho}_{AB}$ is

$$\tilde{\alpha}+k^2\tilde{\beta}, \tag{39}$$

where $\tilde{\alpha}$ is given by Eq. (34). Now we can rewrite Eq. (5) as follows.

$$C_{sq}(\Phi_{k,N_c})=\frac{1}{2}\max_{\beta\in\mathcal{B}_1}g\left(\mathrm{abs}[\Delta^{-1}(\tilde{\alpha}+k^2\tilde{\beta})]-\frac{I_4}{2}\right)-\frac{1}{2}g\left(\mathrm{abs}(\Delta^{-1}\tilde{\alpha})-\frac{I_4}{2}\right), \tag{40}$$

where I_4 is the 4×4 identity matrix and $\mathcal{B}_1$ is a set of real positive 2×2 matrices, satisfying condition (38) and

$$\frac{\beta^{qq}+\beta^{pp}}{2}+N_{sq}\leqslant N_{tr}, \tag{41}$$

which is derived from Eqs. (6) and (35).

In order to compute Eq. (40), we first obtain the eigenvalues of $\Delta^{-1}(\tilde{\alpha}+k^2\tilde{\beta})$. From a general discussion [13], we find that these eigenvalues can be represented as $\pm i\gamma_1$, $\pm i\gamma_2$ with $\gamma_1,\gamma_2\geqslant 1/2$. The characteristic polynomial is calculated as

$$\begin{aligned}\det[\lambda-\Delta^{-1}(\tilde{\alpha}+k^2\tilde{\beta})]&=\lambda^4+\lambda^2[\eta^2-(\beta^{qp})^2-2\zeta^2+(\xi+k^2\beta^{qq})(\xi+k^2\beta^{pp})]\\&\quad-(\beta^{qp})^2\eta^2+(\xi+\beta^{qq})(\xi+\beta^{pp})\eta^2\\&\quad-\zeta^2\eta(2\xi+k^2\beta^{qq}+k^2\beta^{pp})+\zeta^4,\end{aligned} \tag{42}$$

where

$$\begin{aligned}\xi&=k^2N_{sq}+N_c+m(k)+\tfrac{1}{2},\\ \eta&=N_{sq}+\tfrac{1}{2},\\ \zeta&=k[N_{sq}(N_{sq}+1)]^{1/2},\end{aligned} \tag{43}$$

where $m(k)$ is given by Eq. (13). Here let us pay attention to the fact that when the coefficients of λ in Eq. (42) take larger values, the capacity C_{sq} becomes larger. This can be easily shown from the fact that $g(x)$ is a monotonously increasing concave function, and tells us that the optimal values of β^{qq}, β^{pp}, and β^{qp} should satisfy $\beta^{qp}=0$ and $\beta^{pp}=\beta^{qq}$ when $2t=\beta^{qq}+\beta^{pp}$ is fixed. This simplifies Eq. (42) as

$$\begin{aligned}\det[\lambda-\Delta^{-1}(\tilde{\alpha}+k^2\tilde{\beta})]&=[\lambda^2+\eta(\xi+k^2t)-\zeta^2]^2+\lambda^2(\xi+k^2t-\eta)^2.\end{aligned} \tag{44}$$

Solving $\det[\lambda-\Delta^{-1}(\tilde{\alpha}+k^2\tilde{\beta})]=0$, we obtain the solutions $\pm i\gamma_1(t)$ and $\pm i\gamma_2(t)$, with

$$\begin{aligned}\gamma_1(t)&=\frac{\xi+k^2t-\eta}{2}+\sqrt{-\zeta^2+\frac{(\xi+k^2t+\eta)^2}{4}},\\ \gamma_2(t)&=-\frac{\xi+k^2t-\eta}{2}+\sqrt{-\zeta^2+\frac{(\xi+k^2t+\eta)^2}{4}}.\end{aligned} \tag{45}$$

