Radiation damping in classical systems: The role of nonintegrability

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The interaction between matter and fields is a classical problem. Still there are difficulties. Time symmetry is broken. So the radiation damping belongs to a class of phenomena, which includes transport properties. We propose a radically different approach based on the extension of the dynamics of integrable systems. We consider a simple model, a harmonic oscillator interacting with a field. For integrable systems, it is well known that there exists a unitary transformation U. However, in the radiation damping we have resonances between the action of the particle and the actions of the field. This makes the system an example of Poincaré nonintegrable systems. We extend the unitary operator to a new star-unitary operator Λ. This changes the dynamical description of radiation damping. Once we know the Hamiltonian, we can of course write the Hamilton equations. But we have the possibility to go to new descriptions. The invertible Λ transformation gives many new aspects, which are hidden in the initial description. For example, we show that there are fluctuations, and that there exists an irreducible probability description. The transformation to Λ representation corresponds to a transformation to Markovian probability equations. We can always come back to the initial representation by the inverse transformation. We have verified this remarkable prediction by detailed numerical calculations. We need the Λ transformation to obtain the definition of a dressed unstable mode, which has a well-defined lifetime. In the initial representation there are various time scales and there is no strictly exponential lifetime. The situation is the same as the one we have studied in the quantum case in recent papers.

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I. INTRODUCTION

Radiation damping is described by equations breaking time symmetry. We consider a particle (harmonic oscillator) and study the damping of the oscillator through emission of energy to a classical field. Radiation damping is therefore part of the problems we have studied in recent publications [1–6]. We consider here a classical situation, which is very similar to the quantum problem studied in Refs. [1,2]. [Hereafter, we shall indicate Refs. [1] and [2] as $Q1$ and $Q2$, respectively, and cite the equations in these papers, for instance, as Eq. (Q1.2.3) for Eq. (2.3) in $Q1$.] We have extended unitary transformation theory to dissipative systems. Our starting point is Poincaré's distinction between integrable and nonintegrable systems based on resonance singularities. Dissipative systems, which we consider, form a class of nonintegrable systems due to resonances. For integrable systems we can introduce a transformation operator U leading to independent modes and then come back to the initial variables as U is invertible. We achieve now a similar situation for dissipative systems. We can consider the transformation Λ which leads to well-defined equations for dressed modes and, if useful, come back to the initial variables as Λ is also an invertible operator. There is no loss of information in the introduction of the Λ transformation. The importance of Λ in the context of this paper is that we can describe in this way the structure and the time evolution of the unstable mode associated with the radiation damping. It is natural to assume that the excited mode is characterized by a well-defined lifetime independent of initial conditions. This is realized in Λ representation. In the usual representation, this is not so. There are different decay periods such as Zeno time, exponential period, and long time tails. The Λ transformation permits to define dressed modes exactly as in quantum mechanics where it leads to the definition of dressed excited states with well-defined lifetimes.

As is well known, Schrödinger's equation does not allow purely exponential decay of the survival probability of an excited state [7]. The deviation from the exponential decay is specially significant in a short time scale (the so-called "quantum Zeno" period [8]). This phenomenon, while it occurs in an extremely short time scale, leads to some puzzles. Schwinger has written "... with failure of the simple exponential decay law we have reached, not merely the point at which some approximation ceases to be valid, but rather the limit of physical meaningfulness of the very concept of unstable particle" [9]. The corresponding Zeno period also exists in classical unstable systems, as the Hamilton equations of motion also do not allow purely exponential decay in canonical variables. There have been several attempts to improve the classical Lorentz-Abraham equation [10–16]. However, all attempts have failed to predict the existence of the Zeno period.

In the quantum case presented in $Q1$ and $Q2$, the unstable state is described by a nonfactorizable density operator. The situation is similar in radiation damping. Our basic idea was to start with the definition of dressed excited states for an integrable system where the unitary transformation U is well defined in the density matrix space in quantum systems. For the nonintegrable case, we construct a new transformation called Λ which is an analytic extension of the unitary transformation U in such a way that Λ reduces to U when there are no resonance singularities. Due to the analytic continuation, Λ is no more an unitary operator, but has a new symmetry called the "star unitarity." For the classical case discussed in this paper we follow the same basic idea. We require the following properties for Λ: (1) The transformation Λ is obtained by analytic continuation of the unitary transformation U in the continuous spectrum limit for the frequency. (2) The transformation Λ preserves the unit operator, i.e., $\Lambda I = I$. (3) The transformation Λ preserves reality of phase functions. (4) The transformation Λ is analytic with respect to the coupling constant. (5) The transformation Λ is invertible: $\Lambda\Lambda^{-1} = \Lambda^{-1}\Lambda = I$ [17]. (6) The Λ-transformed distribution function that represents an dressed excited unstable mode obeys a Markovian time evolution (strictly exponential law) corresponding to irreversible energy transfer from the excited particle to the decay product. and (7) The transformation Λ is expressed by a suitable combination of the eigenstates of the Liouvillian with complex eigenvalues (the so-called "Gamow states") [18].

In this paper we shall consider the dynamical theory of classical matter-field interacting systems in the Liouvillian formalism, where the Liouvillian L_H is defined as the Poisson bracket with the Hamiltonian with respect to the canonical variables, i.e., $L_H\rho \equiv i\{H,\rho\}$. The Liouvillian formulation is specially intuitive in terms of the Bargmann-Segal (BS) representation for a Hilbert space spanned by entire functions of the normal coordinates [19,20]. The BS representation is a classical analog of the coherent-state representation in quantum systems [21]. Through this representation, one can introduce an (m,n) representation, where m and n are integer power of monomials of the normal coordinates. The (m,n) representation corresponds to the number representation in quantum systems. Since the Liouvillian is a derivative operator with respect to the normal coordinates, the action of the Liouvillian on a monomial generally changes its power. As a result, it is interesting to see that the interaction leads to "transitions" between the states in (m,n) representation in spite of the fact that we deal with classical mechanics. In this representation we find an isomorphism between classical systems in the (m,n) representation and corresponding quantum systems in number representation on the level of the Liouvillian formalism.

Using this formulation, we shall show that a strictly parallel formalism to quantum systems is applicable to the problem of classical radiation damping. As an example, we shall consider the "classical Friedrichs model." In terms of the normal coordinate $\overline{q_a}$ of the particle $a=1$ and the field modes $a=k$ defined in the following section, the Hamiltonian of the classical Friedrichs model [22] is given in Eq. (2.1). This system corresponds to the quantum Friedrichs model discussed in our previous papers $Q1$, $Q2$, and in Ref. [5]. An advantage of this model is that Hamiltonian (2.1) is given in a bilinear form of the normal modes. Thanks to this structure, one can present a complete description of the evolution of the system, not only for the motion of the particle, but also for the radiative field, without any approximation. We have shown that our method can be extended to include virtual processes and nonlinear situations, but this will not be considered in the present paper.

An important consequence of the analytic continuation is that it leads to the transformation Λ which is not distributive when it acts on a product of canonical variables. This is in contrast to the unitary (or canonical) transformation. Because of this property, we shall see that there appear fluctuations in the emitted field from the dressed unstable mode. This implies that the distribution function that represents the dressed unstable mode is a nonlocal ensemble in phase space. Hence, the dressed unstable mode cannot be represented by a trajectory in phase space. Moreover, this property leads to a non-Poissonian algebra incorporating dissipation, which is an extension of the ordinary Poissonian algebra (or Lie algebra).

Also, it is in Λ representation that we can define the thermodynamical aspect of radiation damping and make explicit the entropy production due to the radiation damping. The entropy production (i.e., the dissipation) is a consequence of the fluctuations.

There is another important aspect of our Λ transformation. As noticed, our nonunitary transformation is invertible. As we shall see, the evolution of the original variables associated with the bare mode will be represented by a superposition of the variables associated with the dressed states. In spite of the fact that the dressed states obey stochastic Markov process for $t>0$, we do not loose any information by going to the Λ representation.

Thanks to the isomorphism between the quantum Friedrichs model and the classical Friedrichs model in terms of the BS representation, one can obtain the classical Λ through the results in $Q1$ and $Q2$ by repeating the same calculations. Hence, here we often display the formulas without presenting their detailed derivations. We shall indicate the location of the relevant calculations by citing the equation numbers in $Q1$ and $Q2$ for the reader's convenience,

In Sec. II, we present the Hamiltonian in terms of the normal modes. In Sec. III, we diagonalize the Hamiltonian for both integrable and nonintegrable cases. For the nonintegrable case this can be done in terms of the Gamow (or dual) normal modes [23–26]. The Gamow modes are eigenstates of the Liouvillian L_H with complex eigenvalues. The Gamow modes play an auxiliary role to introduce our star-unitary operator Λ for the nonintegrable system.

In Sec. IV, we introduce the Bargmann-Segal representation of Liouvillian. In Sec. V, we construct explicitly the unitary transformation operator U for the integrable case in terms of the BS representation. We also summarize the correlation dynamics which is the starting point to construct the star-unitary transformation Λ. In Sec. VI, we present Λ for the nonintegrable system. In Sec. VII, we define the distribution function that represents the classical dressed unstable mode through the Λ transformations. We present some properties of the dressed mode. The observables corresponding to the dressed mode obey strictly Markov process with exponential decay. We need the Λ representation to define the unstable mode with well-defined lifetime. Moreover, we discuss the nondistributivity of Λ and its striking consequences on the nonlocality of the distribution function for the dressed unstable mode in phase space. We show that there are intrinsic fluctuations of the normal modes of the field emitted from the dressed unstable mode.

In Sec. VIII, we present time evolution of the initial coordinates. The main points of this section are to analyze non-Markov evolution of an excited bare particle. We show that there are several different time scales in decay process of excited bare particle, i.e., a short time classical Zeno period, intermediate time scale of an approximately exponential decay process, and a long time tail.

In Sec. IX we present several numerical plots to visualize the several different time scales. We also present a numerical verification of the invertibility of our transformation between the initial coordinates and the transformed coordinates, and a numerical calculation of the nondistributibity of the Λ transformation. Whenever we obtain the Λ representation, we can go back to the initial representation through Λ^{-1}.

In Sec. X, we introduce a microscopic analog of Boltzmann's $\mathcal{H}$ function (negative entropy) in statistical mechanics in terms of Λ representation. We show that the entropy production (or the dissipation) is a consequence of the intrinsic fluctuations in the dressed unstable mode. In the last section, we present some concluding remarks. Several useful relations are given in the Appendixes.

II. CLASSICAL FRIEDRICHS MODEL

We consider a classical system that consists of a charged harmonic oscillator with a unit mass coupled with a classical scalar field in one-dimensional space. One may introduce dimensionless variables by using units $J_0=1$ for an action variable J_0 of a typical initial condition of the harmonic oscillator, $\omega_0=1$ for a suitable frequency, and $c=1$ for the speed of light. We assume that the dimensionless Hamiltonian of the system measured by the unit $\omega_0 J_0$ is given by

$$H=H_0+\lambda V=\omega_1\overline{q}_1^{c.c.}\overline{q}_1+\sum_{k=-\infty}^{+\infty}\omega_k\overline{q}_k^{c.c.}\overline{q}_k+\lambda\sum_{k=-\infty}^{+\infty}\overline{V}_k(\overline{q}_1^{c.c.}\overline{q}_k+\overline{q}_1\overline{q}_k^{c.c.}), \tag{2.1}$$

where $\overline{V}_{-k}=\overline{V}_k$, c.c. means complex conjugate, $\overline{q}_1$ and $\overline{q}_k$ are the dimensionless normal coordinates measured by the unit $\sqrt{J_0}, \omega_1>0$ is a dimensionless frequency for the harmonic oscillator, $\omega_k=|k|$ is a dimensionless frequency for the field, and λ is a coupling constant. As a convention we call the harmonic oscillator as "particle." For simplicity we have dropped processes associated with interaction terms proportional to $\overline{q}_1\overline{q}_k$ and $\overline{q}_1^{c.c.}\overline{q}_k^{c.c.}$, which correspond to the "virtual processes" in quantum mechanics. This approximation corresponds to the so-called rotating wave approximation in atomic physics [27].

The normal coordinates $\overline{q}_a$ are related to canonical pairs of variables x_a and p_a for the particle $a=1$ and the field modes $a=k$ as $\overline{q}_a=\sqrt{\omega_a/2}(x_a+ip_a/\omega_a)$. The canonical variables satisfy the Poisson bracket relation with the Kronecker δ normalization,

$$\{\overline{q}_a,\overline{q}_b^{c.c.}\}=-i\delta_{a,b}, \quad \{\overline{q}_a,\overline{q}_b\}=0. \tag{2.2}$$

Here, the Poisson bracket in terms of the normal modes is given by

$$i\{f,g\}=\sum_a\left(\frac{\partial f}{\partial\overline{q}_a}\frac{\partial g}{\partial\overline{q}_a^{c.c.}}-\frac{\partial f}{\partial\overline{q}_a^{c.c.}}\frac{\partial g}{\partial\overline{q}_a}\right), \tag{2.3}$$

where the summation a runs over all indices of the particle 1 and field modes k.

We put the system in a one-dimensional box of size L, and impose the usual periodic boundary condition. Then the spectrum of the field is discrete, i.e., $k=2\pi j/L$ with any integer j. To deal with the continuous spectrum of the field we will take the limit $L\to\infty$ in the appropriate stage of calculations. In this limit we have

$$\frac{1}{\Omega}\sum_{k=-\infty}^{+\infty}\to\int_{-\infty}^{+\infty}dk, \quad \Omega\delta_{k,0}\to\delta(k), \tag{2.4}$$

where $\delta(k)$ is the Dirac δ function and the volume factor $\Omega\equiv L/2\pi$. The volume dependence of the interaction V_k is given by $\overline{V}_k\equiv v_k/\sqrt{\Omega}$, where $v_k\sim O(\Omega^0)$ is a suitable form factor.

We note that the frequencies ω_k in Hamiltonian (2.1) are degenerate as $\omega_k=\omega_{-k}=|k|$. To avoid some complexity coming from this degeneracy, it is better to rewrite the Hamiltonian in terms of the variables $q_k\equiv(\overline{q}_k+\overline{q}_{-k})/\sqrt{2}$ for $k>0$, $q_k\equiv(\overline{q}_k-\overline{q}_{-k})/\sqrt{2}$ for $k\leqslant 0$, and $q_1\equiv\overline{q}_1$ as

$$H=\omega_1 q_1^{c.c.}q_1+\sum_{k=-\infty}^{+\infty}\omega_k q_k^{c.c.}q_k+\lambda\sum_{k=-\infty}^{+\infty}V_k(q_1^{c.c.}q_k+q_1q_k^{c.c.}), \tag{2.5}$$

with $V_k=\sqrt{2}\overline{V}_k$ for $k>0$, and $V_k=0$ for $k\leqslant 0$. In this new form the variable q_k with the negative argument k is decoupled from the other degrees of freedom. In terms of these new canonical variables the Poisson bracket of the phase functions f and g is written in the same expression as Eq. (2.3) but with q_a instead of the original variable $\overline{q}_a$ [see Eq. (3.9)]. We also have relations similar to Eq. (2.2), such as $\{q_a,q_b^{c.c.}\}=-i\delta_{a,b}$.

III. DIAGONALIZATION OF THE HAMILTONIAN

Because Hamiltonian (2.5) is bilinear, one can "diagonalize" it by introducing dressed normal modes through a linear transformation, as in the case of quantum systems. This is true for both integrable and nonintegrable cases. However, it is well known that the diagonalization for the nonintegrable case is not unique, as there is Friedrichs' diagonalization with real spectrum and the Gamow-mode diagonalization (or dual-mode diagonalization) with complex spectrum [22–26]. An advantage of the Gamow-mode diagonalization is that one can unify the diagonalization for both integrable and nonintegrable cases. Moreover, since the Gamow modes naturally appear in our nonunitary transformation, we shall present here this form of diagonalization. Since the algebraic form of the transformation is exactly the same as in quantum mechanics, we shall present here only the relevant formulas of the transformations which will be used in this paper. For the complete result of the transformation including the inverse transformation, see our previous paper $Q1$ for the quantum case.

In the classical system the frequency of the particle (ω_1) is positive. As a result, the integrable case appears only for the discrete spectrum of ω_k, where the size of the box is finite. To avoid un-necessary complexity for the discrete case, we consider a nondegenerate case of the unperturbed frequency; i.e., $\omega_k \neq \omega_1$ for all k. We have the nonintegrable system in the continuous spectrum limit.

In both integrable and nonintegrable cases, one can diagonalize the Hamiltonian in the form

$$H = z_1 Q_1^{c.c.} \tilde{Q}_1 + \sum_{k=-\infty}^{+\infty} \tilde{\omega}_k Q_k^{c.c.} \tilde{Q}_k . \tag{3.1}$$

In order to unify the notation, we use the summation sign over the wave number k of the field instead of integration for the nonintegrable case. If necessary, we shall explicitly indicate the integration sign.

