Relativistic invariant quantum entanglement between the spins of moving bodies

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The entanglement between the spins of a pair of particles may change because the spin and momentum become mixed when viewed by a moving observer [R. M. Gingrich and C. Adami, Phys. Rev. Lett. **89**, 270402 (2002)]. In this paper, it is shown that, if the momenta are appropriately entangled, the entanglement between the spins of the Bell states can remain maximal when viewed by any moving observer. Based on this observation, a relativistic invariant protocol for quantum communication is suggested, with which the nonrelativistic quantum information theory could be invariantly applied to relativistic situations.

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I. INTRODUCTION

Relativistic thermodynamics has been an intriguing problem for decades [1]. It has been shown that the probability distribution can depend on the frame, and thus the entropy and information may change if viewed from different frames [2]. Recently, the effect of Lorentz boosts on quantum states, quantum entanglement, and quantum information has attracted particular interest [3–6]. Relativistic quantum information theory may become necessary in the near future, with possible applications to quantum clock synchronization [7] and quantum-enhanced global positioning [8].

The entanglement of quantum systems forms a vital resource for many quantum information processing protocols [9], including quantum teleportation [10], cryptography [11], and computation [12]. However, it has been shown that fully entangled spin states in the rest frame will most likely decohere due to mixing with momentum if viewed from a moving frame, depending on the initial momentum wave function [4]. Therefore the entanglement between two systems can depend on the frame in which this entanglement is measured. These effects may have important consequences for quantum communication, especially when the communicating parties are in relative movement.

In this paper, we show that for a pair of spin-$\frac{1}{2}$ massive particles, if the momenta are appropriately entangled, the entanglement between the spins can remain the same as in the rest frame when viewed from any Lorentz-transformed frame. We also find a set of states for which the marginal entropy, entanglement, and measurement results of the spins are independent of the frames from which they are observed. Based on this observation, we suggest a relativistic invariant representation of the quantum bit (qubit), and suggest a relativistic invariant protocol for quantum communication, with which the nonrelativistic quantum information theory could be invariantly applied to relativistic situations. In this paper, we restrict ourselves to spin-$\frac{1}{2}$ cases, although a generalization to larger spins could be done analogously. In particular, the generalization to spin-1 massless particles, such as photons, may be of special interest [6], since current experiments in quantum communication are mostly based on photons.

II. ENTANGLEMENT BETWEEN SPINS, WITH THE PRESENCE OF MOMENTUM ENTANGLEMENT

We start by investigating a bipartite state that, in the momentum representation, has the following form viewed from the rest frame:

$$\Psi(\mathbf{p},\mathbf{q}) = g(\mathbf{p},\mathbf{q})|\psi^-\rangle, \tag{1}$$

where $\mathbf{p}$ and $\mathbf{q}$ are the momenta for the first and second particles, respectively (for a review of the definition of the momentum eigenstates for massive particles with spin and the transformations under Lorentz boosts, one may refer to Refs. [3,4,13]). The spin part of the state is the singlet Bell state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \tag{2}$$

where $|\uparrow\downarrow\rangle = |\uparrow\rangle\otimes|\downarrow\rangle$, $|\downarrow\uparrow\rangle = |\downarrow\rangle\otimes|\uparrow\rangle$, with

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}. \tag{3}$$

The momentum distribution $g(\mathbf{p},\mathbf{q})$ is normalized according to

$$\int\int |g(\mathbf{p},\mathbf{q})|^2 \tilde{d}\mathbf{p}\tilde{d}\mathbf{q} = 1, \tag{4}$$

where $\tilde{d}\mathbf{p}$ ($\tilde{d}\mathbf{q}$) is the Lorentz-invariant momentum integration measure given by

$$\tilde{d}\mathbf{p} = \frac{d^3\mathbf{p}}{2\sqrt{\mathbf{p}^2 + m^2}}, \tag{5}$$

where we use natural units: $c = 1$. Note that there is no entanglement between the spin and the momentum parts of

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$\Psi(\mathbf{p},\mathbf{q})$. The spins are maximally entangled, while the entanglement between momenta depends on $g(\mathbf{p},\mathbf{q})$. In what follows, we use $\mathbf{p}$ to represent the momentum four-vector as in Eq. (7) unless it is ambiguous.