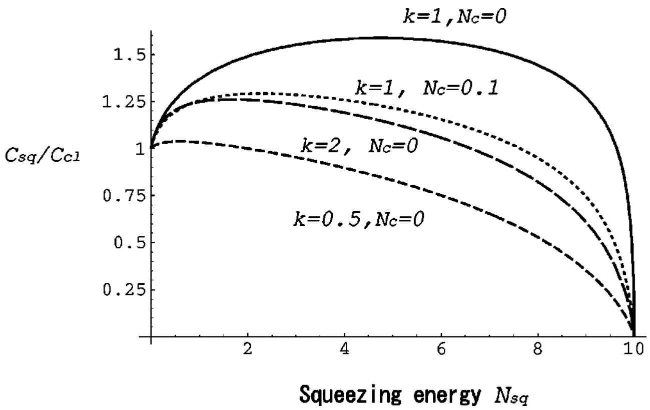

FIG. 1. Dependence of C_{sq}/C_{cl} on the squeezing energy N_{sq} for the ideal channel with $k=1$, $N_c=0$, and $N_{tr}=10$, the noisy channel with $k=1$, $N_c=0.1$, and $N_{tr}=10$, the attenuated channel with $k=0.5$, $N_c=0$, and $N_{tr}=10$, and the amplified channel with $k=2$, $N_c=0$, and $N_{tr}=10$. Here squeezing energy N_{sq} is measured by average photon number.

On the other hand, the eigenvalues of $\Delta^{-1}(\tilde{\alpha}+k^2\tilde{\beta})$ can be obtained as $\pm i\gamma_1(0)$, $\pm i\gamma_2(0)$. Thus, the capacity of attenuated or amplified noisy channel Φ_{k,N_c} assisted by two-mode squeezed states is given by

$$C_{sq}(\Phi_{k,N_c})=\max_{0<t\leq N_{tr}-N_{sq}} g(\gamma_1(t)-1/2)+g(\gamma_2(t)-1/2)-g(\gamma(0)-1/2)+g(\gamma(0)-1/2), \tag{46}$$

$$=g(\gamma_1(N_{tr}-N_{sq})-1/2)+g(\gamma_2(N_{tr}-N_{sq})-1/2)-g(\gamma_1(0)-1/2)-g(\gamma_2(0)-1/2).$$

IV. ABILITY OF TWO-MODE SQUEEZED STATE TO ENHANCE A CLASSICAL COMMUNICATION

In order to evaluate the ability of two-mode squeezed state to enhance classical communication, we compare the ratio of C_{sq} to the unassisted capacity C_{cl} with that of the entanglement-assisted capacity C_{ea} to C_{cl}, where $C_{sq}=C_{sq}(\Phi_{k,N_c})$, $C_{cl}=C_{cl}(\Phi_{k,N_c})$, and $C_{ea}=C_{ea}(\Phi_{k,N_c})$ are given by Eqs. (46), (11), and (14), respectively. In what follows we assume that C_{sq} is optimized with respect to a squeezing energy N_{sq}. In Fig. 1, C_{sq}/C_{cl} are plotted versus N_{sq} for the ideal channel with $k=1$, $N_c=0$, and $N_{tr}=10$, the noisy channel with $k=1$, $N_c=0.1$, and $N_{tr}=10$, the attenuated channel with $k=0.5$, $N_c=0$, and $N_{tr}=10$, and the amplified channel with $k=2$, $N_c=0$, and $N_{tr}=10$. The optimized value of C_{sq}/C_{cl} is given as the peak value of each graph in Fig. 1. In Fig. 2, for the ideal channel with $k=1$ and $N_c=0$, C_{sq}/C_{cl}, and C_{ea}/C_{cl} are plotted versus N_{tr} by spots and by a solid line, respectively. This figure shows that two-mode squeezed states enhance classical communication well when input energy N_{tr} is sufficiently large. In particular, we conjecture that $C_{sq}\approx C_{ea}^{(1)}$ holds for $N_{tr}\gg 1$, where $C_{ea}^{(1)}$ denotes the one-shot entanglement-assisted capacity. For sufficiently large N_{tr}, we have