The transformation is given by

$$Q_1^{c.c.} = N_1^{1/2} \left[q_1^{c.c.} + \lambda \sum_k c_k q_k^{c.c.} \right], \tag{3.2a}$$

$$\tilde{Q}_1 = N_1^{1/2} \left[q_1 + \lambda \sum_k c_k q_k \right], \tag{3.2b}$$

and

$$Q_k^{c.c.} = N_k^{1/2} \left[q_k^{c.c.} + \frac{\lambda V_k}{\eta_{d,k}^+(\tilde{\omega}_k)} \left(q_1^{c.c.} + \sum_{k'} \frac{\lambda V_{k'} q_{k'}^{c.c.}}{\tilde{\omega}_k - \omega_{k'} + i\epsilon} \right) \right], \tag{3.3a}$$

$$\tilde{Q}_k = N_k^{1/2} \left[q_k + \frac{\lambda V_k}{\eta_k^-(\tilde{\omega}_k)} \left(q_1 + \sum_{k'} \frac{\lambda V_{k'} q_{k'}}{\tilde{\omega}_k - \omega_{k'} - i\epsilon} \right) \right]. \tag{3.3b}$$

Here

$$c_k \equiv \frac{\lambda V_k}{(z_1 - \omega_k)^+}, \tag{3.4a}$$

$$\frac{1}{\eta_{d,k}^+(\tilde{\omega}_k)} \equiv \frac{1}{\eta_k^+(\tilde{\omega}_k)} \frac{z_1 - \omega_k}{(z_1 - \omega_k)^+}, \tag{3.4b}$$

where $\eta_k^{\pm}(\omega) \equiv \eta_k(\omega \pm i\epsilon)$ with a possitive infinitesimal $\epsilon \to 0+$ [28]. The function $\eta_k(z)$ is defined by

$$\eta_k(z) \equiv z - \omega_1 - \sum_{k'(\neq k)} \frac{\lambda^2 V_{k'}^2}{z - \omega_{k'}}. \tag{3.4c}$$

For the nonintegrable case we have $\mathrm{Im}\, z_1 < 0$ [see Eq. (3.7)]. Then, the notation $1/(z_1 - \omega_k)^+$ means that the denominator is evaluated on a Riemann sheet that is analytically continued from the upper half plane of z to the lower half plane for the continuous spectrum case. In other words, after we perform the integration over k by keeping z on the upper half plane, we substitute $z = z_1$, which is on the lower half plane. For this reason we often refer this analytic continuation as the "delayed analytic continuation" [5]. Therefore, c_k and $\eta_{d,k}^+$ are not ordinary functions, but are distributions that are defined only under the integration over k. For the discrete spectrum case they reduce to ordinary functions since we have simply $1/(z_1 - \omega_k)^+ = 1/(z_1 - \omega_k)$.

The normalization constants are given by

$$N_1 = (1+\xi)^{-1}, \quad \xi \equiv \lambda^2 \sum_k c_k^2 \tag{3.5a}$$

and

$$N_k = \left[1 + \frac{\lambda^2 V_k^2}{|\eta_k^+(\tilde{\omega}_k)|^2} \left(1 + \sum_{k'(\neq k)} \frac{\lambda^2 V_{k'}^2}{|\tilde{\omega}_k - \omega_{k'} + i\epsilon|^2} \right) \right]^{-1}. \tag{3.5b}$$

The renormalized frequencies z_1 and $\tilde{\omega}_k$ are given by the solutions of the transcendental equations given by

$$\eta_k^+(z_1) = 0, \tag{3.6a}$$

$$\tilde{\omega}_k = \omega_k + \frac{\lambda^2 V_k^2}{\eta_k(\tilde{\omega}_k)}. \tag{3.6b}$$

Because of our assumption $\omega_k \neq \omega_1$ for any k, one can prove that for the discrete spectrum case, $z_1 = \tilde{\omega}_1$ is a real number, and $\tilde{\omega}_1 \neq \omega_k$ and $\tilde{\omega}_k \neq \omega_{k'}$ for any k and k' [29]. Hence, the denominators in the transformation never vanish for the discrete case. For this case the quantities c_k, N_1, and N_k are all real, and we have $\eta_k(\tilde{\omega}_k) = \eta_k^{\pm}(\tilde{\omega}_k) = \eta_{d,k}^{\pm}(\tilde{\omega}_k)$. Moreover, we have $\tilde{Q}_a = Q_a$ for the integrable case.

For the continuous spectrum limit, we have the limits $N_k \to 1$ and $\tilde{\omega}_k \to \omega_k$, as the differences from the limiting values are of order $1/\Omega$ that vanishes for $\Omega \to \infty$. By the same reason, one can remove the restriction of $k' \neq k$ in $\eta_k(z)$ in the continuous spectrum limit. Hence, one may use the simpler notations $\eta(z)$ and $\eta_d(z)$ by dropping the subscript k in the corresponding quantities, when we discuss the nonintegrable case.

Assuming a suitable form factor V_k, the function $1/\eta^+(z)$ has a pole at $z = z_1$ in the lower half plane for the nonintegrable case, where

$$z_1 \equiv \tilde{\omega}_1 - i\gamma. \tag{3.7}$$

This pole corresponds to the "Green's function" pole in the quantum case, and $\tilde{\omega}_1$ corresponds to the Green's function frequency. The imaginary part $\gamma > 0$ leads to the decay rate of the excited particle.

For the integrable case, $\tilde{Q}_a = Q_a$ are the dressed normal modes which diagonalize the Hamiltonian. For the nonintegrable case we have $\tilde{Q}_a \neq Q_a$, and these are the Gamow modes. In both cases they satisfy the Poisson bracket relations given by

$$i\{Q_a, \tilde{Q}_b^{c.c.}\} = \delta_{a,b}, \quad i\{Q_a, \tilde{Q}_b\} = 0. \tag{3.8}$$

Here, we have

$$\begin{aligned} i\{f,g\} &= \sum_a \left(\frac{\partial f}{\partial q_a} \frac{\partial g}{\partial q_a^{c.c.}} - \frac{\partial f}{\partial q_a^{c.c.}} \frac{\partial g}{\partial q_a} \right) \\ &= \sum_a \left(\frac{\partial f}{\partial \tilde{Q}_a} \frac{\partial g}{\partial Q_a^{c.c.}} - \frac{\partial f}{\partial Q_a^{c.c.}} \frac{\partial g}{\partial \tilde{Q}_a} \right). \end{aligned} \tag{3.9}$$

The Hamilton equation of motion for transformed mode $\tilde{Q}_1(t)$ is given in terms of the Liouvillian L_H by

$$i\dot{\tilde{Q}}_1 = -L_H \tilde{Q}_1 = z_1 \tilde{Q}_1. \tag{3.10}$$

This shows that the Gamow mode is an eigenstate of the Liouvillian with a complex eigenvalue $-z_1$ for the nonintegrable case. This leads to the solution

$$\tilde{Q}_1(t) = e^{-iz_1 t} \tilde{Q}_1(0), \tag{3.11}$$

which oscillates in time for the integrable case and decays for the nonintegralbe case.

IV. THE LIOUVILLE SPACE

Following the decomposition of the Hamiltonian in Eq. (2.1), we have the decomposition of the Liouvillian into the unperturbed part L_0 associated to H_0 and the interaction part L_V associated with V as

$$L_H = L_0 + \lambda L_V. \tag{4.1}$$

We have

$$L_0 \rho(\vec{q}) = \sum_a \omega_a \left(q_a^{c.c.} \frac{\partial}{\partial q_a^{c.c.}} - q_a \frac{\partial}{\partial q_a} \right) \rho(\vec{q}) \tag{4.2}$$

and

$$\begin{aligned} L_V \rho(\vec{q}) = \sum_k V_k \bigg(& q_1^{c.c.} \frac{\partial}{\partial q_k^{c.c.}} - q_1 \frac{\partial}{\partial q_k} \\ & + q_k^{c.c.} \frac{\partial}{\partial q_1^{c.c.}} - q_k \frac{\partial}{\partial q_1} \bigg) \rho(\vec{q}). \end{aligned} \tag{4.3}$$

In order to introduce unitary transformations for the integrable case or nonunitary transformations in the nonintegrable case, we have to specify a function space. In our previous works [30,31], we have introduced a Hilbert space that is spanned by a set of eigenstates of the unperturbed Liouvillian L_0. We can start with this Hilbert space to specify our transformation operators.

For the classical Friedrichs model it is easy to see that any monomial $\Pi_a (q_a^{c.c.})^{m_a} q_a^{n_a}$ with non-negative integers m_a and n_a are eigenstates of L_0. However, the integration of these monomials over entire phase space diverges. In order to have a basis of the Hilbert space, we hence multiply converging factors $\exp[-J_a]$ that are invariants of motion for all a for L_0, i.e., $L_0 J_a = 0$, where the action-angle variables (J_a, α_a) are related to the normal modes by $q_a = \sqrt{J_a} \exp[-i\alpha_a]$ with the domains $0 \leqslant J_a < \infty$ and $0 \leqslant \alpha_a < 2\pi$. Then, we obtain a complete basis $f_{\vec{m},\vec{n}}(\vec{q})$ of the Hilbert space, which satisfies

$$L_0 f_{\vec{m},\vec{n}}(\vec{q}) = (\vec{m} - \vec{n}) \cdot \vec{\omega} f_{\vec{m},\vec{n}}(\vec{q}), \tag{4.4}$$

where

$$f_{\vec{m},\vec{n}}(\vec{q}) = \prod_a \frac{(q_a^{c.c.})^{m_a} q_a^{n_a}}{\sqrt{m_a! n_a!}} e^{-|q_a|^2}, \tag{4.5}$$

with $\vec{q} \equiv \{q_1, q_{k_1}, q_{k_2}, \ldots\}$, $\vec{m} \equiv \{m_1, m_{k_1}, m_{k_2}, \ldots\}$, $\vec{m} \cdot \vec{\omega} \equiv \Sigma_b m_b \omega_b$, and $1/\sqrt{m_a! n_a!}$ being a normalization constant. Equation (4.5) is the well-known basis [19,20] of the Hilbert space spanned by entire functions of q_a and $q_a^{c.c.}$, which is closely related to the coherent states in quantum mechanics [21]. In Appendix A we summarize the relations of the BS basis and the coherent states. Let us denote $f_{\vec{m},\vec{n}}$ as

$$f_{\vec{m},\vec{n}}(\vec{q}) = \langle\langle \vec{q}; \vec{q} | \vec{m}; \vec{n} \rangle\rangle = \langle \vec{q} | \vec{m} \rangle \langle \vec{n} | \vec{q} \rangle, \tag{4.6}$$

where $\langle \vec{n} | \vec{q} \rangle$ is the coherent state given in Eq. (A1). The non-negative integers m_a and n_a in the classical abstract state $|\vec{m}; \vec{n}\rangle\rangle$ indicates the powers of the monomial $(q_a^{c.c.})^{m_a} q_a^{n_a}$ in the mode a.

We note

$$L_0 \sum_a J_a = L_H \sum_a J_a = 0. \tag{4.7}$$

with $J_a = |q_a|^2$. Thus, the converging factor in Eq. (4.5) is an invariant of motion not only for the unperturbed case $\lambda = 0$ but also for the perturbed case $\lambda \neq 0$. This is a direct consequence of Hamiltonian (2.1) for classical Friedrichs model where the "virtual processes" have been neglected.

The states $|\vec{m}; \vec{n}\rangle\rangle$ are the basic elements in the Hilbert space which we call the "Liouville space." They satisfy completeness and orthonomality relations

$$\sum_{\vec{m}} \sum_{\vec{n}} |\vec{m}; \vec{n}\rangle\rangle\langle\langle \vec{m}; \vec{n}| = I, \tag{4.8a}$$

$$\langle\langle \vec{m}; \vec{n} | \vec{m'}; \vec{n'} \rangle\rangle = \prod_a \delta_{m_a, m'_a} \delta_{n'_a, n_a}, \tag{4.8b}$$

where I is a unit operator in the Liouville space. As explained in Appendix A, they are dyadic operators $|\vec{m}; \vec{n}\rangle\rangle = |\vec{m}\rangle\langle \vec{n}|$, where $|\vec{m}\rangle$ is a basis of a Hilbert space spanned by entire functions of $\vec{q}$ alone (and not of $\vec{q}^{c.c.}$). In order to distinguish $|\vec{m};\vec{n}\rangle\rangle$ from a vector $|\vec{m}\rangle$, we often call the former as a "supervector," borrowing the terminology from quantum mechanics [32]. The supervector $|\vec{m};\vec{n}\rangle\rangle$ is an eigen-supervector of the unperturbed Liouvillian L_0 in the Liouville space [see Eq. (A13)].

We note that the interaction L_V reproduces a new monomial with a different set of integers $(\vec{m},\vec{n})$ when it acts on a given monomial. In other words, the interaction leads to "transition" between integer powers of monomials [or between the states $|\vec{m};\vec{n}\rangle\rangle$ with different sets of integers, see Eq. (A14) on the level of Liouvillian formulation], in spite of the fact that we deal with classical mechanics.

In the Liouville space one can introduce a set $\{S\}$ of linear "superoperators" that act on the supervectors as

$$S=\sum_{\vec{m},\vec{n}}\sum_{\vec{m'},\vec{n'}}|\vec{m};\vec{n}\rangle\rangle\langle\langle\vec{m};\vec{n}|S|\vec{m'};\vec{n'}\rangle\rangle\langle\langle\vec{m'};\vec{n'}|. \tag{4.9}$$

For a given superoperator S, its Hermitian conjugate operator $S^\dagger$ is defined as usual. Then the Hermitian operators and the unitary operators in the Liouville space are also defined by $S^\dagger=S$ and $S^\dagger=S^{-1}$, respectively. The Liouvillan is an example of Hermitian operators in the Liouville space, i.e.,

$$L_H^\dagger=L_H. \tag{4.10}$$

In Appendix A we show the isomorphism between the classical Friedrichs model and the quantum Friedrichs model on the level of the Liouvillian formalism [see formulas (A13) and (A14)]. Indeed, if $|\vec{m}\rangle$ and $\langle\vec{n}|$ in Eq. (4.6) are regarded as the number states of the unperturbed bosons for the corresponding quantum system, then these formulas are exactly the same as the ones obtained for the quantum Liouvillian for the Friedrichs model. This means that all expressions of the superoperators that are constructed from the matrix elements of the Liouvillian in the number representation for quantum Friedrichs model are also applicable without any modification to the classical Friedrichs model in terms of the (m,n) representation.

However, it should be emphasized that the isomorphism holds only on the level of the Liouvillian, and not on the level of the Hamiltonian. Indeed, since the Hamiltonian is a phase function (or a multiplicative operator acting on any phase functions) in the classical system, there is no such expression in which the Liouvillian is written as a commutation relation with the Hamiltonian as in the case of quantum mechanics. Moreover, the classical "vacuum state" $|0\rangle$ (defined as $|\vec{m}\rangle$ with $\vec{m}=0$) in the (m,n) representation has a very different meaning from the quantum vacuum state. Indeed, the unperturbed energy H_0 of the "classical vacuum state" diverges in the limit of $\Omega\to\infty$ as

$$\begin{aligned}\langle H_0\rangle&=\int\prod_a d\mu(q_a)\sum_b \omega_b q_b^{c.c.}q_b\langle\langle\vec{q};\vec{q}|\vec{0};\vec{0}\rangle\rangle\\&=\sum_b \omega_b\int\prod_a dJ_a\exp\left(-\sum_c J_c\right)J_b\\&=\omega_1+\sum_k \omega_k\sim O(\Omega),\end{aligned} \tag{4.11}$$

where $d\mu(q_a)\equiv\pi^{-1}d^2q_a\equiv\pi^{-1}d(\mathrm{Re}\,q_a)d(\mathrm{Im}\,q_a)=dJ_a d\alpha_a/2\pi$ (see also Appendix A). On the other hand, the quantum vacuum has no energy, since $\langle 0|a_b^+a_b|0\rangle=0$, where a_b is the annihilatioin operator of the quantum mode b.

We note that the transition by the interaction L_V in Eq. (A14) preserves a number defined by

$$N=\sum_a (m_a+n_a), \tag{4.12}$$

in the state $|\vec{m};\vec{n}\rangle\rangle$. This is a direct consequence of our Hamiltonian (2.1) where the interactions proportional to $\overline{q_1}\overline{q_k}$ and $\overline{q_1^{c.c.}}\overline{q_k^{c.c.}}$ are absent. Because of this property, each sector associated to a given integer N in Eq. (4.12) evolves independent of the other sectors. As a result, the description of the time evolution of observables is significantly simplified for the Friedrichs model. For example, if we are interested in the evolution of the unperturbed action variable J_1 for the particle, it is enough to consider the sector with $N=2$.

Moreover, there are disjoint sets of the states that evolve independently from other sets inside the same sector. For example, let us consider the $N=1$ sector. We first introduce a shorthand notation of a state defined by

$$|a\rangle\equiv|1_a,\{\vec{0}\}'\rangle, \tag{4.13}$$

where $(m_a,m_b',\dots,\{\vec{0}\}')$ means all components except $a,b,\dots$ are zero, while $m_a=m$, $m_b'=m'$, and so on. As we shall see later, this shorthand notation is specially convenient to compare the results obtained here for the classical system to the results obtained for the quantum Friedrichs model discussed in our previous papers $Q1$ and $Q2$. Then in the $N=1$ sector we have transitions between the dyadic states $|1;\vec{0}\rangle\rangle$ and $|k;\vec{0}\rangle\rangle$, or between the dyadic states $|\vec{0};1\rangle\rangle$ and $|\vec{0};k\rangle\rangle$, but no transition between the dyadic states $|a;\vec{0}\rangle\rangle$ and $|\vec{0};b\rangle\rangle$ for any a and b.

