Quantum arrival times and operator normalization

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A recent approach to arrival times used the fluorescence of an atom entering a laser illuminated region, and the resulting arrival-time distribution was close to the axiomatic distribution of Kijowski, but not exactly equal, neither in limiting cases nor after compensation of reflection losses by normalization on the level of expectation values. In this paper we employ a normalization on the level of operators, recently proposed in a slightly different context. We show that in this case the axiomatic arrival-time distribution of Kijowski is recovered as a limiting case. In addition, it is shown that Allcock's complex potential model is also a limit of the physically motivated fluorescence approach and connected to Kijowski's distribution through operator normalization.

I. INTRODUCTION

The quantum-mechanical analog of the arrival time of a particle at a given location is physically very interesting, and for wave packets which are spreading in space this is a highly nontrivial subject. It is a particular case of the description of time observables in quantum mechanics, i.e., times as random instants–such as the arrival times–or durations, e.g., dwell or sojourn times. For recent reviews cf. Refs. [1,2]. Difficulties in the formulation of quantum arrival times and attempts to overcome these were presented, e.g., in Refs. [3–28]. In particular, the lack of a self-adjoint arrival-time operator conjugate to the free Hamiltonian lies at the core of these difficulties.

Allcock [3] modeled a simplified detection procedure in the region $x > 0$ by means of a complex absorption potential. Because of reflection, he disregarded strong absorption and only considered the weak absorption limit, in which the detection takes a long time but all particles are eventually detected. Under the assumption that the measured arrival-time distribution was a convolution of an ideal distribution and an apparatus function, he suggested for the unknown ideal distribution an *approximate* positive expression, obtaining the (not semidefinite positive) current density as the exact solution. Somewhat pessimistically he argued that a fully satisfactory, apparatus independent, arrival-time distribution could not be defined.

In contrast, Kijowski [4] (cf. also Ref. [5]) pursued an axiomatic approach modeled on the classical case and obtained as arrival-time distribution at $x = 0$ for a free particle of mass m coming in from the left with initial state $\tilde{\psi}(k)$ (k is the wave number) an expression which, in the one-dimensional case, is given by

$$\Pi_K(t) = \frac{\hbar}{2\pi m} \left| \int dk\, \tilde{\psi}(k) \sqrt{k}\, e^{-i\hbar k^2 t/2m} \right|^2. \quad (1)$$

Surprisingly, this coincides with the approximate expression suggested by Allcock [3].

Much more recently, the distribution Π_K has been related to the positive operator valued measure generated by the eigenstates of the Aharonov-Bohm (maximally symmetric) time-of-arrival operator [1,16,29], and this method emphasizes the fact that self-adjointness is not necessary to generate quantum probability distributions. The distribution has also been generalized for the case where the particle is affected by interaction potentials [17–19] and for multiparticle systems [20].

Yet, the status of Kijowski's distribution has remained unclear and controversial [1,25,26]. As an ideal distribution, some of its properties or of its generalizations have been questioned [25] or considered to be puzzling [1], and its "operational" interpretation, apart from the approximate connection found by Allcock, has remained elusive [27].

In two recent papers [27,28], a procedure to determine arrival times of quantum mechanical particles has been discussed, which is based on the detection of fluorescence photons emitted when a two-level atom enters a laser-illuminated region. In general, due to partial reflection of the atoms by the laser field, not all of them emit photons and hence some go undetected. Therefore, the measured distribution of arrival times is not normalized to one. To normalize the distribution, division by its time integral was considered ("normalization on the level of expectation values"). In some cases this gave good agreement with the axiomatically proposed distribution of Kijowski [4], and parameter regimes where this agreement could be found were described. Analogous to Allcock's absorption model, the current density could be obtained exactly in the weak laser driving limit by deconvolution, and strong driving was problematic because of the atomic reflection. The coincidence between the results of the simplified complex potential model and the more realistic and detailed laser-atom model is not accidental and will be explained below.

Also recently, Brunetti and Fredenhagen [30] have proposed a general construction of an observable measuring the "time of occurrence" of some event. This construction involved a unitary time development and a normalization procedure on the level of operators, not on the level of expec-

tation values. For this purpose they constructed a positive operator on the orthogonal complement of the states on which the time of occurrence is infinite or zero and used its square root for normalization. This normalization procedure was, in particular, applied to sojourn or dwell times.