To an observer in a frame Lorentz transformed by Λ^{-1}, the state $\Psi(\mathbf{p},\mathbf{q})$ appears to be transformed by $\Lambda\otimes\Lambda$. Therefore the state viewed by this observer appears to be

$$\Psi'(\mathbf{p},\mathbf{q})=U(\Lambda\otimes\Lambda)\Psi(\mathbf{p},\mathbf{q})$$
$$=[U_{\Lambda^{-1}\mathbf{p}}\otimes U_{\Lambda^{-1}\mathbf{q}}]\Psi(\Lambda^{-1}\mathbf{p},\Lambda^{-1}\mathbf{q}), \qquad (6)$$

where $U(\Lambda\otimes\Lambda)$ represents the unitary transformation induced by the Lorentz transformation. For compactness of notation, we here define $U_{\mathbf{p}}\equiv D^{(1/2)}(R(\Lambda,\mathbf{p}))$ as the spin-$\frac{1}{2}$ representation of the Wigner rotation $R(\Lambda,\mathbf{p})$ [4,13]. Because $\Psi'(\mathbf{p},\mathbf{q})$ differs from $\Psi(\mathbf{p},\mathbf{q})$ by only local unitary transformations, the entanglement will not change provided we do not trace out a part of the state. However, in looking at the entanglement between the spins, tracing out over the momentum degrees of freedom is implied. In $\Psi'(\mathbf{p},\mathbf{q})$ the spins and momenta may appear to be entangled; therefore the entanglement between the spins may change when viewed by the Lorentz-transformed observer. By writing $\Psi'(\mathbf{p},\mathbf{q})$ as a density matrix and tracing over the momentum degrees of freedom, the entanglement between the spins (viewed by the Lorentz-transformed observer) can be obtained by calculating the Wootters' concurrence [14] of the reduced density matrix for spins.

Any Lorentz transformation can be written as a rotation followed by a boost [13], and tracing over the momentum after a rotation will not change the spin concurrence [4]; therefore we can look only at pure boosts. Without loss of generality we may choose boosts in the z direction and write the momentum four-vector in polar coordinates as

$$\mathbf{p}=(E_{\mathbf{p}},p\cos\varphi_{\mathbf{p}}\sin\theta_{\mathbf{p}},p\sin\varphi_{\mathbf{p}}\sin\theta_{\mathbf{p}},p\cos\theta_{\mathbf{p}}), \qquad (7)$$

with $E_{\mathbf{p}}=\sqrt{p^2+m^2}$, $0\leqslant\theta_{\mathbf{p}}\leqslant\pi$, and $0\leqslant\varphi_{\mathbf{p}}<2\pi$. Let $\Lambda\equiv L(\boldsymbol{\xi})$ be the boost along the z direction (as defined in Ref. [4]), where $\boldsymbol{\xi}$ is the rapidity of the boost and $\xi=|\boldsymbol{\xi}|$. With Eq. (7), we obtain

$$U_{\mathbf{p}}=\begin{pmatrix}\alpha_{\mathbf{p}} & \beta_{\mathbf{p}}e^{-i\varphi_{\mathbf{p}}}\\ -\beta_{\mathbf{p}}e^{i\varphi_{\mathbf{p}}} & \alpha_{\mathbf{p}}\end{pmatrix}, \qquad (8)$$

where

$$\alpha_{\mathbf{p}}=\sqrt{\frac{E_{\mathbf{p}}+m}{E'_{\mathbf{p}}+m}}\left(\cosh\frac{\xi}{2}+\frac{p\cos\theta_{\mathbf{p}}}{E_{\mathbf{p}}+m}\sinh\frac{\xi}{2}\right), \qquad (9)$$

$$\beta_{\mathbf{p}}=\frac{p\sin\theta_{\mathbf{p}}}{\sqrt{(E_{\mathbf{p}}+m)(E'_{\mathbf{p}}+m)}}\sinh\frac{\xi}{2}, \qquad (10)$$