Throughout this paper, we shall treat superoperators S that are functional of L_H (including the unperturbed case $\lambda=0$), i.e., $S=S(L_H)$. For this case and with relation (A15), we see that the expression in Eq. (4.9) that acts to $\langle\langle\vec{m};\vec{n}|$ from the right is equivalent in the normal-mode representation to

$$\prod_a\frac{q_a^{m_a}(q_a^{c.c.})^{n_a}}{\sqrt{m_a!n_a!}}S=\sum_{\vec{m'},\vec{n'}}\langle\langle\vec{m};\vec{n}|S|\vec{m'};\vec{n'}\rangle\rangle\prod_a\frac{q_a^{m_a'}(q_a^{c.c.})^{n_a'}}{\sqrt{m_a'!n_a'!}}. \tag{4.14}$$

This is a very useful relation to obtain an expression with matrix elements of a given superoperator in the (m,n) representation. We shall extensively use this relation when we discuss the nonunitary transformation later.

V. UNITARY TRANSFORMATION AND DYNAMICS OF CORRELATIONS

Before constructing the nonunitary transformation for the nonintegrable case, we first present the unitary transformation for the integrable case. In order to emphasize that the system with which we are dealing is the integrable case, we will use notations with a "bar," such as $\overline{Q}_a \equiv \tilde{Q}_a = Q_a$, as well as $\vec{\overline{\omega}} \equiv \{\overline{\omega}_1, \overline{\omega}_{k_1}, \overline{\omega}_{k_2}, \ldots\}$, and so on. The total Liouvillian is then

$$L_H = \sum_a \overline{\omega}_a \left(\overline{Q}_a^{c.c.} \frac{\partial}{\partial \overline{Q}_a^{c.c.}} - \overline{Q}_a \frac{\partial}{\partial \overline{Q}_a} \right). \tag{5.1}$$

Hence, the solution of the eigenvalue problem of L_H is given by

$$L_H |\overline{\phi}(\vec{m}); \overline{\phi}(\vec{n})\rangle\rangle = \vec{\overline{\omega}} \cdot (\vec{m} - \vec{n}) |\overline{\phi}(\vec{m}); \overline{\phi}(\vec{n})\rangle\rangle, \tag{5.2}$$

where $|\overline{\phi}(\vec{m}); \overline{\phi}(\vec{n})\rangle\rangle = |\overline{\phi}(\vec{m})\rangle\langle \overline{\phi}(\vec{n})|$ is a dyad as usual, and the vector $|\overline{\phi}(\vec{m})\rangle$ is defined in the BS representation by [cf. Eq. (A1)]

$$\langle \vec{q} | \overline{\phi}(\vec{m})\rangle \equiv \prod_a \frac{(\overline{Q}_a^{c.c.})^{m_a}}{\sqrt{m_a!}} \langle \vec{q} | \vec{0}\rangle. \tag{5.3}$$

Similar to Eq. (A2), the states $|\overline{\phi}(\vec{m})\rangle$ satisfy complete and orthonormal relations in the Hilbert space spanned by entire functions of $\vec{q}$. As a result, the supervectors $|\overline{\phi}(\vec{m}); \overline{\phi}(\vec{n})\rangle\rangle$ also satisfy the complete and orthonormal relations of the Liouville space, similar to Eqs. (4.8a) and (4.8b).

For the integrable case, one can introduce a unitary transformation superoperator $U^\dagger = U^{-1}$ in the Liouville space,

$$U^{-1} = \sum_{\vec{m}} \sum_{\vec{n}} |\overline{\phi}(\vec{m}); \overline{\phi}(\vec{n})\rangle\rangle\langle\langle \vec{m}; \vec{n}|, \tag{5.4a}$$

$$U = \sum_{\vec{m}} \sum_{\vec{n}} |\vec{m}; \vec{n}\rangle\rangle\langle\langle \overline{\phi}(\vec{m}); \overline{\phi}(\vec{n})|. \tag{5.4b}$$

It transforms an unperturbed eigenstate of the Liouvillian to a perturbed eigenstate as

$$U^{-1} |\vec{m}; \vec{n}\rangle\rangle = |\overline{\phi}(\vec{m}); \overline{\phi}(\vec{n})\rangle\rangle. \tag{5.5}$$

We note that one can introduce a unitary transformation operator $u^+ = u^{-1}$ in the vector space spanned by $|\vec{m}\rangle$,

$$u \equiv \sum_{\vec{m}} |\vec{m}\rangle\langle \overline{\phi}(\vec{m})|, \quad u^+ \equiv \sum_{\vec{m}} |\overline{\phi}(\vec{m})\rangle\langle \vec{m}|. \tag{5.6}$$

In terms of this unitary operator,

$$U^{-1} = u^{-1} \times u, \tag{5.7}$$

or

$$U^{-1} |\vec{m}; \vec{n}\rangle\rangle = u^{-1} |\vec{m}\rangle\langle \vec{n}| u. \tag{5.8}$$

Here, the factorizable operator $(A \times B)$ in the Liouville space is defined by

$$(A \times B)\rho = A\rho B, \tag{5.9}$$

where A, B, and ρ are linear operators in the vector space spanned by $|\vec{m}\rangle$.

For integrable systems the unitary operator U can be written in terms of "kinetic" operators based on the "correlation dynamics" discussed below. The introduction of the kinetic operators is important, as it allows us to extend the unitary operator to the nonunitary operator Λ. Since the correlation dynamics has been repeatedly presented in our previous papers, Refs. [30–32] and $Q1$, we shall present only a brief discussion of its main idea.

To introduce the kinetic operators, we first introduce the operators $P^{(\nu)}$ which are projectors to orthonormal eigenspaces of L_0,

$$L_0 P^{(\nu)} = P^{(\nu)} L_0 = w^{(\nu)} P^{(\nu)}, \tag{5.10}$$

where

$$\sum_\nu P^{(\nu)} = 1, \quad P^{(\mu)} P^{(\nu)} = P^{(\nu)} \delta_{\mu\nu}, \tag{5.11}$$

and $w^{(\nu)}$ are real eigenvalues of L_0. The complement projectors $Q^{(\nu)}$ are defined by $Q^{(\nu)} \equiv 1 - P^{(\nu)}$, which are orthogonal to $P^{(\nu)}$ and satisfy $[Q^{(\nu)}]^2 = Q^{(\nu)}$.

The unperturbed Liouville equation is then decomposed into a set of independent equations,

$$i \frac{\partial}{\partial t} P^{(\nu)} \rho = w^{(\nu)} P^{(\nu)} \rho. \tag{5.12}$$

In terms of the (m,n) representation, we associate diagonal compoments $P^{(0)} = \Sigma_{\vec{n}} |\vec{n}; \vec{n}\rangle\rangle\langle\langle \vec{n}; \vec{n}|$ of ρ with $w^{(0)} = 0$, which correspond to invariants of motion in the unpertubed case. The off-diagonal components with $\nu \neq 0$ simply oscillate with frequencies $w^{(\nu)}$.

For distribution functions the diagonal components of ρ provide the probability density of the action variable for each degree of freedom, while the off-diagonal components give information on the correlations in angle variables among several degrees of freedom. The interaction changes the state of the correlations. Hence, in the Liouville space formulation, there appears naturally a dynamics of correlations [30]. To formulate this more precisely, let us first introduce the concept of the "vacuum-of-correlations subspace." For the Friedrichs mode, that is a set of the dyadic states $|\vec{m}; \vec{n}\rangle\rangle$, which consists of the elements in which all *field* components are diagonal in the (m,n) representation. For example, using the notation defined in Eq. (4.13), the states $|1;1\rangle\rangle$ and $|k;k\rangle\rangle$ are the vacuum states in the $N=2$ sector. We then introduce an integer d that specifies the "degree of correlation." This is defined as the minimum number d of successive interactions λL_V by which a given dyadic state $|\vec{m}; \vec{n}\rangle\rangle$

can reach the vacuum of correlation. For example, the off-diagonal dyadic states $|1;k\rangle\rangle$ and $|k;1\rangle\rangle$ corresponding to particle-field correlations have $d=1$, while the dyads $|k;k'\rangle\rangle$ corresponding to field-field correlations have $d=2$. The degree of correlation will become important when we specify the analytic continuation of the frequency denominators that appear in the transformation superoperator for the nonintegrable case.

We come now to what we may call the backbone of our approach. For nonintegrable systems, we introduced in our previous work [30,33,34] (see also Refs. [35–40]) the kinetic operators $C^{(\nu)}$, $D^{(\nu)}$, $\chi^{(\nu)}$ corresponding to the dynamics of correlations. The superoperator $C^{(\nu)}$ which will be defined soon is an ''off-diagonal'' superoperator, as it describes off-diagonal transitions $C^{(\nu)}=Q^{(\nu)}C^{(\nu)}P^{(\nu)}$ from the $P^{(\nu)}$ correlation subspace to the $Q^{(\nu)}$ subspace. By operating $C^{(\nu)}$ on the ν correlation subspace $P^{(\nu)}$, this operator creates correlations other than the ν correlation. In particular, $C^{(0)}$ creates higher correlations from the vacuum of correlations. For this reason $C^{(\nu)}$ are generally called ''creation-of-correlations'' superoperators, or creation operators in short. Conversely, the $D^{(\nu)}=P^{(\nu)}D^{(\nu)}Q^{(\nu)}$ are called destruction operators. The superoperator $\chi^{(\nu)}=P^{(\nu)}\chi^{(\nu)}P^{(\nu)}$ is ''diagonal,'' as it describes a diagonal transition between states belonging to the same subspace $P^{(\nu)}$.

In terms of these operators, we may indeed consider dynamics as a dynamics of correlations. For the integrable system the diagonalization of the total Hamiltonian starting with the projectors $P^{(\nu)}$ is equivalent to the dynamics of correlations. For the unitary operator U, the kinetic operators C, D, and χ in the integrable systems are defined thorough the relations

$$U^{-1}P^{(\nu)}=(P^{(\nu)}+\overline{C}^{(\nu)})\overline{\chi}^{(\nu)}, \tag{5.13a}$$

$$P^{(\nu)}U=[\overline{\chi}^{(\nu)}]^{\dagger}(P^{(\nu)}+\overline{D}^{(\nu)}), \tag{5.13b}$$

we use bars to denote operators defined for integrable systems, as before. Using Eq. (5.4a), one can obtain the explicit form of the kinetic operators for the integrable case through the relations $\overline{\chi}^{(\nu)}=P^{(\nu)}U^{-1}P^{(\nu)}$, $\overline{C}^{(\nu)}\overline{\chi}^{(\nu)}=Q^{(\nu)}U^{-1}P^{(\nu)}$, and $\overline{D}^{(\nu)}=[\overline{C}^{(\nu)}]^{\dagger}$ for each subspace ν in the (m,n) representation.

For the integrable case, the perturbed Liouville equation is transformed as

$$i\frac{\partial}{\partial t}\overline{\rho}(t)=\overline{\Theta}\rho(t), \tag{5.14}$$

where $\overline{\rho}(t)\equiv U\rho(t)$ and $\overline{\Theta}\equiv UL_HU^{-1}$. The transformed Liouville operator is diagonal in the unperturbed basis, i.e., we have

$$\overline{\Theta}P^{(\nu)}=P^{(\nu)}\overline{\Theta}\equiv\overline{\theta}^{(\nu)}=\overline{w}^{(\nu)}P^{(\nu)}, \tag{5.15}$$

where $\overline{w}^{(\nu)}$ are the real eigenvalues of the ''collision operator'' $\overline{\Theta}$, corresponding to $w^{(\nu)}$ shifted by the interaction. As a consequence the dynamics is reduced to the set of equations

$$i\frac{\partial}{\partial t}P^{(\nu)}\overline{\rho(t)}=\overline{w}^{(\nu)}P^{(\nu)}\overline{\rho(t)}. \tag{5.16}$$

The collision operator in Eq. (5.15) is expressed in terms of the kinetic operators [32] by

$$\overline{\theta}^{(\nu)}=P^{(\nu)}w^{(\nu)}+[\overline{\chi}^{(\nu)}]^{-1}\lambda L_V\overline{C}^{(\nu)}\overline{\chi}^{(\nu)}. \tag{5.17}$$

The main result is that the Louville space formulation of the classical mechanics can be expressed in terms of the kinetic operators. This is the starting point for our transition from integrable to nonintegrable systems.

VI. Λ TRANSFORMATION

We next consider the nonintegrable case. It is well known that the Friedrichs model is nonintegrable in the sense of Poincaré for the continuous spectum case [41]. More precisely, there are divergences in the perturbation expansion (i.e., expansion in λ^n with $n\geqslant 0$) of invariants of motion other than functions of the Hamiltonian [30]. The divergences are due to vanishing denominators, which occur when the frequencies of the system obey relations called Poincaré resonances, such as $\omega_k=\omega_1$ in $1/(\omega_k-\omega_1)$. Then one cannot diagonalize the Hamiltonian by a unitary (or canonical) transformation that is analytic in λ at $\lambda=0$.

However, we can still deal with the vanishing denominators in the continuous spectrum case by performing a suitable analytic continuation of the denominators which appear in the kinetic operators C, D, and χ. To determine the form of Λ, we use the same formal expressions (5.13a) and (5.13b),

$$\Lambda^{-1}P^{(\nu)}=(P^{(\nu)}+C^{(\nu)})\chi^{(\nu)}, \tag{6.1a}$$

$$P^{(\nu)}\Lambda=[\chi^{(\nu)}]^{*}(P^{(\nu)}+D^{(\nu)}). \tag{6.1b}$$

This guarantees that condition (1) for Λ presented at the introduction is satisfied. The analytic continuation can be achieved by adding $\pm i\epsilon$ with a positive infinitesimal ϵ as $1/w\Rightarrow 1/(w\pm i\epsilon)$. The key point is to choose the sign of $\pm i\epsilon$. Reading each term of the perturbation expansion from right to left, we choose $-i\epsilon$ for a transition to higher, or equal, degree of correlations, as this corresponds to a process oriented towards the future, while we choose $+i\epsilon$ for a transition to lower correlations, as this corresponds to a process oriented towards the past [6,32,33]. As far as the kinetic operators $C^{(\nu)}$ and $D^{(\nu)}$ are concerned, this choice of the analytic continuation completely determines the form of these operators. To determine the operator $\chi^{(\nu)}$, we need the other conditions [(2)–(7)] that are displayed in the Introduction (see $Q1$).

Due to the analytic continuation, Λ is no more unitary operator, but has a new symmetry called the ''star-unitarity'' that was introduced by one of the authors (I.P.), long time ago [33] (see also Ref. [32]), i.e.,

$$\Lambda^{-1}=\Lambda^*, \tag{6.2}$$

where * denotes "star conjugation." This is an extension of the unitary symmetry to nonintegrable systems. We also have $D^{(\nu)}\equiv[C^{(\nu)}]^*$. Star conjugation means a combination of Hermitian conjugation denoted by † and the "prime conjugation" denoted by the prime symbol, $'$ which is defined as an interchange in the role of higher and lower correlations,

$$\Lambda^*\equiv(\Lambda^\dagger)'=(\Lambda')^\dagger. \tag{6.3}$$

Instead of $\overline{\rho(t)}=U\rho(t)$, we now consider $\tilde{\rho}(t)=\Lambda\rho(t)$, which satisfies the same equation as $\bar{\rho}$ in Eq. (5.14) but now $\tilde{\Theta}\equiv\Lambda L_H\Lambda^{-1}$ is the "collision operator" of kinetic theory,

$$i\frac{\partial}{\partial t}P^{(\nu)}\tilde{\rho}(t)=\tilde{\theta}^{(\nu)}P^{(\nu)}\tilde{\rho}(t), \tag{6.4}$$

where $\tilde{\theta}^{(\nu)}\equiv\tilde{\Theta}P^{(\nu)}=P^{(\nu)}\tilde{\Theta}$ and it has the same form as $\overline{\theta}^{(\nu)}$ in Eq. (5.17) without the bar notation. In contrast to the integrable case, one cannot generally write $\tilde{\theta}^{(\nu)}$ as a simple form as $\overline{\theta}^{(\nu)}$ with a shifted frequency $\overline{w}^{(\nu)}$, but it is a non-Hermitian block diagonal operator in $P^{(\nu)}$ subspace. As a result, eigenvalues $z_j^{(\nu)}$ of $\tilde{\theta}^{(\nu)}$ are generally complex numbers, where j characterizes the eigenvalue. This implies that the Liouvillian L_H has the same complex eigenvalues as $\tilde{\theta}^{(\nu)}$, due to the similitude relation $\tilde{\Theta}=\Lambda L_H\Lambda^{-1}$. In other words, Λ^{-1} acting on an eigenstate of the collision operator $\tilde{\theta}^{(\nu)}$ generates an eigenstate of L_H with the same complex eigenvalue $z_j^{(\nu)}$ of $\tilde{\theta}^{(\nu)}$. This is only possible if the eigenstates are not in the Hilbert space. Complex eigenvalues mean time-symmetry breaking and dissipation.

As mentioned before, the isomorphism between the quantum Friedrichs model and the classical Friedrichs model in terms of the (m,n) representation allows us to find the explicit form of Λ, which is essentially the same as the one obtained in the quantum system in our previous papers $Q1$ and $Q2$. Therefore, we here display only the final forms of Λ with an indication of the equation numbers that have been presented in $Q1$ and $Q2$.