In this paper it will be shown that normalization on the level of operators can also be applied to the approach to arrival times of Ref. [27] which uses spontaneous photon emissions and, as a technical device, a ''conditional'' nonunitary time development. As a result we obtain quite simple and explicit expressions. In particular, the physically attractive limit of strong laser field and fast spontaneous emission can be performed and shown to exactly yield the axiomatic distribution of Kijowski [4].

In the following section we briefly review the results of Refs. [27] and [28] and then calculate the operator normalized arrival-time distribution. In Secs. III and IV, fast spontaneous emission and strong laser fields are considered in different limits. Finally, a connection between the fluorescence approach and complex absorption models is exhibited.

II. OPERATOR-NORMALIZED ARRIVAL TIMES

In the one-dimensional model of Ref. [27], a two-level atom wave packet impinges on a perpendicular laser beam at resonance with the atomic transition. Using the quantum jump approach [31] the continuous measurement of the fluorescence photons is simulated by a repeated projection onto no-photon or one-photon subspace every δt, a time interval large enough to avoid the Zeno effect but smaller than any other characteristic time. The amplitude for the undetected atoms in the interaction picture for the internal Hamiltonian obeys, in a time scale coarser than δt, and using the rotating wave and dipole approximations, an effective Schrödinger equation governed by the complex conditional Hamiltonian (the hat is used to distinguish momentum and position operators from the corresponding c numbers)

$$H_c = \frac{\hat{p}^2}{2m} - i\hbar\frac{\gamma}{2}|2\rangle\langle 2| + \frac{\hbar\Omega}{2}\Theta(\hat{x})(|2\rangle\langle 1| + |1\rangle\langle 2|), \quad (2)$$

where the ground state $|1\rangle$ is in vector-component notation $\binom{1}{0}$, the excited state $|2\rangle$ is $\binom{0}{1}$, $\Theta(x)$ is the step function, and Ω is the Rabi frequency, which gives the interaction strength with the laser field.

To obtain the time development under H_c of a wave packet incident from the left, one first solves the stationary equation

$$H_c\Phi_k = E_k\Phi_k, \quad \text{where} \quad \Phi_k(x) \equiv \begin{pmatrix} \phi_k^{(1)}(x) \\ \phi_k^{(2)}(x) \end{pmatrix} \quad (3)$$

for scattering states with real energy

$$E_k = \hbar^2 k^2/2m$$

which are incident from the left ($k > 0$). These are given by [27]

$$\Phi_k(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \begin{pmatrix} e^{ikx} + R_1 e^{-ikx} \\ R_2 e^{-iqx} \end{pmatrix}, & x \leq 0 \\ C_+|\lambda_+\rangle e^{ik_+x} + C_-|\lambda_-\rangle e^{ik_-x}, & x \geq 0, \end{cases} \quad (4)$$

where

$$q = \sqrt{k^2 + im\gamma/\hbar}, \quad (5)$$

$$k_\pm = \sqrt{k^2 - 2m\lambda_\pm/\hbar} \quad (6)$$

with $\mathrm{Im}\, q > 0$, $\mathrm{Im}\, k_\pm > 0$, and

$$\lambda_\pm = (-i\gamma \pm i\sqrt{\gamma^2 - 4\Omega^2})/4, \quad (7)$$

$$|\lambda_\pm\rangle = \begin{pmatrix} 1 \\ 2\lambda_\pm/\Omega \end{pmatrix} \quad (8)$$

are eigenvalues and eigenvectors, respectively, of the matrix $\frac{1}{2}\begin{pmatrix} 0 & \Omega \\ \Omega & -i\gamma \end{pmatrix}$. The coefficients R_1, R_2, C_+, C_- follow from the matching conditions at $x = 0$ as

$$R_1 = [\lambda_+(q+k_+)(k-k_-) - \lambda_-(q+k_-)(k-k_+)]/D, \quad (9)$$

$$R_2 = k(k_- - k_+)\Omega/D, \quad (10)$$

$$C_+ = -2k(q+k_-)\lambda_-/D, \quad (11)$$

$$C_- = 2k(q+k_+)\lambda_+/D \quad (12)$$

with the common denominator

$$D = \lambda_+(q+k_+)(k+k_-) - \lambda_-(q+k_-)(k+k_+). \quad (13)$$

By decomposing an initial state as a superposition of eigenfunctions, one obtains its conditional time development. This is easy for an initial ground-state wave packet coming in from the far left side in the remote past. Indeed,