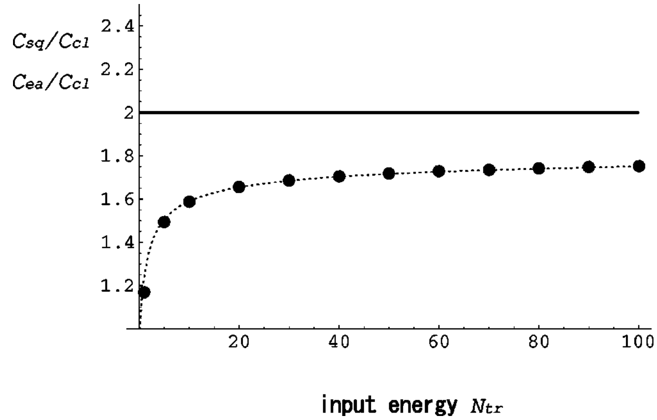

FIG. 2. For the ideal channel with $k=1$ and $N_c=0$, C_{sq}/C_{cl} and C_{ea}/C_{cl} are plotted vs the input energy N_{tr} by dots and a solid line, respectively, where input energy N_{tr} is measured by average photon number. The dotted line shows $\tilde{C}_{sq}/C_{cl}$, where $\tilde{C}_{sq}$ is an approximation of C_{sq} given by assuming the suboptimal squeezing energy $N_{sq}=N_{tr}/2$.

$$C_{sq}\approx -2\ln 2+\ln[1+2N_{tr}\sqrt{(1-\gamma)(1+3\gamma)}+4\gamma(1-\gamma)N_{tr}^2](\equiv\tilde{C}_{sq}), \tag{47}$$

with a ratio of squeezing energy N_{sq} to N_{tr}, $\gamma=N_{sq}/N_{tr}$ $(0\leq\gamma\leq 1)$. As the coefficient $4\gamma(1-\gamma)$ dominates the value of C_{sq} for sufficiently large N_{tr}, we can find that the optimal squeezing energy N_{sq} is equal to $N_{tr}/2$ approximately. In Fig. 2, the dotted line shows the values of C_{sq}/C_{cl} with $N_{sq}=N_{tr}/2$, which seem to give a good approximation of values of C_{sq}/C_{cl} with the optimized squeezing energy. From Eq. (47), we get

$$\lim_{N_{tr}\to\infty}\frac{C_{sq}}{C_{cl}}=2, \tag{48}$$

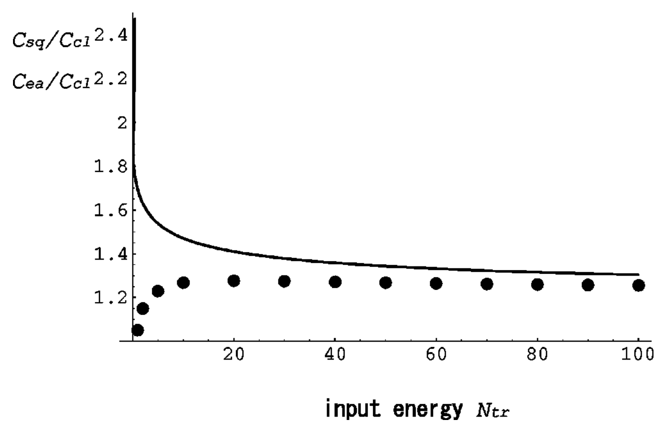

FIG. 3. For the attenuated channel with $k=0.9$ and $N_c=0$, values of C_{sq}/C_{cl} and C_{ea}/C_{cl} are plotted vs the input energy N_{tr} by dots and solid line, respectively, where input energy N_{tr} is measured by average photon number. Note that those values for channels suffering from amplification or small thermal noise show similar behavior.

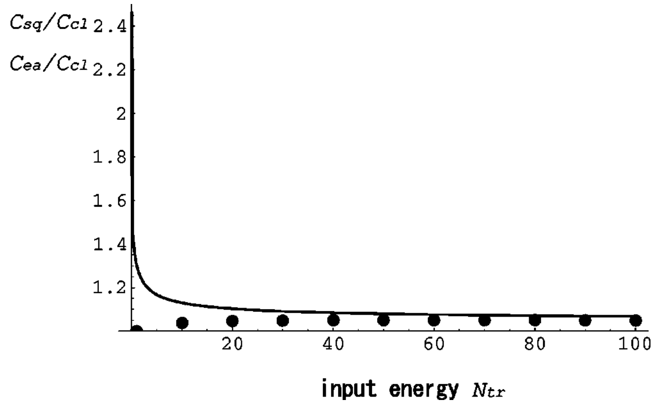

FIG. 4. For the attenuated channel with $k=0.5$ and $N_c=0$, C_{sq}/C_{cl} and C_{ea}/C_{cl} are plotted vs the input energy N_{tr} by dots and solid line, respectively, where input energy N_{tr} is measured by average photon number.