We first present the form of Λ in the $N=1$ sector. To write the explicit form of Λ in this subspace, we first introduce an auxiliary transformation operator u_{1G} in the subspace $I_1\equiv\Sigma_a|a\rangle\langle a|$ in the Hilbert space spanned by the entire functions of $\vec{q}$: [cf. Eqs. (Q1.2.16) and (Q1.2.17)]

$$u_{1G}^{-1}\equiv\sum_a |\phi_a\rangle\langle a|,\ \ u_{1G}\equiv\sum_a |a\rangle\langle\tilde{\phi}_a|, \tag{6.5}$$

which give us

$$|\phi_a\rangle=u_{1G}^{-1}|a\rangle,\quad \langle\tilde{\phi}_a|=\langle a|u_{1G}. \tag{6.6}$$

Here, the "Gamow states" $|\phi_a\rangle$ and $\langle\tilde{\phi}_a|$ are given by [see Eqs. (Q1.2.24)–(Q1.2.29)]

$$|\phi_1\rangle=N_1^{1/2}(|1\rangle+\lambda\sum_k c_k|k\rangle), \tag{6.7a}$$

$$\langle\tilde{\phi}_1|=N_1^{1/2}(\langle 1|+\lambda\sum_k c_k\langle k|), \tag{6.7b}$$

$$|\phi_k\rangle=|k\rangle+\frac{\lambda V_k}{\eta_d^+(\omega_k)}\left(|1\rangle+\sum_{k'}\frac{\lambda V_{k'}}{\omega_k-\omega_{k'}+i\epsilon}|k'\rangle\right), \tag{6.7c}$$

$$\langle\tilde{\phi}_k|=\langle k|+\frac{\lambda V_k}{\eta^-(\omega_k)}\left(\langle 1|+\sum_{k'}\frac{\lambda V_{k'}}{\omega_k-\omega_{k'}-i\epsilon}\langle k'|\right), \tag{6.7d}$$

cf. Eqs. (3.2a)–(3.3b) for the nonintegrable case. These expressions are exactly the same as the Gamow states for the corresponding quantum Friedrichs model constructed in Ref. [5]. They are bicomplete and biorthonormal in the subspace spanned I_1, i.e. [see Eq. (Q1.2.23)],

$$\sum_a |\phi_a\rangle\langle\tilde{\phi}_a|=I_1,\ \ \langle\tilde{\phi}_a|\phi_b\rangle=\delta_{a,b}. \tag{6.8}$$

We also have the relations [see Eq. (Q1.2.32) and Ref. [5]]

$$\langle\phi_1|\phi_1\rangle=|N_1|\left(1+\lambda^2\sum_k c_kc_k^{c.c.}\right)=0, \tag{6.9}$$

which will be used later. The last equality in Eq. (6.9) is possible because c_k is not an ordinary function but is a distribution as mentioned in Eq. (3.4a). Equation (6.9) indicates that $|\phi_1\rangle$ is not an element in the Hilbert space.

Using this auxiliary transformation operator, the Λ transformation in the subspace $I_{10}\equiv\Sigma_a|a;0\rangle\!\rangle\langle\!\langle a;0|$ in the $N=1$ sector is given by

$$\Lambda^{-1}=u_{1G}^{-1}\times 1,\quad \Lambda=u_{1G}\times 1, \tag{6.10}$$

which lead to

$$\Lambda^{-1}|a;0\rangle\!\rangle=|\phi_a;0\rangle\!\rangle,\langle\!\langle a;0|\Lambda=\langle\!\langle\tilde{\phi}_a;0|. \tag{6.11}$$

They satisfy

$$L_H|\phi_a;0\rangle\!\rangle=z_a|\phi_a;0\rangle\!\rangle,$$

$$\langle\!\langle\tilde{\phi}_a;0|L_H=\langle\!\langle\tilde{\phi}_a;0|z_a, \tag{6.12}$$

where we have put $z_k\equiv\omega_k$ to unify the notations.

Similarly, Λ in the $I_{01}\equiv\Sigma_a|0;a\rangle\!\rangle\langle\!\langle 0;a|$ subspace in the $N=1$ sector is given by $\Lambda^{-1}=1\times(u_{1G}^+)^{-1}$ and $\Lambda=1\times(u_{1G}^+)$, where u_{1G}^+ is a Hermitian conjugate operator of u_{1G}.

Next we present the nonunitary transformation in the subspace $I_{11}\equiv\Sigma_{a,b}|a;b\rangle\!\rangle\langle\!\langle a;b|$ in the $N=2$ sector. The nonunitary transformations in this subspace are given by [see Eqs. (Q1.5.2) and (Q2.32)]

$$\Lambda^{-1}|a;a\rangle\rangle=|\rho_a^0\rangle\rangle,\langle\langle a;a|\Lambda=\langle\langle\tilde{\rho}_a^0|, \quad (6.13a)$$

and for $a\neq b$ [see Eq. (Q2.24)],

$$\Lambda^{-1}|a;b\rangle\rangle=|\rho^{ab}\rangle\rangle,\langle\langle a;b|\Lambda=\langle\langle\tilde{\rho}^{ab}|, \quad (6.13b)$$

where [see Eqs. (Q1.7.3) and (Q1.7.7)]

$$|\rho_1^0\rangle\rangle\equiv|F_1^0\rangle\rangle+\sum_k b_k|F_k^0\rangle\rangle, \quad (6.14a)$$

$$|\rho_k^0\rangle\rangle\equiv|F_k^0\rangle\rangle-b_k|F_1^0\rangle\rangle, \quad (6.14b)$$

and

$$\langle\langle\tilde{\rho}_1^0|\equiv\langle\langle\tilde{F}_1^0|+\sum_k b_k\langle\langle\tilde{F}_k^0|, \quad (6.14c)$$

$$\langle\langle\tilde{\rho}_k^0|\equiv\langle\langle\tilde{F}_k^0|-b_k\langle\langle\tilde{F}_1^0|, \quad (6.14d)$$

with

$$|F_1^0\rangle\rangle\equiv|\phi_1;\phi_1\rangle\rangle,\quad |F_k^0\rangle\rangle\equiv|\tilde{\phi}_k;\tilde{\phi}_k\rangle\rangle, \quad (6.15a)$$

$$\langle\langle\tilde{F}_1^0|\equiv\langle\langle\tilde{\phi}_1;\tilde{\phi}_1|,\quad \langle\langle\tilde{F}_k^0|\equiv\langle\langle\tilde{\phi}_k^{c.c.};\tilde{\phi}_k^{c.c.}|. \quad (6.15b)$$

Here, the states $|F_j^\nu\rangle\rangle$ and $\langle\langle\tilde{F}_j^\nu|$ are right and left eigenstates of the Liouvillian L_H. The superscript ν over $|F^\nu\rangle\rangle$ and $|\rho^\nu\rangle\rangle$ is the same index ν in Eq. (5.10) [see also Eq. (Q1.4.23)]. As will be presented later, the first term $|F_1^0\rangle\rangle$ in Eq. (6.14a) is a decaying eigenstate with the complex eigenvalue $-2i\gamma$, while the second term is a superposition of degenerate eigenstates with zero egenvalue of L_H for any value of k [see Eqs. (6.21a) and (6.21b)]. We shall see that the degenerate eigenstates in $|\rho_1^0\rangle\rangle$ play an essential role to obtain a decay product in the decaying process of the state $|\rho_1^0\rangle\rangle$. For $a\neq b$, we present the explicit form of Eq. (6.13b) in Appendix B.

The real function b_k is defined by [see Eq. (Q1.7.5)]

$$b_k\equiv\frac{\lambda^2}{|1+\xi|}[(rc_k^2+\text{c.c.})-c_kc_k^{c.c.}], \quad (6.16)$$

with [see Eq. (Q1.6.17)]

$$r\equiv\frac{1}{2}+\frac{|1+\xi|-1-(\xi+\xi^{c.c.})/2}{\xi-\xi^{c.c.}}. \quad (6.17)$$

We have $b_k\sim O(1/\Omega)$ and [see Eq. (Q1.7.6)]

$$\sum_k b_k=1. \quad (6.18)$$

In Appendix C we show that b_k is the line shape of the field that is emitted from the unstable dressed excited mode. We shall see that b_k will play a central role to describe irreversible process in radiation damping.

To understand the structure of our transformation operator Λ, it is worthwhile to compare Eqs. (6.11) and (6.13a) with the corresponding expressions for the integrable case in Eq. (5.5). For example, we have

$$U^{-1}|1;0\rangle\rangle=|\bar{\phi}_1;0\rangle\rangle,\quad U^{-1}|1;1\rangle\rangle=|\bar{\phi}_1;\bar{\phi}_1\rangle\rangle. \quad (6.19)$$

We see that $\Lambda^{-1}|1;0\rangle\rangle$ is a direct extension of $U^{-1}|1;0\gg$. Since the Gamow state $|\phi_1\rangle$ is analytic with respect to the coupling constant at $\lambda=0$, this extension is possible. In contrast, a simple extension of $U^{-1}|1;1\rangle\rangle$ to the nonintegrable system leads to only the first term $|F_1^0\rangle\rangle=|\phi_1;\phi_1\rangle\rangle$ of Eq. (6.14a), which is not analytic at $\lambda=0$. We can see this singularity by observing the relation $\mathrm{Tr}(|\bar{\phi}_1;\bar{\phi}_1\rangle\rangle)=\langle\bar{\phi}_1|\bar{\phi}_1\rangle=1$ [see Eq. (A7)], while we have $\mathrm{Tr}(|\phi_1;\phi_1\rangle\rangle)=\langle\phi_1|\phi_1\rangle=0$ because of Eq. (6.9) for any λ. Here we impose conditions (4) and (7) displayed in the Introduction to find the form of Λ. As has been shown in Appendix A in $Q2$, the second term with the factor b_k in Eq. (6.14a) has been added to remove this singularity [see Eq. (Q2.A1)]. Indeed, the singular term with the factor $c_kc_k^{c.c.}$ in b_k exactly compensates the singularity in the first term. Thanks to this second term that we now have $\mathrm{Tr}(|\rho_1^0\rangle\rangle)=1$.

Another important consequence of the second term in Eq. (6.14a) is related to degeneracy of the zero eigenstates $|F_k^0\rangle\rangle$ of the Liouvillian L_H. Because of this second term, the state $|\rho_1^0\rangle\rangle$ may decay by producing the decay product $|\rho_k^0\rangle\rangle$ [see Eqs. (6.26a) and (7.8)]. In other words, the existence of the degenerate eigenstates in $|\rho_1^0\rangle\rangle$ guarantees the transition of the state $|\rho_1^0\rangle\rangle$ to $|\rho_k^0\rangle\rangle$. This is in contrast to the Gamow state $|F_1^0\gg$ alone, as $\exp[-iL_Ht]|F_1^0\rangle\rangle$ decays exponentially without producing any decay product [see Eq. (6.21a)]. Furthermore, as has been shown in $Q1$, the number r in Eq. (6.17) reduces to 1/2 in the integrable case [see Eq. (Q1.6.18)], while c_k becomes real as mentioned just below Eq. (3.6b). As a result, b_k vanishes for the integrable case, and our transformed state $|\rho_1^0\rangle\rangle$ reduces to the integrable one $|\bar{\phi}_1;\bar{\phi}_1\rangle\rangle$.

The eigenstates $|F_j^\nu\rangle\rangle$ and $\langle\langle\tilde{F}_j^\nu|$ satisfy bicompleteness and biorthonormal relation in the subspace I_{11} with the complex eigenvalue $z_j^{(\nu)}$ [see Eqs. (Q1.4.1), (Q1.4.27)], and Appendix B5 in $Q1$), i.e.,

$$\sum_{\nu,j}|F_j^\nu\rangle\rangle\langle\langle\tilde{F}_j^\nu|=I_{11},\quad \langle\langle\tilde{F}_i^\mu|F_j^\nu\rangle\rangle=\delta_{\mu,\nu}\delta_{i,j}, \quad (6.20)$$

and

$$L_H|F_1^0\rangle\rangle=-2i\gamma|F_1^0\rangle\rangle,\quad \langle\langle\tilde{F}_1^0|L_H=-2i\gamma\langle\langle\tilde{F}_1^0|, \quad (6.21a)$$

$$L_H|F_k^0\rangle\rangle=\langle\langle\tilde{F}_k^0|L_H=0, \quad (6.21b)$$

where $-2i\gamma=z_1-z_1^{c.c.}$.

To unify the expressions with the ones presented in $Q1$, let us introduce new notations as $|\nu_j\rangle\rangle$ with $\nu=0$ or ab, and $j=1$ or k (we do not write the index j for $\nu=ab$), i.e., $|0_a\rangle\rangle\equiv|a;a\rangle\rangle$ and $|ab\rangle\rangle\equiv|a;b\rangle\rangle$. Then, we may write Λ in the subspace I_{11} as [see Eq. (Q1.6.24)]

$$\Lambda^{-1}=\sum_{\nu,j}|\rho_j^\nu\rangle\rangle\langle\langle\nu_j|,\quad \Lambda=\sum_{\nu,j}|\nu_j\rangle\rangle\langle\langle\tilde{\rho}_j^\nu|. \tag{6.22}$$

This is a star-unitary operator $\Lambda^*=\Lambda^{-1}$ [see Eq. (Q1.6.22)]. As has been shown in $Q1$, the states $|\rho_j^\nu\rangle\rangle$ and $\langle\langle\tilde{\rho}_j^\nu|$ are bicomplete and biorthogonal in the subspace I_{11} [see Eq. (Q1.6.25)],

$$\sum_{\nu,j}|\rho_j^\nu\rangle\rangle\langle\langle\tilde{\rho}_j^\nu|=I_{11},\quad \langle\langle\tilde{\rho}_i^\mu|\rho_j^\nu\rangle\rangle=\delta_{\mu\nu}\delta_{ij}. \tag{6.23}$$

We note that

$$|\rho_j^\nu(t)\rangle\rangle\equiv e^{-iL_Ht}|\rho_j^\nu\rangle\rangle=\Lambda^{-1}e^{-i\tilde{\Theta}t}|\nu_j\rangle\rangle, \tag{6.24}$$

where the relation $\Lambda\exp[-iL_Ht]\Lambda^{-1}=\exp[-i\tilde{\Theta}t]$ has been used. Hence, the time evolution of the states $\Lambda|\rho_j^\nu(t)\rangle\rangle$ is governed by the collision operator $\tilde{\Theta}$ mentioned in Eq. (6.4). Using Eqs. (6.13a)–(6.21b) and (B7), one can find the matrix elements of $\tilde{\theta}^{(\nu)}$ (for $\tilde{\theta}^{(\nu)}_{ab;cd}\equiv\langle\langle a;b|\tilde{\theta}^{(\nu)}|c;d\rangle\rangle$) in this sector as, e.g.,

$$\tilde{\theta}^{(0)}_{11;11}=-2i\gamma+O(\Omega^{-2}), \tag{6.25a}$$

$$\tilde{\theta}^{(0)}_{11;kk}=\tilde{\theta}^{(0)}_{kk;11}=2i\gamma b_k+O(\Omega^{-2}), \tag{6.25b}$$

$$\tilde{\theta}^{(0)}_{kk;pp}=-2i\gamma b_kb_p+O(\Omega^{-3}), \tag{6.25c}$$

as well as the explicit forms of the time evolution of $|\rho_j^\nu(t)\rangle\rangle$ as [see Eqs. (Q1.7.79) and (Q1.7.10)],

$$|\rho_1^0(t)\rangle\rangle=e^{-2\gamma t}|\rho_1^0\rangle\rangle+(1-e^{-2\gamma t})\sum_k b_k|\rho_k^0\rangle\rangle, \tag{6.26a}$$

$$|\rho_k^0(t)\rangle\rangle=|\rho_k^0\rangle\rangle+(1-e^{-2\gamma t})b_k[|\rho_1^0\rangle\rangle-\sum_l b_l|\rho_k^0\rangle\rangle], \tag{6.26b}$$

and

$$|\rho^{ab}(t)\rangle\rangle=e^{-i(z_a-z_b)t}|\rho^{ab}\rangle\rangle. \tag{6.26c}$$

Applying the Hermitian-conjugation to Eq. (6.22), we see that the prime-conjugation operator of Λ is given by [see Eq. (6.3)]

$$(\Lambda')^{-1}=\sum_{\nu,j}|\tilde{\rho}_j^\nu\rangle\rangle\langle\langle\nu_j|,\quad \Lambda'=\sum_{\nu,j}|\nu_j\rangle\rangle\langle\langle\rho_j^\nu|. \tag{6.27}$$

We note the relation

$$\int d\Gamma Ae^{-iL_Ht}\Lambda^{-1}\rho=\int d\Gamma[\Lambda'A(t)]\rho, \tag{6.28}$$

where $d\Gamma\equiv\Pi_a dJ_ad\alpha_a$ is a volume element in phase space, and $A(t)=\exp[+iL_Ht]A$ is an observable, such as the coordinate $x_a(t)$ associated with a normal mode a. We have a transformation $\hat{A}(t)=\Lambda'A(t)$ of the observable. It is interesting to compare this to a transformation in the integrable systems. Corresponding to this transformation, we have $\overline{A}(t)=UA(t)$ for the observable. For integrable systems, the unitary operator U generates canonical transformation $\overline{x_a}=Ux_a$ and $\overline{p_a}=Up_a$ for the phase variables. Our star-unitary transformation gives a generalization of canonical transformation for the nonintegrable systems.

VII. DRESSED UNSTABLE MODE AND RADIATION DAMPING

In this section we shall define the distribution function that represents a dressed unstable excited mode in the Λ representation. We shall see that the evolution of the action variable over this distribution function is purely exponential. Hence the lifetime of the dressed excited mode is well defined. This requires a consistent dressing of the field surrounding the bare particle. To introduce the dressed mode, we first note that the unperturbed excited mode at $t=0$ is represented in terms of the distribution function by

$$\rho(\vec{J},\vec{\alpha},0)=\rho_1^\delta(\vec{J},\vec{\alpha})\equiv\delta(J_1-J_{10})\delta(\alpha_1-\alpha_{10})\prod_k\delta(J_k)\delta(\alpha_k-\alpha_{k0}), \tag{7.1}$$

where J_{10} and α_{10} are the action-angle variables of the particle, and the superscript δ on ρ indicates that the distribution function is a δ function. The subscript 1 in ρ_1^δ indicates that only the particle is excited. All field modes are in vacuum at $t=0$. The state $\rho_1^\delta(\vec{J},\vec{\alpha})$ is local in phase space and gives an expectation value ω_1J_{10} for H_0. The dynamics preserves the δ function that represents a trajectory.