$$\Psi(x,t) = \int_0^\infty dk\, \tilde{\psi}(k)\Phi_k(x)\, e^{-i\hbar k^2 t/2m} \quad (14)$$

describes the conditional time development of a state which in the remote past behaves like a wave packet in the ground state coming in from the left, with $\tilde{\psi}(k)$, $k > 0$, the momentum amplitude it would have at $t = 0$ as a freely moving packet. The probability N_t of no photon detection up to time t is given by [31]

$$N(t) = ||\Psi_t||^2, \quad (15)$$

and the probability density $\Pi(t)$ for the first photon detection is given by

$$\Pi(t) = -\frac{dN(t)}{dt}. \quad (16)$$

For the two-level system under consideration one has $H_c - H_c^\dagger = -i\gamma\hbar|2\rangle\langle 2|$, and thus

$$\Pi(t) = \gamma||\psi_t^{(2)}||^2. \quad (17)$$

The integral of the distribution $\Pi(t)$ is, in general, smaller than 1, in fact

$$\int_{-\infty}^{\infty} dt \ \Pi(t) = 1 - N(\infty), \tag{18}$$

and this was used in Ref. [27] for normalization on the level of expectation values.

In order to employ operator normalization we rewrite Eq. (16) in operator form, and to do this we go to the interaction picture with respect to $H_0 = \hat{p}^2/2m$,

$$H_c^I = e^{iH_0 t/\hbar}(H_c - H_0)e^{-iH_0 t/\hbar},$$

$$U_c^I(t, t_0) = e^{iH_0 t/\hbar} e^{-iH_c(t-t_0)/\hbar} e^{-iH_0 t_0/\hbar}, \tag{19}$$

where U_c^I is the conditional time development corresponding to H_c^I. Then, Eq. (14) can be written as

$$\boldsymbol{\Psi}_t = e^{-iH_0 t/\hbar} U_c^I(t, -\infty)|\psi\rangle|1\rangle, \tag{20}$$

and Eq. (15) as

$$N(t) = \langle 1|\langle\psi|\hat{N}_t|\psi\rangle|1\rangle, \tag{21}$$

with

$$\hat{N}_t = U_c^I(t, -\infty)^\dagger U_c^I(t, -\infty). \tag{22}$$

Similarly,

$$\Pi(t) = \langle 1|\langle\psi|\hat{\Pi}_t|\psi\rangle|1\rangle, \tag{23}$$

with

$$\hat{\Pi}_t = -\frac{d\hat{N}(t)}{dt} \tag{24}$$

$$= \gamma U_c^I(t, -\infty)^\dagger|2\rangle\langle2|U_c^I(t, -\infty). \tag{25}$$

In analogy to Eq. (18) we consider the integral

$$\int_{-\infty}^{\infty} dt \ \hat{\Pi}_t = 1 - \hat{N}_\infty, \tag{26}$$

and define the operator $\hat{B}$ on the incoming states (with internal ground state) through its matrix elements as

$$\langle 1|\langle\varphi|\hat{B}|\psi\rangle|1\rangle = \langle 1|\langle\varphi|1 - \hat{N}_\infty|\psi\rangle|1\rangle. \tag{27}$$

The operator $\hat{B}$ can be easily calculated as follows. From Eq. (14) one sees that for large t the second component of $\boldsymbol{\Psi}(x,t)$ is damped away and therefore only the reflected wave remains:

$$\boldsymbol{\Psi}(x,t) \simeq \int_0^\infty dk \, \tilde{\psi}(k) R_1(k) e^{ikx} e^{-i\hbar k^2 t/2m}|1\rangle \tag{28}$$

for large t. Pulling $e^{-i\hbar k^2 t/2m}$ out from the integral as $e^{-iH_0 t/\hbar}$ one sees, from Eqs. (20) and (22), that
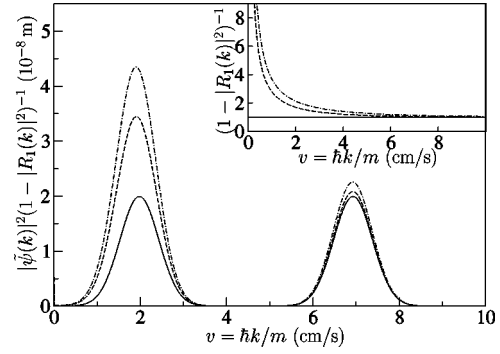