while C_{ea}/C_{cl} is equal to 2 for any value of N_{tr}. Note that Eq. (48) holds not only for $\gamma=1/2$ but also for any value of $0<\gamma\leqslant 1$.

In Fig. 3, values of C_{sq}/C_{cl} and C_{ea}/C_{cl} are plotted versus N_{tr} by spots and by a solid line, respectively, for the attenuated channel with $k=0.9$ and $N_c=0$. In addition, it is found that those values for the channel with a small thermal noise also have similar behavior. These show that the two-mode squeezed states enhance classical communication well, when the channel with a large input energy does not suffer strongly from attenuation or thermal noise.

In Fig. 4, values of C_{sq}/C_{cl} and C_{ea}/C_{cl} are plotted versus N_{tr} by spots and by a solid line, respectively, for the attenuated channel with $k=0.5$ and $N_c=0$. This figure shows that two-mode squeezed states are useless when the effect of attenuation is large. The behavior of C_{sq}/C_{cl} and C_{ea}/C_{cl} for the nonideal attenuated noisy channel is different from that for the ideal channel. In fact, we have

$$\lim_{N_{tr}\to\infty}\frac{C_{sq}}{C_{cl}}=\lim_{N_{tr}\to\infty}\frac{C_{ea}}{C_{cl}}=1, \tag{49}$$

for any nonideal attenuated noisy channel. Like for the ideal channel, C_{sq} achieves C_{ea} approximately when input energy N_{tr} is sufficiently large, but the gain of entanglement-assisted capacity in itself vanishes in the limit of $N_{tr}\to\infty$ in the nonideal case. On the other hand, in the amplified case, where Eq. (49) also holds, the speed of convergence of C_{ea}/C_{cl} and C_{sq}/C_{cl} as $N_{tr}\to\infty$ is very slow for any value of $k>1$, and the gain of entanglement-assisted capacity does not vanish.

In order to estimate the effect of two-mode squeezed state analytically, let us consider the case $N_c\to\infty$. It is known that

$$\lim_{N_c\to\infty}\frac{C_{ea}}{C_{cl}}=(N_{tr}+1)\ln\left(1+\frac{1}{N_{tr}}\right), \tag{50}$$

which tends to infinity when N_{tr} tends to zero. Note that Eq. (50) is stated in Ref. [6] using the value $C_{shan}=\ln(1+k^2N_{tr}/N_c)$ instead of C_{cl}; these two statements are equivalent because $C_{cl}/C_{shan}\to 1$ as $N_c\to\infty$. On the other hand, we can show by straightforward calculation

$$\lim_{N_c\to\infty}\frac{C_{sq}}{C_{cl}}=\frac{N_{tr}-N_{sq}}{N_{tr}}\leqslant 1, \tag{51}$$

which means that two-mode squeezed state is useless in this case. Note that Eqs. (50) and (51) hold for any value of k.

V. DISCUSSION

We have obtained the capacity of attenuated or amplified noisy channel assisted by two-mode squeezed state. As a result, we have found that C_{sq} approaches the entanglement-assisted capacity C_{ea} for sufficiently large input energy, and that two-mode squeezed states enhance classical communication well for nearly ideal channels. On the other hand, C_{sq} does not achieve C_{ea} at all for small input energies. This is because the two-mode squeezed state necessarily needs large energy in order to be entangled well. Unfortunately, for the channel suffering strong attenuation or thermal noise, the gain of entanglement assistance takes large values only when the input energy is small. This means that any entangled state with large input energy is useless in this case. Thus, we conclude that two-mode squeezed state cannot enhance classical communication through such a channel. The problem of finding a good two-mode entangled state for a small input energy will be the subject of further investigation.

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