Before introducing the dressed unstable excited mode, let us give a comment on our classical distribution function of the unperturbed excited mode, Eq. (7.1). In quantum mechanics the quantum state that represents an unperturbed excited mode is given by a quantum dyadic state $|1;1\rangle\rangle$ [see Eq. (Q1)]. The expectation value of H_0 over this quantum state is rightfully given by $\hbar\omega_1$. However, we cannot use the corresponding classical BS state $|1;1\rangle\rangle$ to represent the unperturbed excited mode for our classical system. Indeed, the expectation value of H_0 over this BS state diverges with order Ω in the continuous spectrum limit $\Omega\to\infty$ by the same reason that led to Eq. (4.11). This is the reason why we must use the state Eq. (7.1) instead of $|1;1\rangle\rangle$ to specify the classical unperturbed excited mode.

Let us now consider the dressed excited mode. In the integrable case the dressed mode would be represented by the unitary transformation as

$$\overline{\rho_1^\delta}(\vec{J},\vec{\alpha},0)\equiv U^{-1}\rho_1^\delta(\vec{J},\vec{\alpha}). \tag{7.2}$$

The time evolution $\overline{\rho_1^{\delta}}(t)=\exp[-iL_Ht]\overline{\rho_1^{\delta}}(0)$ of this transformed state still represents a trajectory, since any unitary transformation preserves a trajectory.

We extend this definition to the nonintegrable case, and now consider evolution in Λ representation. A distribution function that represents a dressed unstable mode is given in contrast by the star-unitary Λ transformation as

$$\hat{\rho}_1(\vec{J},\vec{\alpha},t)\equiv e^{-iL_Ht}\Lambda^{-1}\rho_1^{\delta}(\vec{J},\vec{\alpha}). \tag{7.3}$$

As could be expected, the transformed distribution function $\hat{\rho}_1(\vec{J},\vec{\alpha},t)$ does not any more represent a trajectory, but corresponds to a nonlocal ensemble in phase space. Let us now present some properties of the time evolution of the unstable mode in the Λ representation.

A. Dressing of the field cloud and Markov process

Denoting the expectation value of an observable A over the state $\hat{\rho}_1$ by $\langle A\rangle_{\hat{\rho}_1}$, the evolution of the expectation value of q_a in the $N=1$ sector is given by

$$\begin{aligned}\langle q_a\rangle_{\hat{\rho}_1(t)}&=\int d\Gamma\, q_a e^{-iL_Ht}\Lambda^{-1}\rho_1^{\delta}(\vec{J},\vec{\alpha})\\ &=\sum_b \langle\langle a;0|e^{-iL_Ht}\Lambda^{-1}|b;0\rangle\rangle\int d\Gamma\, q_b\rho_1^{\delta}(\vec{J},\vec{\alpha})\\ &=\langle\langle a;0|e^{-iL_Ht}|\phi_1;0\rangle\rangle q_{10},\end{aligned} \tag{7.4}$$

where $q_{a0}\equiv\sqrt{J_{a0}}\exp[-\alpha_{a0}]$ is the initial value of the normal mode q_a, and Eq. (4.14) has been used to get the second equality. Using Eq. (6.12), we have

$$\langle q_a\rangle_{\hat{\rho}_1(t)}=e^{-iz_1t}\langle a|\phi_1\rangle q_{10}. \tag{7.5}$$

This gives the evolution of the normal mode.

For $t=0$, we have

$$\langle q_1\rangle_{\hat{\rho}_1(0)}=N_1^{1/2}q_{10}, \tag{7.6a}$$

$$\langle q_k\rangle_{\hat{\rho}_1(0)}=N_1^{1/2}\frac{\lambda V_k}{(z_1-\omega_k)^+}q_{10}. \tag{7.6b}$$

The interaction leads to nonvanishing dressing field $\langle q_k\rangle_{\hat{\rho}_1}$, which disappears when the interaction is switched off, $\lambda=0$.

Similar to Eq. (7.4), the evolution of the expectation value of $q_aq_b^{c.c.}$ in the $N=2$ sector is given by

$$\begin{aligned}\langle q_aq_b^{c.c.}\rangle_{\hat{\rho}_1(t)}&=\int d\Gamma\, q_aq_b^{c.c.}e^{-iL_Ht}\Lambda^{-1}\rho_1^{\delta}(\vec{J},\vec{\alpha})\\ &=\sum_{c,d}\langle\langle a;b|e^{-iL_Ht}\Lambda^{-1}|c;d\rangle\rangle\\ &\quad\times\int d\Gamma\, q_cq_d^{c.c.}\rho_1^{\delta}(\vec{J},\vec{\alpha})\\ &=\langle\langle a;b|e^{-iL_Ht}|\rho_1^0\rangle\rangle J_{10},\end{aligned} \tag{7.7}$$

where Eq. (4.14) has again been used to get the second equality. Using Eq. (6.26a), we have

$$\langle q_aq_b^{c.c.}\rangle_{\hat{\rho}_1(t)}=\Big[e^{-2\gamma t}\langle\langle a;b|\rho_1^0\rangle\rangle+(1-e^{-2\gamma t})\times\sum_k b_k\langle\langle a;b|\rho_k^0\rangle\rangle\Big]J_{10}, \tag{7.8}$$

or equivalently [see Eqs. (6.14a), (6.21a), and (6.21b)],

$$\langle q_aq_b^{c.c.}\rangle_{\hat{\rho}_1(t)}=\Big[\langle\langle a;b|e^{-iL_Ht}|\phi_1;\phi_1\rangle\rangle+\sum_k b_k\langle\langle a;b|e^{-iL_Ht}|\tilde{\phi}_k;\tilde{\phi}_k\rangle\rangle\Big]J_{10}. \tag{7.9}$$

Equations (7.5) and (7.8) show that these expectation values obey strictly the Markov process. For example, for $a=b=1$, Eq. (7.8) [or Eq. (7.9) with Eqs. (6.21a) and (6.21b)] leads to

$$\langle J_1\rangle_{\hat{\rho}_1(t)}=e^{-2\gamma t}|N_1|J_{10}+\sum_k\frac{\lambda^2V_k^2b_k}{|\eta^-(\omega_k)|^2}J_{10}. \tag{7.10}$$

Since V_k^2 and b_k are both proportional to $1/\Omega$, the second term in the right-hand side gives a negligible contribution of order $1/\Omega$ in the continuous spectrum limit $\Omega\to\infty$. Equation (7.10) shows that $\langle J_1\rangle_{\hat{\rho}_1}$ decays strictly obeying the exponential law (see also Appendix C on the evolution of $\langle J_k\rangle_{\hat{\rho}_1}$). In contrast to our dressed excited mode, we will see in the following section that there is a classical "Zeno" phenomenon in a short time evolution of the bare excited state, i.e., a significant deviation from the exponential decay (non-Markovian process) which corresponds to the quantum Zeno effect for unstable quantum states [8]. The bare excited mode is not intrinsic to the system, as it has a memory effect of the initial condition, that is, its behavior strongly depends on its initial preparation.

B. Nondistributivity of Λ

As mentioned before, our dressed excited state $\hat{\rho}_1(\vec{J},\vec{\alpha})$ does not any more represent a trajectory, but represents a nonlocal ensemble in phase space. This is a striking difference of the star-unitary Λ transformation in nonintegrable systems from the unitary transformation U in integrable systems. This difference comes from the nondistributivity of our transformation operator Λ. In a nonlocal ensemble, there must be fluctuations. We consider

$$\Delta_a(t)\equiv\langle q_aq_a^{c.c.}\rangle_{\hat{\rho}_1(t)}-\langle q_a\rangle_{\hat{\rho}_1(t)}\langle q_a^{c.c.}\rangle_{\hat{\rho}_1(t)}. \tag{7.11}$$

We now prove that

$$\Delta_1(t)=0, \tag{7.12}$$

while

$$\Delta_k(t) = b_k J_{10}. \tag{7.13}$$

There are no fluctuations for the unstable mode but there are three fluctuations for the field.

We note that the fluctuation Δ_a can also be written as

$$\Delta_a(t) = \int d\Gamma[(\Lambda' q_a^{c.c.} q_a) - (\Lambda' q_a^{c.c.})(\Lambda' q_a)]\rho_1^{\delta}(\vec{J},\vec{\alpha},t), \tag{7.14}$$

where we have used relation (6.28) and the fact that $\rho_1^{\delta}(\vec{J},\vec{\alpha})$ defined in Eq. (7.1) is the δ function. Hence, our result (7.13) shows that the star-unitary transformation is *not distributive*. This result shows that the line shape $b_k J_{10}$ in Eq. (7.13) of the emitted field from the dressed excited particle represents not only the intensity of the fluctuation but also the intensity of the nondistributivity of Λ.

The proofs of Eqs. (7.12) and (7.13) are as follows. Equation (7.9) leads to

$$\langle q_a q_a^{c.c.}\rangle_{\hat{\rho}_1(t)} = \left(e^{-2\gamma t}|\langle a|\phi_1\rangle|^2 + \sum_k b_k |\langle a|\tilde{\phi}_k\rangle|^2 \right) J_{10}. \tag{7.15}$$

On the other hand, Eqs. (7.4) and (7.5) give us

$$\langle q_a\rangle_{\hat{\rho}_1(t)} \langle q_a^{c.c.}\rangle_{\hat{\rho}_1(t)} = \langle\langle a;a|e^{-iL_H t}|\phi_1;\phi_1\rangle\rangle J_{10} = e^{-2\gamma t}|\langle a|\phi_1\rangle|^2 J_{10}. \tag{7.16}$$

Hence, we obtain the fluctuations

$$\Delta_a(t) = \sum_k b_k |\langle a|\tilde{\phi}_k\rangle|^2 J_{10}, \tag{7.17}$$

which are invariants of motion.

For $a=1$, $\Delta_1(t)$ is just the second term in the right-hand side of Eq. (7.10), which is of order $1/\Omega$. For $a=k$, Eq. (7.17) gives Eq. (7.13) plus higher-order contributions with order Ω^{-2} that can be neglected in the continuous spectrum limit $\Omega\to\infty$. For both $a=1$ and k, the dominant contribution of $\Delta_a(t)$ is proportional to $1/\Omega$. We first note that the order of magnitude of the action variable of the particle is 1. As a result, one can neglect the fluctuation $\Delta_1(t)\sim 1/\Omega$ as compared with the action variable of the particle. This gives us Eq. (7.12) in the limit $\Omega\to\infty$.

In contrast, one cannot neglect the fluctuation of each field mode Δ_k, even in the limit $\Omega\to\infty$, since $b_k J_{10}$ is the line shape of the emitted field from the dressed excited state (see Appendix C). The existence of non-negligible fluctuations of the field implies that $\hat{\rho}_1(\vec{J},\vec{\alpha})$ does not represent a trajectory. As mentioned above, the fluctuations are basically due to the nondistributivity of the Λ operator.

It is well known in nonequilibrium statistical physics that there is a close relation between fluctuation and dissipation. In Sec. IX we shall discuss this relation in detail after introducing the $\mathcal{H}$ function in Λ representation.

C. Non-Poissonian algebra

Another important consequence of the nondistributivity is that it leads to a non-Poissonian algebra. To see this we note that Eq. (6.11) lead to

$$\Lambda^{-1} q_1 = Q_1, \quad \Lambda^{-1} q_1^{c.c.} = Q_1^{c.c.}, \tag{7.18}$$

where $Q_1^{c.c.}$ is the Gamow mode defined in Eq. (3.2a). Since Λ preserves any constant because of condition (3) listed at Sec. I, we have for a Poisson bracket,

$$\Lambda^{-1}\{q_1, q_1^{c.c.}\} = -i. \tag{7.19}$$

On the other hand, we have

$$\{\Lambda^{-1}q_1, \Lambda^{-1}q_1^{c.c.}\} = \{Q_1, Q_1^{c.c.}\} = -i|N_1|\left(1 + \lambda^2 \sum_k c_k c_k^{c.c.}\right), \tag{7.20}$$

where we have used the first equailty in Eq. (3.9) for the Poisson bracket. Comparing this with Eq. (6.9), we obtain

$$\{\Lambda^{-1}q_1, \Lambda^{-1}q_1^{c.c.}\} = \{Q_1, Q_1^{c.c.}\} = 0. \tag{7.21}$$

This is in contrast to the unitary transformation for the integrable case, since we would have $\{U^{-1}q_1, U^{-1}q_1^{c.c.}\} = U^{-1}\{q_1, q_1^{c.c.}\} = -i$ because of the distributivity of U. Hence, for the nonintegrable case we have a new non-Poissonian algebra incorporating flcutluation as well as dissipation, which is an extention of the ordinary Poissonian algebra (or Lie algebra).

Another interesting quantity associated with the excited dressed state $\hat{\rho}_1(\vec{J},\vec{\alpha})$ is the expectation value of the energy $\langle H\rangle_{\hat{\rho}_1}$. As one can easily verify, its expression is the same as $(\tilde{\omega}_1 + \delta\tilde{\omega}_1)J_{10}$ shown in Eq. (Q1.6.13) in $Q1$, except that $\hbar$ is replaced by J_{10}. Here, $\delta\tilde{\omega}_1$ is a deviation from the Green's frequency $\tilde{\omega}_1$, the difference of which starts with order λ^4 [see Eq. (Q1.7.30)]. In quantum mechanics, the frequency shift from the unperturbed value ω_1 due the coupling of the particle with the field is known to be the Lamb shift (see, for example, Ref. [42]). As could be expected, the ratio of the frequency shift with respect to ω_1 in classical systems is much smaller than the corresponding ratio in quantum systems because of heavier mass and smaller ω_1 in classical systems. However, since we need a careful calculation to estimate the frequency shift in order to remove the ultraviolet divergence that also appears in classical systems, we shall discuss this problem in a separate paper.

VIII. NON-MARKOV EVOLUTION OF THE EXCITED BARE MODE AND INVERTIBILITY OF Λ

We now come to the time evolution of the original coordinates. We consider the same initial condition $\rho_1^{\delta}(\vec{J},\vec{\alpha})$ as Eq. (7.1), where only the bare particle is excited and all field modes are in vacuum at $t=0$. In contrast to the dressed excited state, we will see that it obeys a non-Markov process where the lifetime is *not well defined*. The non-Markov process consists of three time scales in the decay of the bare particle, showing the classical Zeno effect in a short time scale, an *approximately* exponential decay in an intermediate time scale, and a power-law decay in a long time scale. The traditional approach to radiation damping is mainly focused on this intermediate time scale. As emphasized by Schwinger (see the citation in the Introduction), the main problem of radiation damping is to identify an object that has a well-defined lifetime. That can be only realized by our Λ transformation.

Let us first consider the bare normal mode q_a that is in the $N=1$ sector. We have

$$\langle q_a\rangle_t=\int d\Gamma q_a e^{-iL_H t}\rho(\vec{J},\vec{\alpha},0)$$
$$=\sum_b \langle\langle a;0|e^{-iL_H t}|b;0\rangle\rangle\int d\Gamma q_b\rho_1^{\delta}(\vec{J},\vec{\alpha}). \quad (8.1)$$

We now apply the invertibility of our star-unitary transformation $\Lambda^{-1}\Lambda=1$. For the initial condition (7.1), this leads to

$$\langle q_a\rangle_t=\sum_b \langle\langle a;0|e^{-iL_H t}\Lambda^{-1}|b;0\rangle\rangle\langle\langle b;0|\Lambda|1;0\rangle\rangle q_{10}. \quad (8.2)$$

This shows that the time evolution of the unperturbed normal mode is given as a superposition of the time evolution of the transfromed states $\Lambda^{-1}|b;0\gg$ in Eq. (6.11), i.e.,

$$\langle q_a\rangle_t=\langle\langle a;0|e^{-iL_H t}|\phi_1;0\rangle\rangle\langle\langle\tilde{\phi}_1;0|1;0\rangle\rangle q_{10}$$
$$+\sum_k \langle\langle a;0|e^{-iL_H t}|\phi_k;0\rangle\rangle\langle\langle\tilde{\phi}_k;0|1;0\rangle\rangle q_{10}. \quad (8.3)$$

This with relation (6.12) leads to

$$\langle q_a\rangle_t=g_a(t)q_{10}, \quad (8.4)$$

where

$$g_a(t)\equiv e^{-iz_1 t}\langle a|\phi_1\rangle\langle\tilde{\phi}_1|a\rangle+\sum_k e^{-i\omega_k t}\langle a|\phi_k\rangle\langle\tilde{\phi}_k|a\rangle. \quad (8.5)$$

Using Eqs. (6.7a)–(6.7d), we obtain

$$g_1(t)=N_1 e^{-iz_1 t}+\int_0^{+\infty}dk\frac{2\lambda^2 v_k^2 e^{-i\omega_k t}}{\eta_d^+(\omega_k)\eta^-(\omega_k)}, \quad (8.6a)$$

$$g_k(t)=\sqrt{\frac{2}{\Omega}}\left[N_1\frac{\lambda v_k e^{-iz_1 t}}{(z_1-\omega_k)^+}+\frac{\lambda v_k e^{-i\omega_k t}}{\eta^-(\omega_k)}\right.$$
$$\left.+\int_0^{+\infty}dl\frac{2\lambda v_l^2 e^{-i\omega_l t}}{\eta_d^+(\omega_l)\eta^-(\omega_l)}\frac{\lambda v_k}{\omega_l-\omega_k+i\epsilon}\right], \quad (8.6b)$$

where we have replaced the summation sign by the integration sign and used the definition of V_k in Eq. (2.5).

