Comment on "Solution of the Schrödinger equation for the time-dependent linear potential"

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We present the correct way to obtain the general solution of the Schrödinger equation for a particle in a time-dependent linear potential following the approach used in the paper of Guedes [Phys. Rev. A **63**, 034102 (2001)]. In addition, we show that, in this case, the solutions (wave packets) are described by the Airy functions.

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Recently, there has been particular interest in the solution of the one-dimensional Schrödinger equation with a time-dependent linear potential [1–3]. In Ref. [1], Guedes used the technique of Lewis and Riesenfeld [4] based on the invariant and proposed a solution which is not, however, the general one. Feng [2] followed a method based on spatio-temporal transformations of the Schrödinger equation and got a new solution where the one in Ref. [1] constitutes a particular case which corresponds to the so-called "standing" particle case in a linear potential [2]. In his Comment [3], Bauer explained that the solution found by Guedes [1] is simply a special case of the Volkov solution to the time-dependent Schrödinger equation describing a nonrelativistic charged particle moving in an electromagnetic field. Although in Refs. [2] and [3] the authors commented on the type of solution proposed in Ref. [1], they did not, however, debate on the origin of the problem.

In the present work, we show (i) how to correctly use the invariant method to obtain the general solution of the Schrödinger equation with a time-dependent linear potential, (ii) that the procedure used by Guedes is not the right one to find a general solution and that the result found in Ref. [1] is a particular case, and (iii) that the solutions can be described by the Airy functions.

The problem is to find the solutions of the Schrödinger equation,

$$\left(\frac{\partial}{\partial t}-\frac{1}{i\hbar}H\right)\psi(x,t)=0, \tag{1}$$

for the Hamiltonian

$$H=\frac{p^2}{2m}+f(t)x. \tag{2}$$

where $f(t)$ is a time-dependent function.

According to the theory of Lewis and Riesenfeld [4], a solution of the Schrödinger equation with a time-dependent Hamiltonian is easily found if a nontrivial Hermitian operator $I(t)$ exists and satisfies the invariant equation

$$\frac{dI}{dt}=\frac{\partial I}{\partial t}-\frac{i}{\hbar}[I,H]=0. \tag{3}$$

Indeed, this equation is equivalent to saying that if $\varphi_\lambda(x,t)$ is an eigenfunction of I with a time-independent eigenvalue λ, we can find a solution of the Schrödinger equation in the form $\psi_\lambda(x,t)=e^{\mu_\lambda(t)}\varphi_\lambda(x,t)$, where $\mu_\lambda(t)$ satisfies the eigenvalue equation for the Schrödinger operator,

$$\left(\frac{1}{i\hbar}H-\frac{\partial}{\partial t}\right)\varphi_\lambda(x,t)=\dot{\mu}_\lambda(t)\varphi_\lambda(x,t). \tag{4}$$

In Ref. [1], the author chose a linear invariant with respect to canonical variables,

$$I(t)=A(t)p+B(t)x+C(t). \tag{5}$$

The invariant equation is satisfied if the time-dependent coefficients are such that

$$\left(\dot{A}+\frac{1}{m}B\right)p+\dot{B}x+(\dot{C}-Af)\equiv 0, \tag{6}$$

which implies

$$B(t)=B_0, \tag{7a}$$

$$A(t)=-\frac{B_0}{m}t+A_0, \tag{7b}$$

$$C(t)=A_0\int_0^t f(\tau)d\tau-\frac{B_0}{m}\int_0^t \tau f(\tau)d\tau+C_0, \tag{7c}$$

where A_0, B_0, and C_0 are arbitrary real constants.

The eigenstates of $I(t)$ corresponding to time-independent eigenvalues are the solutions of the equation

$$I\varphi_\lambda(x,t)=\lambda\varphi_\lambda(x,t). \tag{8}$$

At this point, had Guedes solved Eq. (8) and found the eigenfunctions, he would have obtained the general solution. In the following, we will use the right procedure and obtain the general solution.