FIG. 1. Operator normalization viewed as change of initial momentum distribution. Two Gaussian momentum wave packets with $\langle v\rangle_1 = 2$ cm/s, $\langle v\rangle_2 = 7$ cm/s, $\Delta v_1 = \Delta v_2 = 0.48$ cm/s, without (solid line) and with operator normalization for $\Omega = 0.66\gamma$ (dashed line) and $\Omega = \gamma$ (dot-dashed line). All figures are for the transition $6^2 P_{3/2} - 6^2 S_{1/2}$ of cesium with $\gamma = 33.3 \times 10^6$ s^{-1}.

$$U_c(\infty, -\infty)|\psi\rangle|1\rangle = \int_0^\infty dk \, \tilde{\psi}(k) R_1(k)|-k\rangle|1\rangle. \tag{29}$$

Taking the scalar product with $U_c(\infty, -\infty)|\varphi\rangle|1\rangle$ one finds from Eq. (27), and in k space,

$$\langle 1|\langle k|\hat{B}|k'\rangle|1\rangle = (1 - \overline{R_1(k)}R_1(k'))\delta(k-k'). \tag{30}$$

Hence, on the incoming states, one can define the operator

$$\hat{\Pi}_t^{ON} = \hat{B}^{-1/2}\hat{\Pi}_t\hat{B}^{-1/2}. \tag{31}$$

From Eqs. (27) and (24) one sees that $\int_{-\infty}^{\infty}dt\,\hat{\Pi}_t^{ON} = 1$ and so the probability distribution

$$\Pi^{ON}(t) \equiv \langle 1|\langle\psi|\hat{\Pi}_t^{ON}|\psi\rangle|1\rangle \tag{32}$$

is normalized to 1. From Eqs. (24) and (30) one finally obtains

$$\Pi^{ON}(t) = \gamma \int_{-\infty}^{\infty} dx \int dk\,dk' \, \overline{\tilde{\psi}(k)}\tilde{\psi}(k')$$

$$\times [1 - |R_1(k)|^2]^{-1/2}[1 - |R_1(k')|^2]^{-1/2}$$

$$\times e^{i\hbar(k^2 - k'^2)t/2m}\overline{\phi_k^{(2)}(x)}\phi_{k'}^{(2)}(x). \tag{33}$$

Since $|R_1(k)| < 1$, $\hat{B}$ is not only a positive operator but also its inverse square-root exists.

Operator normalization can be viewed as a change in the incident momentum distribution $\tilde{\psi}(k)$ by a factor of $[1 - |R_1(k)|^2]^{-1/2}$. The effect of this factor on a Gaussian wave packet is shown in Fig. 1. For mean initial velocities of the order of cm/s, a single wave packet is multiplied by a nearly constant factor. Only for very slow particles and $\Omega \gg \gamma$, a distortion of the packet occurs. In this region the amplification of the slow components by operator normalization leads to an additional delay of $\Pi^{ON}(t)$ compared to $\Pi(t)$.

III. THE AXIOMATIC ARRIVAL-TIME DISTRIBUTION AS A LIMIT

In this section it will be shown that the operator-normalized distribution $\Pi^{ON}(t)$ approaches Kijowski's axiomatic distribution for large γ and Ω, with $\gamma^2/\Omega^2 = $ const. We put $\alpha \equiv \sqrt{1 - 4\Omega^2/\gamma^2}$ and find, for large γ,

$$\lambda_\pm = \frac{i\gamma}{4}(-1 \pm \alpha),$$

$$q \simeq \sqrt{\frac{im\gamma}{\hbar}},$$

$$k_\pm \simeq q \sqrt{\frac{1 \mp \alpha}{2}},$$

$$R_1 \simeq -1 - \frac{2ik}{\gamma^{1/2}} \sqrt{\frac{i\hbar}{m}} C_1(\alpha),$$