For the particle mode $a=1$, the first term in Eq. (8.5) [or in Eq. (8.6a)] decays exponentially. The existence of the second term in Eq. (8.6a) indicates that the decay process of the bare particle is not purely exponential. For a short time scale the second term leads to the classical Zeno effect. In Appendix D, we shall discuss this effect in terms of a short time expansion of the action variables.

Next, we consider the time evolution of the unperturbed action variables $J_a(t)$ that are in the $N=2$ sector. As far as we are interested in the evolution of original variables over a given trajectory represented by a δ function, it is enough to evaluate the evolution of each normal mode in the $N=1$ sector. Then, any original variable in an arbitrary N sector is given by a function of each normal mode. For example, the average of a product $q_a q_b^{c.c.}$ of two a modes is given by a product of averages,

$$\langle q_a q_b^{c.c.}\rangle_t=\langle q_a\rangle_t\langle q_b^{c.c.}\rangle_t=g_a(t)g_b^{c.c.}(t)J_{10}. \quad (8.7)$$

However, this is not the case if we evaluate expectation values of the observables over the dressed unstable mode, because of the fluctuation due to the nondistributivity of Λ discussed in the preceding section. For this reason, it is instructive to show that one can recover the factorizable result (8.7) for the bare variables, by starting with the time evolution of the Λ^{-1} transformed states and then inverting by applying Λ, similarly as calculated before in Eq. (8.2).

For the observable $q_a q_b^{c.c.}$ in the $N=2$ sector, we have

$$\langle q_a q_b^{c.c.}\rangle_t=\int d\Gamma q_a q_b^{c.c.}e^{-iL_H t}\rho(\vec{J},\vec{\alpha},0)$$
$$=\sum_{c,d}\langle\langle a;b|e^{-iL_H t}|c;d\rangle\rangle\int d\Gamma q_c q_d^{c.c.}\rho_1^{\delta}(\vec{J},\vec{\alpha}), \quad (8.8)$$

Applying the invertibility just as in Eq. (8.2), we obtain

$$\langle q_a q_b^{c.c.}\rangle_t=\sum_{c,d}\langle\langle a;b|e^{-iL_H t}\Lambda^{-1}|c;d\rangle\rangle\langle\langle c;d|\Lambda|1;1\rangle\rangle J_{10}. \quad (8.9)$$

This shows that the time evolution of the observable $q_a^{c.c.}q_b$ is given as a superposition of the time evolution of the transformed states $|\rho_j^{\nu}\gg$ in Eqs. (6.13a) and (6.13b),

$$\langle q_a q_b^{c.c.}\rangle_t = \langle\langle a;b|e^{-iL_H t}|\rho_1^0\rangle\rangle\langle\langle\tilde{\rho}_1^0|1;1\rangle\rangle J_{10} + \sum_k \langle\langle a;b|e^{-iL_H t}|\rho_k^0\rangle\rangle\langle\langle\tilde{\rho}_k^0|1;1\rangle\rangle J_{10} + {\sum_{k,k'}}' \langle\langle a;b|e^{-iL_H t}|\rho^{kk'}\rangle\rangle\langle\langle\tilde{\rho}^{kk'}|1;1\rangle\rangle J_{10} + \sum_k [\langle\langle a;b|e^{-iL_H t}|\rho^{1k}\rangle\rangle\langle\langle\tilde{\rho}^{1k}|1;1\rangle\rangle + \text{c.c.}]J_{10}, \tag{8.10}$$

where the prime over the summation sign denotes the restriction $k\neq k'$.

Using Eqs. (6.14a)–(6.21b) and (B1)–(B7) with the explicit expressions for the Gamow states in Eqs. (6.7a)–(6.7d), and using the volume dependence $b_k\sim O(1/\Omega)$, one can verify with the straightforward calculation that the right-hand side of Eq. (8.10) leads to

$$\langle q_a q_b^{c.c.}\rangle_t = e^{-i(z_1 - z_1^{c.c.})t}\langle\langle a;b|\phi_1;\phi_1\rangle\rangle\langle\langle\tilde{\phi}_1;\tilde{\phi}_1|1;1\rangle\rangle J_{10} + {\sum_{k,k'}}' e^{-i(\omega_k-\omega_{k'})t}\langle\langle a;b|\phi_k;\phi_{k'}\rangle\rangle \times\langle\langle\tilde{\phi}_k;\tilde{\phi}_{k'}|1;1\rangle\rangle J_{10} + \sum_k [e^{-i(z_1-\omega_k)t}\langle\langle a;b \times|\phi_1;\phi_k\rangle\rangle\langle\langle\tilde{\phi}_1;\tilde{\phi}_k|1;1\rangle\rangle J_{10} + \text{c.c.}]J_{10}, \tag{8.11}$$

where we have kept predominant contribution of the volume dependence by neglecting higher-order contribution in $1/\Omega$. Adding negligible terms with $k=k'$ in the last term in Eq. (8.11), we finally obtain Eq. (8.7). This illustrates the invertibility of Λ.

Let us now analyze the results obtained above. Hereafter, we use new notations $q_a(t)\equiv\langle q_a\rangle_t$ and $J_a(t)\equiv\langle J_a\rangle_t$ to emphasize that we are dealing with variables over a trajectory. We first consider the particle mode $a=1$. The complex conjugate $g_1^{c.c.}(t)$ of Eq. (8.6a) has exactly the same form as the survival amplitude $\langle\psi(0)|\psi(t)\rangle$ of the excited quantum bare state when the initial condition is given by $|\psi(0)\rangle=|1\rangle$ [5]. Hence, the action variable $J_1(t)/J_{10}$ obtained for $a=b=1$ in the Eq. (8.7) corresponds to the survival probability $|\langle\psi(0)|\psi(t)\rangle|^2$ in the quantum system.

The first term in Eq. (8.6a) decays exponentially with the lifetime $1/\gamma$. Hence, the exponential part of $J_1(t)$ decays with the relaxation time $t_r\equiv 1/2\gamma$. This term gives a predominant contribution in a time scale $t\sim t_r$.

The second term in Eq. (8.6a) gives a nonexponential behavior of the decaying process of the excited bare mode. Indeed, due to the delayed analytic continuation in Eq. (3.4c), the pole at $\omega_k=z_1$ does not contribute in the integration over k in the second term. As a result, we have only the branch-point contribution at $k=0$, which leads to a power-law decay as $1/(\omega_1 t)^\alpha$ with $\alpha>0$. The value of α depends on the choice of the form factor v_k. The time scale t_z of the Zeno period is therefore given by $t_z\equiv 1/\omega_1$. Since $\gamma\sim\lambda^2$ for $\lambda\ll 1$ while $\omega_1\sim 1$, we have the well-separated time scales,

$$t_z\ll t_r. \tag{8.12}$$

The second term in Eq. (8.6a) is essential to satisfy Eqs. (D1) and (D2) that lead the classical Zeno effect in the short time scale for the bare excited mode. For extremely long time scales with $t\gg t_r$, the branch-point contribution again gives a predominant contribution.

In summary, we obtain three time scales in time evolution of the classical excited bare mode, just as in quantum mechanics; (1) a short-time Zeno period of the deviation from the exponential decay, (2) an intermediate time scale of the exponential decay, and (3) a long time scale of a power-law decay [43]. Strictly speaking there is no well-defined lifetime for the bare excited mode because of the existence of nonexponential behavior in the short and long time scales. This is in contrast to the dressed excited mode as it obeys strictly the exponential law shown in Eq. (7.10). Only through our Λ we may rigorously define the lifetime of the excited mode.

Next we consider the emission of the field from the excited unstable bare mode. The evolution of the action variable J_k for the field is given by Eq. (8.10) for $a=b=k$ as $J_k(t)=|g_k(t)|^2 J_{10}$. The first term of Eq. (8.6b) vanishes in the limit $t\to+\infty$. The contour of the integration in the last term is located below all singularities that come from the denominators. Hence, assuming that the form factor is chosen in such a way that the contribution coming from its singularities vanish in the limit $t\to+\infty$, the integration in the last term of Eq. (8.6b) also vanishes in this limit. Consequently, the dominant contribution comes from the second term in Eq. (8.6b), and we obtain

$$\lim_{t\to+\infty} J_k(t)=\frac{2}{\Omega}\frac{\lambda^2 v_k^2}{|\eta^-(\omega_k)|^2}J_{10}\approx\frac{2}{\pi\Omega}\frac{\gamma}{(\omega_k-\tilde{\omega}_1)^2+\gamma^2}J_{10}. \tag{8.13}$$

The approximation in the last line is valid for a weakly coupling case $\lambda\ll 1$ and in the vicinity of the resonance frequency $\omega_k=\tilde{\omega}_1$. The right-hand side is the well-known Lorentzian line shape.

IX. NUMERICAL PLOTS

In order to visualize the result obtained in the previous sections, we have peformed numerical calculations. The main results of this section are (1) a visualization of the different time scales discussed in the preceding section, (2) a numerical verification of the invertibility of our transformation between the initial coordinates and the transformed coordinates, and (3) a numerical calculation of the nondistributibity of the Λ transformation. As has been shown in the preceding section, Λ-transformed variables have nonvanishing dispersion. Hence, it is worthwhile to demonstrate by the numerical simulation that information of the system, such as a memory effect of the initial condition of the bare excited mode, is not lost by the star-unitary transformation Λ.

In order to visualize the different time scales mentioned

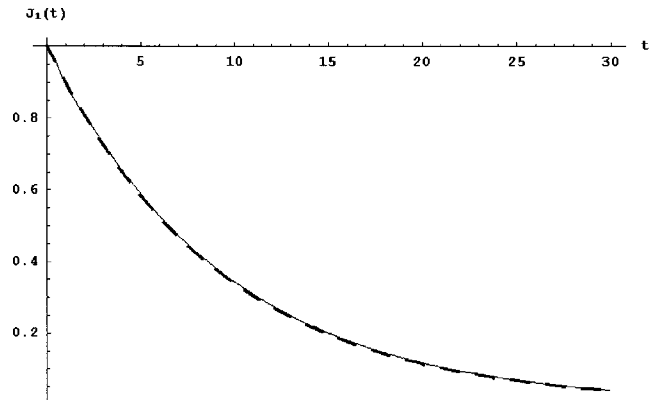

FIG. 1. Numerical results of the time evolution of the original variable $J_1(t)$ (the solid line) and the transformed variable $\langle J_1\rangle_{\hat{\rho}_1(t)}$ (the broken line). The abscissa is t, which is measured by a unit $1/\omega_1$. On this scale we do not see any difference between $J_1(t)$ and $\langle J_1\rangle_{\hat{\rho}_1(t)}$.

above, we have plotted $J_1(t)$ obtained by numerical integrations of the equation of motions and compared them with our theoretical results. To perform the numerical integrations we have used the discretized form of Hamiltonian (2.5) with a given size L of the box and with a cuttoff wave number k_{max}. The equations of motion have been solved for $q_a(t)$ [and also for $q_a(-t)$] by the numerical diagonalization of the Hamiltonian into the form (3.1) (see Ref. [29] for a description of the numerical method). This numerical method is more reliable than other numerical methods, such as the Runge-Kutta method. To compare the numerical results with the theoretical results for the continuous spectrum, we have restricted the time scale as $t \ll t_{box}$, where $t_{box} \equiv L/c$, so that the light cannot cross the box in this time scale. Since the agreement between the theoretical results and the numerical results is excellent, we show only the plots by the numerical results in all figures presented in this paper [44]. In Figs. 1–3, we plot $J_1(t)$ as a function of time t. In all plots in this

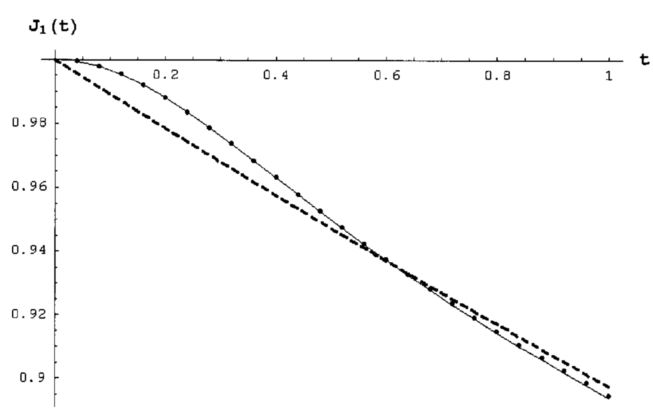

FIG. 2. A magnification of a short-time portion of Fig. 1. The solid line is $J_1(t)$ and the broken line is $\langle J_1\rangle_{\hat{\rho}_1(t)}$. One can see a deviation from exponential decay in $J_1(t)$ (the classical Zeno effect). The dots on the solid line are the results of the inverse transformation from the transformed variables, by applying Λ to $\langle J_1\rangle_{\hat{\rho}_1(t)}$ we go back to $J_1(t)$.

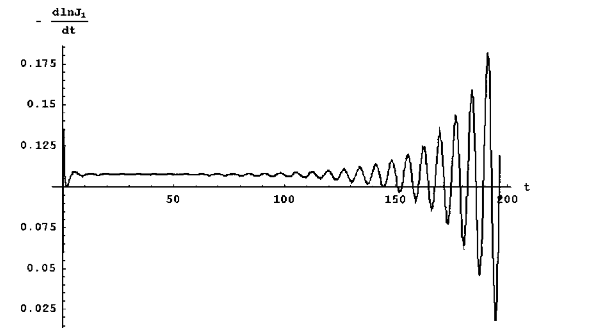

FIG. 3. A numerical result of $-dJ_1(t)/dt$ as a function of time t, which is measured by a unit $1/\omega_1$. One can see both a short-time deviation and a long-time deviation from the exponential decay.

paper, we put $\omega_1 = 1$, $\lambda = 0.1$, and we use the form factor $v_k = \sqrt{\omega_k/\pi}\,\theta(k_{max} - |k|)$ with the cutoff wave number $k_{max} = 2\pi$. We also use $\Delta k = 0.005$, which gives the size of the box as $L = 1256.6$. For this case we obtain the decay rate $2\gamma = 0.108$ (the relaxation time $t_r = 9.5$) and the shifted frequency $\tilde{\omega}_1 = 0.878$. In all figures presented in this paper, time is measured by the unit $1/\omega_1 = 1$.

In Fig. 1 we show $J_1(t) = |q_1(t)|^2$ by the solid line (see also Ref. [45]). This has been obtained by a numerical integration of motion for the original normal mode $q_1(t)$. In order to compare the evolution of the original variable $J_1(t)$ to the transformed variable, we also plot $\langle J_1\rangle_{\hat{\rho}_1(t)}$ by the broken line. On this scale we do not see any difference.

The numerical value of $\langle J_1\rangle_{\hat{\rho}_1(t)}$ has been calculated by using the data of $q_a(-t)$ obtained from the numerical integration of the equations of motion, as follows. In Eq. (7.9), we first note that

$$|\phi_1;\phi_1\rangle\rangle = N_1\Bigg[|1;1\rangle\rangle + \lambda^2 \sum_{k,l} c_k c_l^{c.c.} |k;l\rangle\rangle + \lambda \sum_k \left(c_k|k;1\rangle\rangle + c_k^{c.c.}|1;k\rangle\rangle\right)\Bigg]. \tag{9.1}$$

This gives us for the first term in Eq. (7.9),

$$\langle\langle 1;1|e^{-iL_H t}|\phi_1;\phi_1\rangle\rangle J_{10} = N_1\Bigg[|q_1(-t)|^2 + \left|\lambda \sum_k c_k \times q_k^{c.c.}(-t)\right|^2 + \lambda \sum_k [c_k q_k^{c.c.}(-t) \times q_1(-t) + \text{c.c.}]\Bigg], \tag{9.2}$$

where we have used the relation

$$\begin{aligned}\langle\langle 1;1|e^{-iL_H t}|a;b\rangle\rangle &= \langle\langle 1;0|e^{-iL_H t}|a;0\rangle\rangle\langle\langle 0;1|e^{-iL_H t}|0;b\rangle\rangle\\ &= [\langle\langle a;0|e^{+iL_H t}|1;0\rangle\rangle]^{c.c.}[\langle\langle 0;b| \times e^{+iL_H t}|0;1\rangle\rangle]^{c.c.}\\ &= \frac{q_a(-t)^{c.c}}{q_{10}^{c.c.}}\frac{q_b(-t)}{q_{10}}.\end{aligned} \tag{9.3}$$

For the second term in Eq. (7.9), one can repeat a similar calculation to Eq. (9.2) and obtain an expression of $\langle\langle 1;1|\exp[-iL_H t]|\tilde{\phi}_k;\tilde{\phi}_k\rangle\rangle J_{10}$, which is a function of $q_a(-t)$. Hence, from the numerical data of $q_a(-t)$, one can evaluate the numerical value of $\langle J_1\rangle_{\hat{\rho}_1(t)}$.