It is easy to see that the solutions of Eq. (8) are of the form

$$\varphi_\lambda(x,t)=\exp[\eta_\lambda(t)x+\rho(t)x^2], \tag{9}$$

where

$$\rho(t) = -\frac{i}{\hbar}\frac{B_0}{2A(t)}, \tag{10a}$$

$$\eta_\lambda(t) = \frac{i}{\hbar}\frac{\lambda - C(t)}{A(t)}. \tag{10b}$$

It is obvious that $\eta_\lambda(t)$ and $\rho(t)$ are purely imaginary functions. These terms are chosen so that $\{\varphi_\lambda(x,t)\}$ forms an orthonormal basis. The phase μ_λ is given by the differential equation

$$\dot{\mu}_\lambda = \frac{i\hbar}{2m}(\eta_\lambda^2 + 2\rho), \tag{11}$$

and is not a purely imaginary function for $\rho \neq 0$. Since the physical solutions $\psi_\lambda(x,t)$ of the Schrödinger equation must be a square integrable with a time-bounded norm [5], one must have $\rho \equiv 0$, which corresponds to $B_0 = 0$.

Note that the author of Ref. [1] seems to imply that $B_0 = 0$ is a simple choice but not a constraint that must be taken to get physical solutions. Besides, the author had not solved the eigenvalue equation of the invariant operator as we have done above, but found the eigenfunction (for $B_0 = 0$) in the form $e^{\eta(t)x}$ without reference to the eigenvalue λ. The function $\eta(t)$ and the phase $\mu(t)$ are then obtained by imposing that $e^{\eta(t)x+\mu(t)}$ must satisfy the Schrödinger equation. Consequently, the obtained solution corresponds to the particular case with $\lambda = 0$. This solution is then, as mentioned in Ref. [2], a particular one corresponding to a "standing" particle in a linear potential.

Thus the physically acceptable solution for the Hamiltonian (2) reads

$$\psi_\lambda(x,t) = \exp\left[-\frac{i\hbar}{2m}\int^t d\tau \left(\frac{[\lambda - C(\tau)]}{A_0}\right)^2\right] \times \exp\left[\frac{i}{\hbar}\frac{[\lambda - C(t)]}{A_0}x\right]. \tag{12}$$

Note that one gets the same result as in Ref. [2]. The most general form of the solutions can be chosen as a wave packet, i.e.,

$$\psi(x,t) = \int_{-\infty}^{+\infty} d\lambda\, g(\lambda)\, \psi_\lambda(x,t), \tag{13}$$

where $g(\lambda)$ is an arbitrary amplitude constant. Any suitable choice of $g(\lambda)$ yields a conventional solution as the Airy functions. Let us now choose $g(\lambda) = e^{i\lambda^3/3}$ by using the integral representation of the Airy function,

$$\mathrm{Ai}(z) = \int_{-\infty}^{+\infty} d\lambda\, e^{i\lambda z} e^{i\lambda^3/3}, \tag{14}$$

and by changing the integration variable $\lambda \to \lambda/B + (B^2 S)/2$, where B is an arbitrary constant and $S = t/m$. We find after integrating Eq. (13),

$$\psi(x,t) = A_0^{1/3} \exp\left\{-\frac{i}{A_0}\left[C(t)x - \frac{i}{2m}\left(\int^t d\tau\, C(\tau)\right)^2\right]\right\} \times \exp\left[\frac{iB^3 S}{2A_0}\left(x + \frac{1}{m}\int^t d\tau\, C(\tau) - \frac{B^3 S}{6}\right)\right] \times \mathrm{Ai}\left[\frac{B}{A_0^{2/3}}\left(x + \frac{1}{m}\int^t d\tau\, C(\tau) - \frac{B^3 S^2}{4}\right)\right]. \tag{15}$$

Note that in Ref. [2] the author gave the Airy function [Eq. (8), Ref. [2]]. It should be pointed out, however, that is not a new solution. In fact, it is just the wave packet found by integrating ψ_λ with weight $g(\lambda)$ as shown above.

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