$$R_2 \simeq -\frac{k}{\gamma^{1/2}} \sqrt{\frac{i\hbar}{m}} C_2(\alpha), \tag{34}$$

to leading order in γ, where the constants C_i are given explicitly in the Appendix. From this one obtains

$$[1 - |R_1(k)|^2]^{-1/2}[1 - |R_1(k')|^2]^{-1/2} \simeq \frac{1}{4C_1} \sqrt{\frac{2m\gamma}{\hbar kk'}} \tag{35}$$

and

$$\gamma \overline{\Phi_k^{(2)}(x)} \Phi_{k'}^{(2)}(x) \simeq \frac{\hbar kk'}{2\pi m} \left\{ \Theta(-x) C_2^2 e^{-i(q-\bar{q})x} \right.$$

$$+ \Theta(x) \frac{16}{C_3^2} \frac{\Omega^2}{\gamma^2} \left| \left(1 + \sqrt{\frac{1+\alpha}{2}}\right) e^{ik_+ x} \right.$$

$$\left. \left. - \left(1 + \sqrt{\frac{1-\alpha}{2}}\right) e^{ik_- x} \right|^2 \right\}. \tag{36}$$

Then $\Pi^{ON}(t)$ becomes, for large γ and $\Omega^2/\gamma^2 = $ const,

$$\Pi^{ON}(t) \simeq \frac{\hbar}{2\pi m} \int dk dk' \overline{\tilde{\psi}(k)} \tilde{\psi}(k') e^{i\hbar(k^2 - k'^2)t/2m} \sqrt{kk'}$$

$$\times \frac{1}{4C_1} \sqrt{\frac{2m\gamma}{\hbar}} \int_{-\infty}^{\infty} dx \left\{ \Theta(-x) C_2^2 e^{-i(q-\bar{q})x} \right.$$

$$+ \Theta(x) \frac{16}{C_3^2} \frac{\Omega^2}{\gamma^2} \left| \left(1 + \sqrt{\frac{1+\alpha}{2}}\right) e^{ik_+ x} \right.$$

$$\left. \left. - \left(1 + \sqrt{\frac{1-\alpha}{2}}\right) e^{ik_- x} \right|^2 \right\}. \tag{37}$$

Inserting q and $k_\pm$ from Eq. (34) one sees that the expression after $\times$ is independent of k and k'. One can insert C_i from
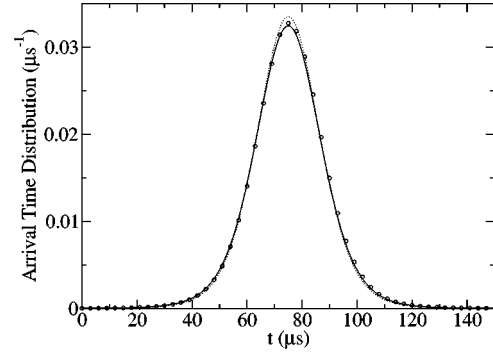

FIG. 2. Good agreement of Π^{ON} (circles) with Π_K (solid line) for large but finite γ, $\gamma = 10\gamma_{\text{Cesium}}$, $\Omega = 0.33\gamma$. The initial Gaussian wave packet is chosen to become minimal when its center arrives at $x = 0$ (in the absence of the laser) to enhance the difference between Π_K and the flux (dotted line); $\langle v \rangle = 0.9$ cm/s, $\Delta x = 0.106 \ \mu$m.

the Appendix and explicitly calculate the integral over x, but it is easier to note that the term before $\times$ is just Kijowski's distribution, which is normalized to 1, and therefore the expression after $\times$ must equal 1.

Thus, it follows that

$$\Pi^{ON}(t) \to \Pi_K(t) \quad \text{for} \quad \gamma \to \infty, \quad \gamma^2/\Omega^2 = \text{const.} \tag{38}$$

In Fig. 2 it is shown how $\Pi_{ON}(t)$ approaches Π_K for large but finite γ.