Figure 2 is a magnification of a portion of Fig. 1 for a short Zeno time scale of order t_z. Again, the solid line is $J_1(t)$, while the broken line is $\langle J_1\rangle_{\hat{\rho}_1(t)}$. The difference is clear. $J_1(t)$ deviates from exponential decay. One can see that our result satisfies relations (D1) and (D2) for $J_1(t)$. The Zeno time scale is not intrinsic to the system, but depends on the initial condition. Since the traditional approaches of the radiation damping have ignored the existence of this short time scale, the basic formulas, such as Larmors radiation formula [11], to derive radiation damping do not hold in the Zeno period.

The most interesting result in Fig. 2 is the bold dots on the solid line for $J_1(t)$. The dots are the result of the inverse transformation from the transformed variables. By applying Λ to $\langle J_1\rangle_{\hat{\rho}_1(t)}$, we go back to $J_1(t)$. No information is lost.

As mentioned above, the solid line in Fig. 2 has been obtained by a numerical integration of the equations of motion for the initial normal modes $q_a(t)$, while the broken line for the transformed variable has been plotted using the numerical data of $q_a(-t)$. In Fig. 2, one can see that there is indeed no Zeno period in the transformed variable $\langle J_1\rangle_{\hat{\rho}_1(t)}$. The dots have been obtained by performing the inverse transformation over the numerical data for the transformed variables at several points of time. The result shows that there is a good agreement between the inverse transformed variable and the initial variable $J_1(t)$. This numerically verifies the invertibility of our nonunitary transformation Λ.

In the intermediate time scale with $t\sim t_r$, the $|\rho_1^0\rangle\rangle$ component in Eq. (8.10) predominates and the excited bare mode decays approximately obeying the exponential law (see Fig. 1). The exponential decay is a Markov process that is intrinsic to the unstable particle independent of its initial condition.

For the long time scale with $t\gg t_r$, the field-field correlation component $|\rho^{kk'}\rangle\rangle$ in Eq. (8.10) now predominates, i.e., the branch-point effect coming from the second term in Eq. (8.6a) again gives a predominant contribution. This is also the time scale that has been neglected by traditional approaches to radiation damping. In Fig. 3 we have plotted $-d\ln J_1/dt$ for this long time scale as a function of time t (see also Ref. [45]). We can again observe the deviation from the exponential behavior (the long time tail). For the long time scale the exponential part becomes so small that the power decaying component predominates. However, the absolute value of the action variable is already very small in this long time scale.

In Fig. 4 we show plots of the fluctuation $\Delta_k(t)$ in Eq. (7.11) for $a=k$. In this figure we show a numerical result of $\langle J_k\rangle_{\hat{\rho}_1}$ as well as $\langle q_k\rangle_{\hat{\rho}_1}\langle q_k^{c.c.}\rangle_{\hat{\rho}_1}$ at $k=0.9$ as a function of the time *t*. The thin solid line above in the figure is $\langle J_k\rangle_{\hat{\rho}_1}$ and the thick solid line below is $\langle q_k\rangle_{\hat{\rho}_1}\langle q_k^{c.c.}\rangle_{\hat{\rho}_1}$. In order to find numerically the values of $\langle J_k\rangle_{\hat{\rho}_1}$ and $\langle q_k\rangle_{\hat{\rho}_1}\langle q_k^{c.c.}\rangle_{\hat{\rho}_1}$, we have individually calculated the first and the second terms in Eq. (7.9) for $a=b=k$, as well as $\langle\langle k;k|\exp[-iL_H t]|\phi_1;\phi_1\rangle\rangle J_{10}$ in Eq. (7.16) by a similar numerical method presented in Eqs. (9.1)–(9.3). The broken line is the fluctuation $\Delta_k(t)$ that corresponds to the second term in Eq. (7.9) for $a=b=k$. As has been predicted by our theoretical calculation, $\Delta_k(t)$ remains a constant in time and its value is consistent with our theoretical prediction of $\Delta_k(t)=0.6147$ for our specific values of k and L. In contrast to $\Delta_k(t)$, $\langle J_k\rangle_{\hat{\rho}_1}$ (the thin solid line) and $\langle q_k\rangle_{\hat{\rho}_1}\langle q_k^{c.c.}\rangle_{\hat{\rho}_1}$ (the thick solid line) change in time.

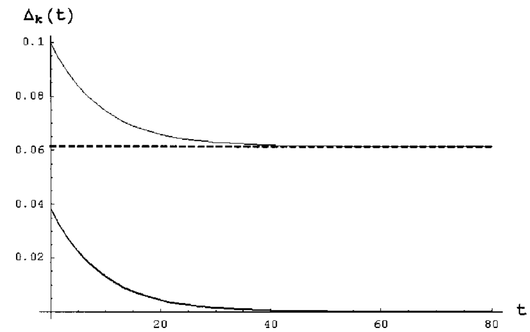

FIG. 4. Numerical results of the fluctuation $\Delta_k(t)$ (the broken line), $\langle J_k\rangle_{\hat{\rho}_1}$ (the thin solid line above) and $\langle q_k\rangle_{\hat{\rho}_1}\langle q_k^{c.c.}\rangle_{\hat{\rho}_1}$ (the thick solid line below) at $k=0.9$ as a function of time *t*, which is measured by a unit $1/\omega_1$. As has been predicted by our theoretical calculation, $\Delta_k(t)$ remains a constant in time and its value is consistent with our theoretical prediction of $\Delta_k(t)=0.6147$.

X. $\mathcal{H}$ FUNCTION AND FLUCTUATION VS DISSIPATION

An important consequence of the star-unitary transformation is that it allows us to introduce a microscopic analog of Boltzmann's $\mathcal{H}$ theorem in statistical mechanics by constructing a Lyapunov operator that decays monotonically for all times [32–34,46]. We can describe radiation damping in terms of $\mathcal{H}$ function that is defined as an expectation value of a Lyapunov operator in the Liouville space. The $\mathcal{H}$ function corresponds to negative "entropy" in statistical mechanics. In the preceding section we have shown that there are flucturations $\Delta_k\propto b_k$ of the field in the dressed state $\hat{\rho}_1(t)$. As is well known in nonequilibrium statistical physics, there is a close relation between fluctuation and dissipation, which is often expressed in the form of the fluctuation-dissipation theorem [47]. In this section we shall show this relation from the point of "entropy production" defined through our $\mathcal{H}$ function.

Let us introduce a Lyapunov operator associated with a phase function A_n that depends on a *finite* number n of the modes of the system [32],

$$M_{A_n}\equiv\Lambda^\dagger|A_n\rangle\rangle\langle\langle A_n|\Lambda. \tag{10.1}$$

Then we have a $\mathcal{H}$ function defined by

$$\mathcal{H}_{A_n}[\rho(t)]\equiv\langle\langle\rho(t)|M_{A_n}|\rho(t)\rangle\rangle, \tag{10.2}$$

which is a nonlinear functional of $\rho(t)$.

For example, the $\mathcal{H}$ function associated to the particle mode q_1 is given by

$$\mathcal{H}_{q_1}[\rho(t)] \equiv |\langle\langle q_1|\Lambda|\rho(t)\rangle\rangle|^2 = \left|\int d\Gamma q_1 \Lambda e^{-iL_H t}\rho(\vec{J},\vec{\alpha},0)\right|^2$$

$$= \left|\sum_a \langle\langle 1;0|\Lambda e^{-iL_H t}|a;0\rangle\rangle\langle q_a\rangle_0\right|^2, \qquad (10.3)$$

where $\langle q_a\rangle_0$ is the expectation value of q_a over an arbitrary initial ensemble $\rho(\vec{J},\vec{\alpha},0)$. Applying Eqs. (6.11) and (6.12) to this, we obtain

$$\mathcal{H}_{q_1}[\rho(t)] = e^{-2\gamma t}\langle\psi(0)|\tilde{\phi}_1\rangle\langle\tilde{\phi}_1|\psi(0)\rangle, \qquad (10.4)$$

where $|\psi(0)\rangle \equiv \Sigma_a|a\rangle\langle q_a\rangle_0$. This has exactly the same structure of the $\mathcal{H}$ function as the one introduced in the quantum Friedrichs model in our previous work $Q2$ and in Refs. [5,48,49]. For example, if we consider the trajectory given by Eq. (7.1), we obtain $|\psi(0)\rangle = |1\rangle q_{10}^{c.c.}$, which corresponds to the case discussed in detail in our previous paper [5]. Instead, if we consider the case $q_{10}=0$, but with nonvanishing field modes $q_{k0}\neq 0$ at $t=0$, our expression (10.4) reduces to the form in the scattering case of the field, which has been investigated in detail in $Q2$ and in [48].

Let us now discuss the relation between the fluctuations b_k of the field in the dressed state $\hat{\rho}_1(t)$ and the dissipation associated with the entropy production. To this end, we consider the $\mathcal{H}$ function associated to the action variables which are evaluated over the transformed state $\hat{\rho}_1(t)$. We have

$$\sum_a \mathcal{H}_{J_a}[\hat{\rho}_1(t)] = \sum_a \left|\int d\Gamma J_a \Lambda e^{-iL_H t}\Lambda^{-1}\rho_1^{\delta}(\vec{J},\vec{\alpha})\right|^2$$

$$= \sum_a |f_{aa}(t)|^2 J_{10}^2, \qquad (10.5)$$

where

$$f_{aa}(t) \equiv \langle\langle a;a|e^{-i\tilde{\theta}^{(0)}t}|1;1\rangle\rangle. \qquad (10.6)$$

Taking the time derivative and using Eqs. (6.25a)–(6.25c) with (6.18), we have a set of equations,

$$\frac{\partial}{\partial t} f_{11}(t) = 2\gamma \sum_k b_k[f_{kk}(t) - f_{11}(t)], \qquad (10.7a)$$

$$\frac{\partial}{\partial t} f_{kk}(t) = 2\gamma b_k \sum_p b_p[f_{11}(t) - f_{pp}(t)]. \qquad (10.7b)$$

From these equations we obtain

$$f_{11}(t) + \sum_k f_{kk}(t) = 1, \qquad (10.8)$$

which is an invariant of motion. One may notice that Eqs. (10.7a)–(10.7b) have exactly the same structure as the Pauli master equation that is well known in quantum nonequilibrium statistical mechanics, in spite of the fact that we deal with classical mechanics. From these equations we obtain

$$-\frac{d}{dt}\sum_a \mathcal{H}_{J_a}[\hat{\rho}_1(t)] = 4\gamma J_{10}^2\left|f_{11}(t) - \sum_k b_k f_{kk}(t)\right|^2 \geqslant 0. \qquad (10.9)$$

Hence, the "entropy production" per unit time defined as the left-hand side is greater or equal to zero. There is a dissipation. The dissipation is a result of nonvanishing collision kernels b_k of master equation (10.7a) and (10.7b). On the other hand, the collision kernels are just the fluctuations of the field in Eq. (7.13). Hence the fluctuations lead to the dissipation and vice versa.

It should be emphasized that our approach to entropy production is purely based on microscopic dynamics, which is valid for arbitrary strength of the coupling constant and arbitrarily far from equilibrium without any approximations. In this sense we now establish a deeper relation between the fluctuations and the dissipation both coming from the resonace singularities in nonintegrable systems.

XI. CONCLUDING REMARKS

The fundamental result in this paper is that we found fluctuations in the Λ transformed variables in the classical radiation damping problem. The reason for the fluctuations is that information concentrated in the excited mode 1 goes to many modes k of the field through the resonance interaction. The resonance thus leads to the fluctuations as well as the dissipation. The line shape b_k of the emitted classical field gives the probability to find the mode k, which is somewhat similar to Born's probablistic interpretation in quantum mechanics. Hence, our theory is a theory of fluctuations and dissipation in classical electrodynamcis. We should emphasize that the fluctuations are explicit in the Λ-transformed variables, not in the initial variables. There are no trajectories in the Λ representation, as the states are nonlocal in phase space. Trajectories exist in the original variables. But we need Λ transformation to identify the exponentially decaying modes. Through our star-unitary transformation, we have a physics of resonances, not of forces, and not by a description in terms of space-time points. The deep change by going to the Λ representation comes from the fact that resonances play an essential role. This is quite different from the idea of point events used in classical field theory.

In this paper we have focused our attention on the case where the fields surrounding the particle are not in the thermodynamic condition, as they has a finite energy. Indeed this is a typical situation of radiation damping problem. If we take the thermodynamic limit of the fields, the linear transformations that leads to diagonalization of the Hamiltonian leads to divergence [50]. Nevertheless, one can still construct an exact form of Λ transformation for the classical Friedrichs model. In the thermodynamic limit, the star-unitary transformation leads to Gaussian white noise in the Λ representation without relying upon any phenomenological argument and approximations. This result has been presented in a separate paper [51].

ACKNOWLEDGMENTS

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APPENDIX A: THE BARGMANN-SEGAL REPRESENTATION

The Bargmann-Segal basis (4.6) of the Hilbert space spanned by entire functions of q_a and $q_a^{c.c.}$ is related to the ''coherent states'' in analogy to quantum mechanics by

$$\langle\vec{m}|\vec{q}\rangle=\prod_a \frac{(q_a)^{m_a}}{\sqrt{m_a!}}e^{-|q_a|^2/2}. \tag{A1}$$

In the Hilbert space spanned by entire functions of $\vec{q}$, the states $|\vec{m}\rangle$ statisfy complete and orthonomal relations

$$\sum_{\vec{m}=\vec{0}}|\vec{m}\rangle\langle\vec{m}|=1,\quad \langle\vec{m}|\vec{n}\rangle=\prod_a \delta_{m_a,n_a}, \tag{A2}$$

where the summation goes over all non-negative integers m_a for all a.

As usual, the ''coherent states'' defined by

$$|\vec{q}\rangle=\sum_{\vec{m}}|\vec{m}\rangle\langle\vec{m}|\vec{q}\rangle, \tag{A3}$$

are overcomplete in the Hilbert space, i.e.,

$$\int\prod_a d\mu(q_a)|\vec{q}\rangle\langle\vec{q}|=1, \tag{A4}$$

and

$$\langle\vec{q}|\vec{q}\,'\rangle=\prod_a e^{[-(1/2)|q_a|^2+q_a^{c.c.}q_a'-(1/2)|q_a'|^2]}, \tag{A5}$$

where $d\mu(q_a)\equiv\pi^{-1}d^2q_a\equiv\pi^{-1}d(\mathrm{Re}\,q_a)d(\mathrm{Im}\,q_a)$.

Any linear operator A acting on a vector $|\alpha\rangle$ in this Hilbert space may be written as a superposition of dyadic operators $\{|\vec{m}\rangle\langle\vec{n}|\}$ as

$$A=\sum_{\vec{m}}\sum_{\vec{n}}A_{\vec{m};\vec{n}}|\vec{m}\rangle\langle\vec{n}|. \tag{A6}$$

Similar to quantum mechanics, one can introduce the Liouville space spanned by linear operators in this Hilbert space [see Eqs. (4.8a) and (4.8b)]. To represent vectors in the Liouville space, we use the double bra-ket notation as $|A\gg$ and $\ll B|$ where A and B, are linear operators in the Hilbert space spanned by $|\vec{m}\rangle$. The Liouville space is also a Hilbert space where the inner product between the supervectors are defined by

$$\ll B|A\gg=\mathrm{Tr}(B^\dagger A)\equiv\sum_{\vec{m}}\langle\vec{m}|B^\dagger A|\vec{m}\rangle, \tag{A7}$$

where $B^\dagger$ is a Hermitian conjugate of the linear operator B. Then the Hilbert norm of the vector $|A\gg$ in the Liouville space is defined by $||A||\equiv\sqrt{\ll A|A\gg}$, if it exists.

In analogy with quantum mechanics, we introduce the super ''bra-ket'' notations for dyads $|\vec{m};\vec{n}\gg=|\vec{m}\rangle\langle\vec{n}|$. Then we have $A_{\vec{m};\vec{n}}=\ll\vec{m};\vec{n}|A\gg$. Hence, A can be written as an element of the Liouville space as

$$A=\sum_{\vec{m}}\sum_{\vec{n}}|\vec{m};\vec{n}\rangle\rangle\langle\langle\vec{m};\vec{n}|A\rangle\rangle. \tag{A8}$$

Relations (A4) and (A5) lead to

$$\int\int\prod_{a,b}d\mu(q_a)d\mu(q_b')|\vec{q};\vec{q}\,'\rangle\rangle\langle\langle\vec{q};\vec{q}\,'|=I \tag{A9}$$

and

$$\langle\langle\vec{q};\vec{q}\,'|\vec{q}\,'';\vec{q}\,'''\rangle\rangle=\langle\vec{q}|\vec{q}\,''\rangle\langle\vec{q}\,'''|\vec{q}\,'\rangle. \tag{A10}$$

In terms of the BS representation, the Liouvillian L_H is written by

$$\langle\langle\vec{q};\vec{q}\,'|L_0|\vec{q}\,'';\vec{q}\,'''\rangle\rangle=\sum_a\omega_a\left[q_a^{c.c.}\left(\frac{\partial}{\partial q_a^{c.c.}}+\frac{q_a}{2}\right)-q_a'\left(\frac{\partial}{\partial q_a'}+\frac{q_a'^{c.c.}}{2}\right)\right]\langle\langle\vec{q};\vec{q}\,'|\vec{q}\,'';\vec{q}\,'''\rangle\rangle, \tag{A11}$$

and

$$\begin{aligned}\langle\langle\vec{q};\vec{q}\,'|L_V|\vec{q}\,'';\vec{q}\,'''\rangle\rangle &=\sum_k V_k\left[q_1^{c.c.}\left(\frac{\partial}{\partial q_k^{c.c.}}+\frac{q_k}{2}\right)-q_1'\left(\frac{\partial}{\partial q_k'}+\frac{q_k'^{c.c.}}{2}\right)\right.\\ &\quad\left.+q_k^{c.c.}\left(\frac{\partial}{\partial q_1^{c.c.}}+\frac{q_1}{2}\right)-q_k'\left(\frac{\partial}{\partial q_1'}+\frac{q_1'^{c.c.}}{2}\right)\right]\\ &\quad\times\langle\langle\vec{q};\vec{q}\,'|\vec{q}\,'';\vec{q}\,'''\rangle\rangle.\end{aligned} \tag{A12}$$

Putting $\vec{q}\,'=\vec{q}$ in the above expressions, we recover the expressions for the Liouvillian in Eqs. (4.2) and (4.3). In the (m,n) representation, Eqs. (A11) and (A12) lead to

$$L_0|\vec{m};\vec{n}\rangle\rangle=\vec{\omega}\cdot(\vec{m}-\vec{n})|\vec{m};\vec{n}\rangle\rangle \tag{A13}$$

and

$$L_V|\vec{m};\vec{n}\rangle\rangle=\sum_k V_k[\sqrt{m_1(m_k+1)}|m_1-1,m_k+1,\{\vec{m}\}';\vec{n}\rangle\rangle$$
$$+\sqrt{(m_1+1)m_k}|m_1+1,m_k-1,\{\vec{m}\}';\vec{n}\rangle\rangle$$
$$-\sqrt{(n_1+1)n_k}|\vec{m};n_1+1,n_k-1,\{\vec{n}\}'\rangle\rangle$$
$$-\sqrt{n_1(n_k+1)}|\vec{m};n_1-1,n_k+1,\{\vec{n}\}'\rangle\rangle], \quad \text{(A14)}$$

where the prime on $\{\vec{m}\}'$ denotes that the particle 1 and the field mode k are excluded from the set of the components in $\vec{m}$.