and $E'_{\mathbf{p}}=E_{\mathbf{p}}\cosh\xi+p\cos\theta_{\mathbf{p}}\sinh\xi$. Similar equations can be obtained for the second particle with momentum $\mathbf{q}$. Substituting Eq. (8) into Eq. (6), we obtain the state viewed by the Lorentz-boosted observer as

$$\Psi'(\Lambda\mathbf{p},\Lambda\mathbf{q})=\frac{g(\mathbf{p},\mathbf{q})}{\sqrt{2}}\begin{pmatrix}\alpha_{\mathbf{p}}\beta_{\mathbf{q}}e^{-i\varphi_{\mathbf{q}}}-\alpha_{\mathbf{q}}\beta_{\mathbf{p}}e^{-i\varphi_{\mathbf{p}}}\\ \alpha_{\mathbf{p}}\alpha_{\mathbf{q}}+\beta_{\mathbf{p}}\beta_{\mathbf{q}}e^{-i(\varphi_{\mathbf{p}}-\varphi_{\mathbf{q}})}\\ -\alpha_{\mathbf{p}}\alpha_{\mathbf{q}}-\beta_{\mathbf{p}}\beta_{\mathbf{q}}e^{i(\varphi_{\mathbf{p}}-\varphi_{\mathbf{q}})}\\ \alpha_{\mathbf{p}}\beta_{\mathbf{q}}e^{i\varphi_{\mathbf{q}}}-\alpha_{\mathbf{q}}\beta_{\mathbf{p}}e^{i\varphi_{\mathbf{p}}}\end{pmatrix}. \qquad (11)$$

At the present stage, we use an "entangled Gaussian" with width σ for the momentum distribution, as follows:

$$g(\mathbf{p},\mathbf{q})=\sqrt{\frac{1}{N}\exp\left[-\frac{\mathbf{p}^2+\mathbf{q}^2}{4\sigma^2}\right]\exp\left[-\frac{\mathbf{p}^2+\mathbf{q}^2-2x\mathbf{p}\cdot\mathbf{q}}{4\sigma^2(1-x^2)}\right]}, \qquad (12)$$

where $x\in[0,1)$ and N is the normalization. In Eq. (12), for a given σ, x can be reasonably regarded as a measure of the entanglement between the momenta. When $x=0$, the momentum part of the state is separable, i.e., the momentum entanglement is zero. However, in the limit $x\to1$, we have

$$\lim_{x\to1}g(\mathbf{p},\mathbf{q})=\sqrt{\frac{1}{N'}\exp\left[-\frac{\mathbf{p}^2}{2\sigma^2}\right]\delta^3(\mathbf{p}-\mathbf{q})}, \qquad (13)$$

where N' is the normalization. Equation (13) indicates a perfect correlation between the momenta. Note that in Eq. (13) the momenta are not necessarily maximally entangled.

By integrating over the momenta, we obtain the reduced density matrix for spins, viewed by the Lorentz-boosted observer, as

$$\rho=\int\int\Psi'(\mathbf{p},\mathbf{q})\Psi'(\mathbf{p},\mathbf{q})^{\dagger}\tilde{d}\mathbf{p}\tilde{d}\mathbf{q}. \qquad (14)$$

The entanglement between the spins viewed by the Lorentz-boosted observer is obtained by calculating the Wootters' concurrence [14], denoted as $C(\rho)$. The change in the Lorentz-transformed concurrence $C(\rho)$ depends on σ/m, x, and ξ. Figure 1 shows the concurrence as a function of rapidity ξ, for different values of σ/m and x. As in Ref. [4], the decrease from the maximum value [$C(\rho)=1$ for Bell states] documents the boost-induced decoherence of the spin entanglement [4]. However, it is interesting to see that for fixed σ/m and ξ the concurrence decreases less for nonzero x. Further, it is surprising that at the limit $x\to1$ the concurrence does not decrease, no matter what σ/m and ξ are. Indeed, in the limit $x\to1$, not only the concurrence but also the reduced density matrix for spins is independent of σ/m and ξ.