IV. LIMIT OF LARGE Ω AND DECONVOLUTION

Experimentally, Ω is easier to adjust than γ. Therefore, we also consider the limit of large Ω, with γ held fixed. In this case, one obtains

$$\lambda_\pm \simeq \mp \frac{\Omega}{2} - \frac{i\gamma}{4}, \tag{39}$$

$$q = \sqrt{k^2 + im\gamma/\hbar}, \quad \text{Im } q > 0, \tag{40}$$

$$k_\pm \simeq \sqrt{\pm \frac{m\Omega}{\hbar} \pm \frac{1}{2}\left(k^2 + \frac{im\gamma}{2\hbar}\right) \sqrt{\frac{\pm\hbar}{m\Omega}}}, \tag{41}$$

$$R_1 \simeq -1 + (1-i)k\sqrt{\frac{\hbar}{m\Omega}}, \tag{42}$$

$$R_2 \simeq -(1+i)k\sqrt{\frac{\hbar}{m\Omega}}, \tag{43}$$

to leading order in Ω. This yields

$$[1 - |R_1(k)|^2]^{-1/2}[1 - |R_1(k')|^2]^{-1/2} \simeq \frac{1}{2}\sqrt{\frac{m\Omega}{\hbar kk'}} \tag{44}$$

and

$$\gamma\overline{\Phi_k^{(2)}(x)}\Phi_{k'}^{(2)}(x) \simeq \frac{\hbar\gamma}{2\pi m}\frac{kk'}{\Omega}\{\Theta(-x)2e^{i(\bar{q}-q')x}+\Theta(x)$$

$$\times(-ie^{-i\bar{k}_+x}-e^{-i\bar{k}_-x})(ie^{ik'_+x}-e^{ik'_-x})\}. \tag{45}$$

When integrating over x, only the term $e^{-i(\bar{k}_+-k'_+)x}$ contributes in leading order of Ω, and this gives

$$\Pi^{\text{ON}}(t) \to \frac{\hbar}{2\pi m}\int dk dk'\,\overline{\tilde{\psi}(k)}\tilde{\psi}(k')e^{i(\hbar/2m)(k^2-k'^2)t}$$

$$\times\sqrt{kk'}\,\frac{\gamma}{\gamma+(i\hbar/m)(k^2-k'^2)}. \tag{46}$$

For $\gamma\to\infty$ one again obtains Kijowski's distribution, but for finite γ one has a delay in the arrival times. One can try to eliminate this, as in Ref. [27], by a deconvolution (denoted by $*$) with the first-photon distribution $W(t)$ of an atom at rest in the laser field, making the ansatz

$$\Pi^{\text{ON}}(t) = \Pi_{\text{id}}(t)*W(t) \tag{47}$$

for an ideal distribution $\Pi_{\text{id}}(t)$. Clearly, $W(t)$ denotes apparatus resolution function, similar to Ref. [3]. In terms of Fourier transforms one obtains from the ansatz

$$\tilde{\Pi}_{\text{id}}(\nu) = \frac{\tilde{\Pi}^{\text{ON}}(\nu)}{\tilde{W}(\nu)}, \tag{48}$$

where [32]

$$\frac{1}{\tilde{W}(\nu)} = 1 + \left(\frac{\gamma}{\Omega^2}+\frac{2}{\gamma}\right)i\nu + \frac{3}{\Omega^2}(i\nu)^2 + \frac{2}{\gamma\Omega^2}(i\nu)^3. \tag{49}$$

From Eq. (33) one obtains

$$\tilde{\Pi}^{\text{ON}}(\nu) = \gamma\int_{-\infty}^{\infty}dx\int dk dk'\,\overline{\tilde{\psi}(k)}\tilde{\psi}(k')[1$$

$$-|R_1(k)|^2]^{-1/2}[1-|R_1(k')|^2]^{-1/2}2\pi$$

$$\times\delta\left(\nu-\frac{\hbar}{2m}(k^2-k'^2)\right)\overline{\phi_k^{(2)}(x)}\phi_{k'}^{(2)}(x). \tag{50}$$