In Eq. (A14) the first term in the right-hand side vanishes for $m_1=0$. The same is true for $m_k=0$ in the second term, for $n_k=0$ in the third term, and for $n_1=0$ in the last term. Hence, one cannot have any $(\vec{m},\vec{n})$ states that have a negative value of m_a or n_b for a certain value of a or b in the right-hand side of Eq. (A14).

As a special case, we have

$$L_H|\vec{0};\vec{0}\rangle\rangle=L_0|\vec{0};\vec{0}\rangle\rangle=0, \quad \text{(A15)}$$

where

$$\langle\langle\vec{q};\vec{q}|\vec{0};\vec{0}\rangle\rangle=e^{-\sum_a |q_a|^2}. \quad \text{(A16)}$$

These are consistent as $\Sigma_a J_a$ is an invariant of motion for both unperturbed and perturbed cases shown in Eq. (4.7).

In Eqs. (A13) and (A14), we see an isomorphism between the classical Friedrichs model and the quantum Friedrichs model on the level of the Liouvillian formalism. Indeed, if $|\vec{m}\rangle$ and $\langle\vec{n}|$ in Eq. (4.6) are regarded as the number states of the unperturbed bosons for the corresponding quantum system, then these formulas are exactly the same as the ones obtained for the quantum Liouvillian for the Friedrichs model.

APPENDIX B: EXPLICIT FORMS OF ρ^{ab} AND $\tilde{\rho}^{ab}$

The explicit forms of ρ^{ab} and $\tilde{\rho}^{ab}$ are given [see Eqs. (Q1.7.2) and (Q1.B20)–(Q1.B29)] by

$$|\rho^{ab}\rangle\rangle\equiv|F^{ab}\rangle\rangle, \quad \langle\langle\tilde{\rho}^{ab}|\equiv\langle\langle\tilde{F}^{ab}|. \quad \text{(B1)}$$

Here,

$$|F^{kk'}\rangle\rangle=|\phi_k;\phi_{k'}\rangle\rangle,$$

$$|F^{1k}\rangle\rangle=|\phi_1;\phi_k\rangle\rangle-\sum_l |l;l\rangle\rangle f(k,l), \quad \text{(B2)}$$

and

$$\langle\langle\tilde{F}^{kk'}|=\langle\langle\tilde{\phi}_k;\tilde{\phi}_{k'}|,$$

$$\langle\langle\tilde{F}^{1k}|=\rangle\rangle\tilde{\phi}_1;\tilde{\phi}_k|-\sum_l \langle\langle l;l|\tilde{f}(k,l), \quad \text{(B3)}$$

with

$$f(k,l)\equiv\langle\langle l;l|\phi_1;\phi_k\rangle\rangle-N_1^{1/2}\frac{\lambda V_k}{z_1-\omega_k}\left[\delta_{l,k}-\frac{\lambda^2V_l^2}{\eta_d^-(\omega_k)}\right.$$
$$\left.\times\left(\frac{1}{\omega_l-\omega_k+i\epsilon}+\frac{1}{(z_1-\omega_l)^+}\right)\right], \quad \text{(B4)}$$

$$\tilde{f}(k,l)\equiv\langle\langle l;l|\tilde{\phi}_1;\tilde{\phi}_k\rangle\rangle-N_1^{1/2}\frac{\lambda V_k}{z_1-\omega_k}\left[\delta_{l,k}-\frac{\lambda^2V_l^2}{\eta^+(\omega_k)}\right.$$
$$\left.\times\left(\frac{1}{\omega_l-\omega_k-i\epsilon}-\frac{1}{(z_1-\omega_l)^+}\right)\right] \quad \text{(B5)}$$

and

$$\langle\langle a;b|F^{k1}\rangle\rangle=[\langle\langle a;b|F^{1k}\rangle\rangle]^{c.c.},$$

$$\langle\langle\tilde{F}^{k1}|a;b\rangle\rangle=[\langle\langle\tilde{F}^{1k}|a;b\rangle\rangle]^{c.c.}. \quad \text{(B6)}$$

These are the right and left eigenstates of the Liouvillian,

$$L_H|F^{ab}\rangle\rangle=(z_a-z_b^{c.c.})|F^{ab}\rangle\rangle,$$

$$\langle\langle\tilde{F}^{ab}|L_H=(z_a-z_b^{c.c.})\langle\langle\tilde{F}^{ab}|. \quad \text{(B7)}$$

APPENDIX C: LINE SHAPE OF THE EMITTED FIELD FROM $\hat{\rho}_1$

In this appendix we show that the function b_kJ_{10} is the line shape of the field that is emitted from the dressed excited mode.

For $a=b=k$ in Eq. (7.8), we obtain

$$\lim_{t\to\infty}\langle J_k\rangle_{\hat{\rho}_1(t)}=\sum_l b_l\langle\langle k;k|\rho_l^0\rangle\rangle J_{10}=b_kJ_{10}+O(\Omega^{-2}). \quad \text{(C1)}$$

This shows that the function b_kJ_{10} is just the line shape of the field that is emitted from the dressed excited mode. In $Q1$ we have shown that for a weakly coupling case with $\lambda \ll 1$, we can approximate b_k as [see Eq. (Q1.7.15)]

$$b_k\approx\frac{1}{\Omega}\frac{1}{\pi}\frac{(\lambda^2\gamma_2)^3}{[(\omega_k-\tilde{\omega}_1)^2+\lambda^4\gamma_2^2]^2}, \quad \text{(C2)}$$

where $\lambda^2\gamma_2$ is the lowest-order approximation of γ in the expansion of λ given by

$$\lambda^2\gamma_2\equiv\Omega\int_0^{+\infty}dk\lambda^2V_k^2\pi\delta(\omega_k-\omega_1)=2\pi\lambda^2v_{\omega_1}^2. \quad \text{(C3)}$$

Therefore, this line shape is narrower than the Lorentzian line shape (8.13) emitted by the bare excited mode. As has been discussed in $Q1$, the difference between the line shape $b_k J_{10}$ and the Lorentzian line shape is consistent with an elimination of the short-time event in the Zeno period in the dressed excited particle. Indeed, during the Zeno period the bare excited state reorganizes space by producing a consistent dressing field attracted around the particle, and after that the particle starts to decay obeying the exponential law. In contrast, since our dressed excited mode already includes a consistent dressing field, the dressed excited particle decays strictly exponentially from the beginning. The dispersion of the action variable is of order γ in the line shape $b_k J_{10}$, and there is no large fluctuation of the action variable as in the case of the Lorentzian line shape.

APPENDIX D: CLASSICAL ZENO PERIOD

We first prove the theorems

$$\lim_{t\to 0}\frac{dJ_a(t)}{dt}=0, \tag{D1}$$

while

$$\lim_{t\to 0}\frac{d^2J_a(t)}{dt^2}\neq 0. \tag{D2}$$

These theorems are important when we consider the evolution of the system in a short time scale. The proofs are as follows: we have

$$\begin{aligned} i^n\frac{d^nJ_a(t)}{dt^n} &= \int d\Gamma J_a L_H^n e^{-iL_Ht}\rho_1^{\delta}(\vec{J},\vec{\alpha}) \\ &= \sum_{c,d} \langle\langle a;a|L_H^n e^{-iL_Ht}|c;d\rangle\rangle \\ &\quad \times \int d\Gamma q_c q_d^{c.c.}\rho_1^{\delta}(\vec{J},\vec{\alpha}). \end{aligned} \tag{D3}$$

Substituting Eq. (7.1) into this expression, we obtain

$$\lim_{t\to 0}\frac{d^nJ_a(t)}{dt^n}=(-i)^n\langle\langle a;a|L_H^n|1;1\rangle\rangle J_{10}. \tag{D4}$$

We have

$$\langle\langle a;a|L_H|1;1\rangle\rangle=0, \tag{D5}$$

while

$$\langle\langle a;a|L_H^2|1;1\rangle\rangle=\lambda^2\langle\langle a;a|L_V^2|1;1\rangle\rangle\neq 0, \tag{D6}$$

which can be easily verified from the expressions in Eqs. (A13) and (A14). Thus we obtain theorems (D1) and (D2).

These theorems show that in a short time scale, there is a classical ''Zeno'' period that corresponds to the quantum Zeno period for unstable quantum states [8]. From theorems (D1) and (D2), we see that a short-time expansion of the solution of $J_a(t)-J_a(0)$ in the series of t should start with t^2 term. In other words, the short-time evolution is time symmetric for an operation to change the sign of t, which is in contrast to the case in the exponential decay.

Using Eq. (A14), we have

$$\langle\langle 1;1|L_V^2|1;1\rangle\rangle=2\sum_k |V_k|^2, \tag{D7}$$

$$\langle\langle k;k|L_V^2|1;1\rangle\rangle=-2|V_k|^2. \tag{D8}$$

The quantity $\lambda^2\Sigma_k|V_k|^2$ is of the order of the square of the ultraviolet cutoff frequency ω_M of the interaction and is generally much larger than ω_1. Hence, in an extremely short time scale $t\sim 1/\omega_M \ll 1/\omega_1$, the action variable $J_1(t)$ of the particle decreases as

$$J_1(t)=\left(1-\lambda^2\sum_k |V_k|^2t^2\right)J_{10}, \tag{D9}$$

while $J_k(t)$ increases as

$$J_k(t)=\lambda^2|V_k|^2t^2J_{10}. \tag{D10}$$

This implies that during this extremely short time scale, there is an emission of the fields from the particle through a reversible process.

We have proven theorems (D1) and (D2) for the classical Friedrichs model that is a simplified model of radiation damping in classical matter-field coupling systems. These theorems are not restricted to our model, but hold for very general situation of the radiation damping including three-dimensional case of the electromagnetic field coupled with a charged particle.

[1] G. Ordonez, T. Petrosky, and I. Prigogine, Phys. Rev. A **63**, 052106 (2001). We refer to this paper as $Q1$ throughout the present paper.
[2] T. Petrosky, G. Ordonez, and I. Prigogine, Phys. Rev. A **64**, 062101 (2001). We refer to this paper as $Q2$ throughout the present paper.
[3] T. Petrosky, G. Ordonez, and I. Prigogine, Phys. Rev. A **62**, 042106 (2000).
[4] E. Karpov, T. Petrosky, I. Prigogine, and G. Pronko, J. Math. Phys. **41**, 118 (2000).
[5] T. Petrosky, I. Prigogine, and S. Tasaki, Physica A **173**, 175 (1991).
[6] I. Prigogine and T. Petrosky, Physica A **147**, 461 (1988).
[7] L.A. Khalfin, Zh. Éksp. Theor. Fiz. **33**, 1371 (1957) [Sov. Phys. JETP **6**, 1053 (1958)].
[8] E.C.G. Sudarshan and B. Misra, J. Math. Phys. **18**, 756 (1977).
[9] J. Mehra and K. Milton, *Climbing the Mountain. The Scientific Biography of Julian Schwinger* (Oxford University Press, Oxford, 2000).
[10] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 2nd ed. (Pergamon, London, 1962).
[11] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New

York, 1998).
[12] H.A. Lorentz, Arch. Néer. Sci. Exact. Nat. **XXV**, 363 (1892).
[13] M. Abraham, *Theorie der Elektrizitat* (Teubner, Leipzig, 1905), Vol. II.
[14] P.A.M. Dirac, Proc. R. Soc. London, Ser. A **167**, 148 (1938).
[15] H. Spohn, Europhys. Lett. **50**, 287 (2000).
[16] F. Rohrilich, Phys. Lett. A **283**, 276 (2001).
[17] We add this important condition in addition to the conditions enumerated in $Q1$ [see Eq. (Q1.5.11)].
[18] In $Q2$ we have used condition (7), which is a replacement of another condition on the energy uncertainty in our other paper $Q1$. Since the energy fluctuation that is proportional to $\hbar$ vanishes in the classical case, we rather use Eq. (7) to specify the classical Λ.
[19] V. Bargmann, Commun. Pure Appl. Math. **14**, 187 (1961).
[20] I.E. Segal, Ill. J. Math. **6**, 500 (1962); *Mathematical Problems of Relativistic Physics* (American Mathematical Society, Providence, RI, 1963).
[21] J.R. Klauder and E.C.G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).
[22] K. Friedrichs, Commun. Pure Appl. Math. **1**, 361 (1948).
[23] N. Nakanishi, Prog. Theor. Phys. **19**, 607 (1958).
[24] E.C.G. Sudarshan, C.B. Chiu, and V. Gorini, Phys. Rev. D **18**, 2914 (1978).
[25] A. Böhm and M. Gadella, *Dirac Kets, Gamow Vectors and Gelfand Triplets*, Springer Lecture Notes on Physics Vol. 348 (Springer, New York, 1989).
[26] S. Tasaki, Phys. Rep. **219**, 110 (1992).
[27] C. Cohen-Tannouji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions. Basic Processes and Applications* (Wiley, New York, 1992).
[28] In order to obtain consistent form of distributions in the integrand over k, we have to take the limit $\Omega\rightarrow\infty$ first, and then take the limit $\epsilon\rightarrow 0+$ in the continuous spectrum limit.
[29] F. Gailoli, E.T. Garcia-Alverez, and J. Guevara, Int. J. Theor. Phys. **36**, 2167 (1997).
[30] I. Prigogine, *Nonequilibrium Statistical Mechanics* (Wiley-Interscience, New York, 1962).
[31] T. Petrosky and I. Prigogine, Chaos, Solitons Fractals **7**, 441 (1996).
[32] T. Petrosky and I. Prigogine, Adv. Chem. Phys. **99**, 1 (1997).
[33] I. Prigogine, C. George, F. Henin, and L. Rosenfeld, Chem. Scr. **4**, 5 (1973).
[34] I. Prigogine, *From Being to Becoming* (Freeman, New York, 1980).
[35] I. Prigogine, in *Fundamental Problems in Elementary Particle Physics, XIV Conseil de Physique Solvay* (Interscience, New York, 1968), p. 155.
[36] I. Prigogine, F. Henin, and C. George, Proc. Natl. Acad. Sci. U.S.A. **59**, 7 (1968).
[37] F. Henin, Physica (Amsterdam) **39**, 599 (1968).
[38] P. Mandel, Physica (Amsterdam) **54**, 241 (1971).
[39] C. George, F. Henin, F. Mayne, and I. Prigogine, Hadronic J. **7**, 520 (1978).
[40] T. Petrosky and I. Prigogine, Physica A **175**, 146 (1991).
[41] T. Petrosky and I. Prigogine, Physica A **147**, 439 (1988).
[42] G. Compagno, R. Passante, and F. Persico, *Atom-Field Interactions and Dressed Atoms* (Cambridge University Press, Cambridge, 1995).
[43] Precisely speaking, there is another time scale corresponding to the t^2 term for an extremely short time scale of order $1/\omega_M$ related to the ultraviolet cutoff frequency, as discussed in Appendix D.
[44] All figures presented in this paper have been produced by C. O. Ting.
[45] T. Petrosky and I. Prigogine, Can. J. Phys. **68**, 670 (1990).
[46] M. de Haan, C. George, and F. Mayne, Physica A **92**, 584 (1978).
[47] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, 2nd ed. (Pergamon, London, 1962).
[48] T. Petrosky and G. Ordonez, in *Quantum Limits of the Second Law*, edited by D. P. Sheehan, AIP Conf. Proc. No. 643 (AIP, Melville, NY, 2000), p.59.
[49] E. Karpov, G. Ordonez, T. Petrosky, and I. Prigogine, Phys. Rev. A **66**, 012109 (2002).
[50] T. Petrosky and V. Barsegov, Phys. Rev. E **65**, 046102 (2002).
[51] S. Kim and G. Ordonez, Phys. Rev. E **67**, 056117 (2003).