One possible explanation might be the following. By boosting the state, we move some of the spin entanglement to the momentum [4], and simultaneously the momentum entanglement appears to be moved to the spins. The transfer of momentum entanglement to spins hence compensates the decrease of spin entanglement, and the Lorentz-transformed concurrence decreases less. When the momenta of the two particles are perfectly correlated, even though they may not be maximally entangled, the transfer of entanglement from momenta to spins happens to fully compensate the decrease of spin entanglement, and the entanglement of the reduced

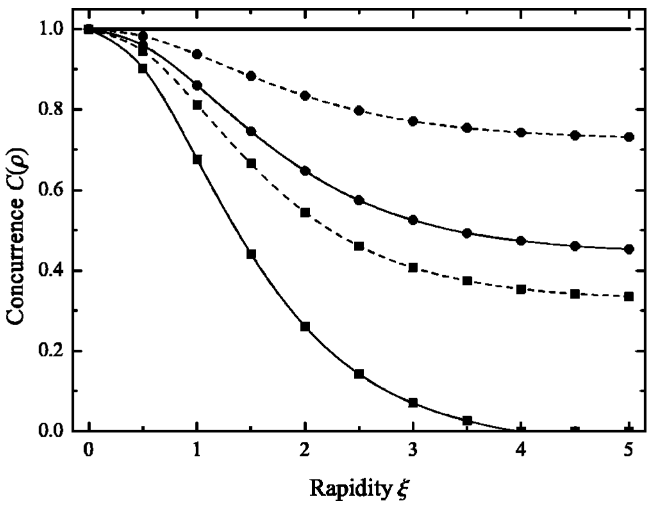

FIG. 1. Spin concurrence $C(\rho)$ as a function of rapidity ξ, for an initial Bell state with momentum in an "entangled Gaussian." Data shown as dots (squares) are for $\sigma/m=1$ ($\sigma/m=4$), with solid (dashed) line for $x=0$ ($x=0.8$). The solid line at $C(\rho)=1$ represents the spin concurrence in the limit $x\rightarrow 1$ for any value of σ/m.

spin state remains maximal when viewed by any Lorentz-boosted observer. For the singlet Bell states with momentum distribution given in Eq. (13) [generally for those given in Eq. (15) in the following], the Lorentz boost does not affect the reduced spin state, only transforms $\mathbf{p}$ ($\mathbf{q}$) to $\Lambda\mathbf{p}$ ($\Lambda\mathbf{q}$). The momentum and spin parts of such states always appear to be separate viewed from any Lorentz-boosted frame.

That the spin concurrence remains maximal in the limit $x\rightarrow 1$ when viewed from any Lorentz-boosted frame can be generalized, without using the "entangled Gaussian" in Eq. (12). Directly from Eq. (11), we see that if the momentum distribution takes the form

$$g'(\mathbf{p},\mathbf{q})=\sqrt{f(\mathbf{p})\,\delta^3(\mathbf{p}-\mathbf{q})}, \tag{15}$$

where $f(\mathbf{p})$ can be any distribution as long as $g'(\mathbf{p},\mathbf{q})$ is normalized according to Eq. (4), the boosted state can be written as

$$\Psi'(\Lambda\mathbf{p},\Lambda\mathbf{q})=\frac{g'(\mathbf{p},\mathbf{q})}{\sqrt{2}}\begin{pmatrix}0\\ \alpha_{\mathbf{p}}^2+\beta_{\mathbf{p}}^2\\ -\alpha_{\mathbf{p}}^2-\beta_{\mathbf{p}}^2\\ 0\end{pmatrix}=\Psi(\mathbf{p},\mathbf{q}), \tag{16}$$

with $\alpha_{\mathbf{p}}^2+\beta_{\mathbf{p}}^2\equiv 1$ due to the unitarity of $U_{\mathbf{p}}$. For the singlet Bell state shown in Eq. (1) with momentum distribution given in Eq. (15), the reduced density matrix remains the same as in the rest frame and the entanglement between the spins remains maximal when viewed from any Lorentz-transformed frame. Indeed, the following four "Bell" states all have invariant reduced density matrices for spins viewed from any frame Lorentz boosted along the z axis:

$$\Phi_f^+=\sqrt{f(\mathbf{p})\,\delta(p-q)\,\delta_{\theta_{\mathbf{p}},\theta_{\mathbf{q}}}\delta_{\varphi_{\mathbf{p}}+\varphi_{\mathbf{q}},0}}|\phi^+\rangle, \tag{17a}$$

$$\Phi_f^-=\sqrt{f(\mathbf{p})\,\delta(p-q)\,\delta_{\theta_{\mathbf{p}},\theta_{\mathbf{q}}}\delta_{\varphi_{\mathbf{p}}+\varphi_{\mathbf{q}},\pi}}|\phi^-\rangle, \tag{17b}$$

$$\Psi_f^+=\sqrt{f(\mathbf{p})\,\delta(p-q)\,\delta_{\theta_{\mathbf{p}},\theta_{\mathbf{q}}}\delta_{\varphi_{\mathbf{p}}-\varphi_{\mathbf{q}},\pi}}|\psi^+\rangle, \tag{17c}$$

$$\Psi_f^-=\sqrt{f(\mathbf{p})\,\delta^3(\mathbf{p}-\mathbf{q})}|\psi^-\rangle. \tag{17d}$$

Here $|\phi^\pm\rangle=(|\uparrow\uparrow\rangle\pm|\downarrow\downarrow\rangle)/\sqrt{2}$ and $|\psi^\pm\rangle=(|\uparrow\downarrow\rangle\pm|\downarrow\uparrow\rangle)/\sqrt{2}$ are the conventional Bell states, and we define $\delta_{x,y}\equiv\delta((x-y)\bmod 2\pi)$ for compactness of notation. In Eqs. (17), $f(\mathbf{p})$ can be any distribution as long as the state is normalized. Further, the states in Eqs. (17), together with those differing by only rotations, constitute a set of states of which the entanglement between the spins remains invariant when viewed from any Lorentz-transformed frame. This invariance leads to possible applications to relativistic quantum information processing.

Here we shall note that, in Eqs. (17) as well as in the remaining part of this paper, the δ functions should be regarded as limits of analytical functions under certain conditions, e.g., Eq. (13) is the limit of Eq. (12) at $x\rightarrow 1$. The only restriction on $f(\mathbf{p})$ is that the states should be normalized. Although it is known how standard fermionic Bell states can be manufactured, an important question still remains, i.e., how to produce momentum-correlated states. We might imagine such states arising from a particular particle decay process, possibly with further manipulations. The difficulty involved in producing these states depends on the specific physical system and process.

III. RELATIVISTIC INVARIANT PROTOCOL FOR QUANTUM INFORMATION PROCESSING

A possible application of the above results is to suggest a relativistic invariant protocol for quantum communication. The conventional use of a single spin-$\frac{1}{2}$ particle as a qubit may not be appropriate in relativity theory, because the reduced density matrix for its spin is generally not covariant under Lorentz transformations [3]. If and only if we consider momentum eigenstates (plane waves), the reduced density matrix for the spin of a single particle can be covariant under Lorentz transformations, but momentum eigenstates are not localized and may be difficult in feasible applications.

However, two spin-$\frac{1}{2}$ particles that are appropriately entangled, as in Eqs. (17), without being momentum eigenstates, could indeed have a reduced density matrix for spins that is invariant under Lorentz transformation. Such invariance provides us the possibility to feasibly represent a single qubit using two appropriately entangled spin-$\frac{1}{2}$ particles, in a Lorentz-invariant manner. Taking into account that in many practical situations of communication one may need to maintain the particles along desired directions, here we assume the ideal case where the momenta of the pair of particles have deterministic directions and the two particles are moving along the same deterministic direction. We may also choose the boost Λ to be along the z axis and the momenta to lie in the x-z plane, i.e., $\theta_{\mathbf{p}}\equiv\theta_{\mathbf{q}}\equiv\theta$ and $\varphi_{\mathbf{p}}\equiv\varphi_{\mathbf{q}}\equiv 0$, without loss of generality. In this protocol we use a momentum distribution that has the following form in the rest frame:

$$\tilde{g}(\mathbf{p},\mathbf{q}) = \sqrt{f(\mathbf{p})\,\delta(p-q)\,\delta_{\theta_{\mathbf{p}},\theta}\delta_{\theta_{\mathbf{q}},\theta}\delta_{\varphi_{\mathbf{p}},0}\delta_{\varphi_{\mathbf{q}}0}}, \qquad (18)$$

with $f(\mathbf{p})$ being arbitrary as long as $\tilde{g}(\mathbf{p},\mathbf{q})$ is normalized as in Eq. (4). Because Eq. (18) is a simultaneous instance of the momentum distributions of the states in both Eq. (17a) and Eq. (17d), both $\tilde{g}(\mathbf{p},\mathbf{q})|\phi^+\rangle$ and $\tilde{g}(\mathbf{p},\mathbf{q})|\psi^-\rangle$ have invariant reduced density matrices for spins when viewed from any Lorentz-boosted frame. This enables us to use these two states as the orthonormal bases, namely, $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$, of a qubit, as follows:

$$|\tilde{0}\rangle \sim \tilde{g}(\mathbf{p},\mathbf{q})|\phi^+\rangle, \qquad (19a)$$

$$|\tilde{1}\rangle \sim \tilde{g}(\mathbf{p},\mathbf{q})|\psi^-\rangle. \qquad (19b)$$

Equations (19) can be regarded as a representation of a single ‘‘Lorentz-invariant’’ qubit, in the sense that we look only at the spin part of the state. The representation of ‘‘ Lorentz-invariant’’ multiple qubits can be obtained straightforwardly. Note that in multiqubit states the momentum distributions of individual qubits are not necessarily the same. We can further find an operator acting upon a single qubit, in terms of the ‘‘Lorentz-invariant’’ bases, as

$$\tilde{\mathcal{O}} = \sum_{\sigma,\tau=0,1} \lambda_{\sigma\tau}|\tilde{\sigma}\rangle\langle\tilde{\tau}|. \qquad (20)$$

The operators acting upon multiple qubits can be obtained analogously. We refer to these operators as ‘‘Lorentz invariant’’ in the sense that, if we look only at spins, the action of the operator on the state $a|\tilde{0}\rangle + b|\tilde{1}\rangle$ ($\forall\ a,b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$) remains the same when viewed in any Lorentz-boosted frame.

Within the set of these ‘‘Lorentz-invariant’’ qubits and operators, the entropy, entanglement, and measurement results all have invariant meanings in different frames, despite the fact that these quantities may have no invariant meanings for a single quantum spin and some other situations [3,4]. Therefore it is guaranteed that, by using such states and operators, the nonrelativistic quantum information theory can be invariantly applied to relativistic situations.

IV. CONCLUSION

As observed in Ref. [4], because Lorentz boosts entangle the spin and momentum degrees of freedom, the entanglement between the spins may change if viewed from a moving frame. In particular, maximally entangled spin states will most likely decohere due to mixing with the momentum degrees of freedom, depending on the initial momentum wave function [4].

In this paper, we investigate the quantum entanglement between the spins of a pair of spin-$\frac{1}{2}$ massive particles in moving frames, for the case that the momenta of the particles are entangled. We show that, if the momenta of the pair are appropriately entangled, the entanglement between the spins of the Bell states remains maximal when viewed from any Lorentz-transformed frame. Further, we suggest a relativistic invariant protocol for quantum communication with which the nonrelativistic quantum information theory could be invariantly applied to relativistic situations.

Although the investigations are based on spin-$\frac{1}{2}$ particles, we believe that similar results for larger spins could be obtained analogously. In particular, we hope our work will help to find a relativistic invariant protocol for quantum information processing based on photons, i.e., the case of massless spin-1 particles.

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