For large Ω one has $1/\tilde{W}(\nu)\simeq 1+2i\nu/\gamma$. Inserting this into Eq. (48) and using Eq. (46) yields

$$\tilde{\Pi}_{\text{id}}(\nu) = \frac{\hbar}{2\pi m}\int dk dk'\,\overline{\tilde{\psi}(k)}\tilde{\psi}(k')$$

$$\times\sqrt{kk'}2\pi\delta\left(\nu-\frac{\hbar}{2m}(k^2-k'^2)\right), \tag{51}$$

and therefore, for any value of γ and in the limit of strong driving,

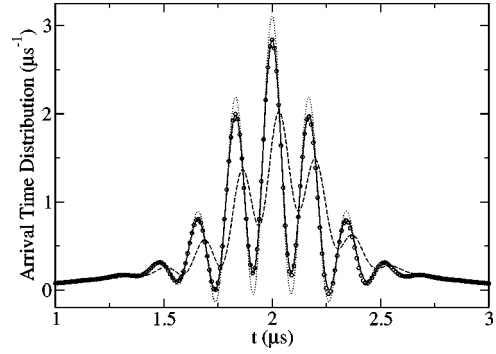$$\Pi_{\text{id}}(t) = \Pi_K(t). \tag{52}$$

FIG. 3. Excellent agreement between the deconvoluted operator-normalized distribution Π_{id} (white circles) and Π_K (solid line) for large $\Omega=500\gamma$. Π^{ON} is also shown before deconvolution (dashed line). The initial wave packet is a coherent combination, $\psi = 2^{-1/2}(\psi_1+\psi_2)$, of two Gaussian states for the center-of-mass motion of a single cesium atom that become separately minimal uncertainty packets (with $\Delta x_1=\Delta x_2=0.031\ \mu$m, and average velocities $\langle v\rangle_1=18.96$ cm/s, $\langle v\rangle_2=5.42$ cm/s at $x=0$ and $t=2\ \mu$s). The flux (dotted) becomes negative at some place.

The convergence of Π_{id} to Kijowski's distribution is shown in Fig. 3. In this example the flux, which is a limit of a deconvoluted fluorescence distribution without operator normalization [27], becomes negative.

V. CONNECTION WITH COMPLEX POTENTIALS

The above approach to arrival times, which was based on photon emissions, has another interesting limit which establishes a connection with the complex-potential approach proposed by Allcock [3]. Consider now large γ and Ω, but with $\Omega^2/\gamma=$const instead of Ω^2/γ^2 as before. Then, a little calculation shows that in Eq. (14) the second component $\psi_t^{(2)} \sim \gamma^{-1/2}$ while the first component goes to

$$\psi^{(1)}(x,t) = \int_0^{\infty}dk\,\tilde{\psi}(k)e^{-i\hbar k^2 t/2m}\phi_k(x), \tag{53}$$

where

$$\phi_k(x) = \begin{cases} e^{ikx}+Re^{-ikx}, & x\leq 0 \\ Te^{i\kappa x}, & x\geq 0, \end{cases}$$

$$R = \frac{k-\kappa}{k+\kappa},$$

$$T = \frac{2k}{k+\kappa},$$

$$\kappa = \sqrt{k^2+\frac{2imV_0}{\hbar^2}},\quad \text{Im }\kappa>0,$$

$$V_0 = \frac{\hbar\Omega^2}{2\gamma}. \tag{54}$$

From Eq. (53) one obtains that $\psi_t^{(1)}$ satisfies the one-dimensional Schrödinger equation

$$i\hbar \frac{d}{dt}\psi_t^{(1)} = [\hat{p}^2/2m - iV_0\Theta(\hat{x})]\psi_t^{(1)} \tag{55}$$

with the complex potential $-iV_0\Theta(\hat{x})$. Since $\psi_t^{(2)} \to 0$ one has, from Eq. (15),

$$N(t) = ||\psi_t^{(1)}||^2 \tag{56}$$

and so, from $\Pi(t) = -dN/dt$ together with Eq. (55),

$$\Pi(t) = \frac{2V_0}{\hbar}\int_0^\infty dx |\psi^{(1)}(x,t)|^2. \tag{57}$$

This is consistent with Eq. (17) since $\gamma|\psi_t^{(2)}|^2$ remains finite.

Equations (55) and (57) provide a connection with the complex-potential model of Allcock where the particle absorption rate is taken as a measure for the arrival time. This model is here seen to arise as a limiting case from the approach of Ref. [27]. It is also obtained by considering, somewhat artificially, a position-dependent Einstein coefficient $\gamma(x) = \gamma\Theta(x)$, and using an incoming state in the upper level, or from the irreversible detector model put forward by Halliwell [11].

The distribution in Eq. (57) is again not normalized to 1, and it is therefore natural to employ an operator normalization. With the same arguments as in Sec. III, the operator-normalized distribution is obtained as

$$\Pi^{ON}(t) = \frac{2V_0}{\hbar}\int_0^\infty dx \int dk dk' \overline{\tilde{\psi}(k)}\tilde{\psi}(k')[1-|R(k)|^2]^{-1/2}$$
$$\times [1-|R(k')|^2]^{-1/2}\overline{T(k)}T(k')$$
$$\times e^{i\hbar(k^2-k'^2)t/2m}e^{-i(\bar{\kappa}-\kappa')x}. \tag{58}$$

In the limit of strong interaction, $V_0 \to \infty$, one again finds that this goes to Kijowski's distribution:

$$\Pi^{ON}(t) \to \Pi_K(t) \quad \text{for } V_0 \to \infty. \tag{59}$$

The advantage of the one-channel model is that it provides a simple calculational tool for further, more complicated, arrival-time problems, and that, by simple limits and operator normalization, it is related to the operational fluorescence approach as well as to the axiomatic distribution of Kijowski.

VI. DISCUSSION

In Ref. [27] it had been pointed out that from the algebraic structure of the arrival-time distribution in the operational fluorescence model, it seemed impossible to obtain Kijowski's distribution by considering a suitable limit since one could not produce the necessary term $\sqrt{k}$. This term now arises in the model through an operator normalization which corresponds to the normalization approach of Ref. [30]. In simple, operational terms, this normalization can also be viewed as a modification of the initial state in such a way that the detection losses, due, in particular, to a strong laser driving, are compensated. Our results provide a crucial step towards understanding and clarifying the physical content of Kijowski's distribution, and, more precisely, establish a set of operations and limits in which such a distribution could exactly be measured. In addition, it has been shown in this paper that Allcock's one-channel model, which was based on a somewhat *ad hoc* complex absorption potential, is in fact a limiting case of the fluorescence model and also related to Kijowski's distribution through operator normalization.

Instead of the operator-normalized expression of Eq. (31) one could also consider the expectation value of the not manifestly positive expression $\hat{\Pi}_t^J \equiv \frac{1}{2}(\hat{B}^{-1}\hat{\Pi}_t + \hat{\Pi}_t\hat{B}^{-1})$ whose time integral is also 1. Interestingly, in the limit $\gamma \to \infty$ and $\Omega^2/\gamma^2 = $ const, this yields for the distribution the quantum-mechanical flux J, discussed in Ref. [27].

In this paper we have concentrated on initial states with a definite momentum sign, and freely moving particles. However, the approach can be carried over to a more general setting and this will be investigated elsewhere.

APPENDIX: EXPLICIT EXPRESSIONS FOR $C_i(\alpha)$

The constants $C_i(\alpha)$ in Eqs. (34) and (36) are given by

$$C_1 = \frac{2\sqrt{2}\alpha + (1+\alpha)^{3/2} - (1-\alpha)^{3/2}}{\sqrt{2}\alpha\sqrt{1-\alpha^2} + \sqrt{\alpha+1}(\alpha-1) + \sqrt{1-\alpha}(1+\alpha)}, \tag{A1}$$

$$C_2 = \frac{2\sqrt{2}\sqrt{1-\alpha^2}(\sqrt{1+\alpha} - \sqrt{1-\alpha})}{\sqrt{1+\alpha}(\sqrt{2} + \sqrt{1-\alpha})(\alpha-1) + \sqrt{1-\alpha}(\sqrt{2} + \sqrt{1+\alpha})(\alpha+1)}, \tag{A2}$$

$$C_3 = \frac{1}{2}[\sqrt{1+\alpha}(\sqrt{2} + \sqrt{1-\alpha})(\alpha-1) + \sqrt{1-\alpha}(\sqrt{2} + \sqrt{1+\alpha})(\alpha+1)], \tag{A3}$$

with $\alpha \equiv \sqrt{1 - 4\Omega^2/\gamma^2}$.

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