

Effective Hamiltonians for periodically driven systemsSaar Rahav,¹ Ido Gilary,² and Shmuel Fishman¹¹*Department of Physics, Technion, Haifa 32000, Israel*²*Department of Chemistry, Technion, Haifa 32000, Israel*

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The dynamics of classical and quantum systems, which are driven by a high-frequency (ω) field, is investigated. For classical systems, the motion is separated into a slow part and a fast part. The motion for the slow part is computed perturbatively in powers of ω^{-1} to the order ω^{-4} , and the corresponding time independent Hamiltonian is calculated. Such an effective Hamiltonian for the corresponding quantum problem is computed to the order ω^{-4} in a high-frequency expansion. Its spectrum is the quasienergy spectrum of the time dependent quantum system. The classical limit of this effective Hamiltonian is the classical effective time independent Hamiltonian. It is demonstrated that this effective Hamiltonian gives the exact quasienergies and quasienergy states of some simple examples, as well as the lowest resonance of a nontrivial model for an atom trap. The theory that is developed in this paper is useful for the analysis of atomic motion in atom traps of various shapes.

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I. INTRODUCTION

The interaction of cold atoms with strong electromagnetic fields results in many novel, interesting experimental observations [1–3]. The relevant systems are characterized by an extremely high degree of control that enables one to explore various problems of general physical interest. The response to a rapid oscillating force is such an issue, and will be the subject of the present paper.

Recently, in a series of experiments atomic billiards were realized [4,5]. In these billiards atoms were confined by a standing wave of light to move in planes. The boundary of the billiards was generated by a laser beam, perpendicular to the plane of motion. This beam rapidly traverses a closed curve, which acts as the boundary of the billiard. The boundary of the billiard is assumed to be approximated by the time average of this beam, and the force applied by the boundary on the particles is approximately the mean force applied by the beam. One expects that this approximation is valid when the motion of the beam is fast relative to the typical velocities of the atoms in the billiard. The billiards generated by the rapidly moving light beam have motivated the present work. The more general physical problem, which is explored here, is the description of the classical and quantum dynamics in presence of fields that oscillate with high frequency.

In traditional atomic physics, one typically assumes that the fields which affect the atoms have an amplitude which is constant in space and is time independent. This assumption is justified since the wavelength of the light field is much larger than the size of the atom and the electronic (internal) degrees of freedom react to the periodic change in the field much faster than the external ones (center-of-mass coordinate and momentum). The main subject of traditional atomic physics is the response of the internal degrees of freedom to this field. Atomic spectroscopy is the most spectacular result of this line of research. The center-of-mass motion of the atom can be ignored in most laboratory experiments that explore the dynamics of the internal degrees of freedom.

For the field of atom optics, the effect of the internal

degrees of freedom on the center-of-mass motion is important, in particular, near resonance of the external field with the internal motion (level spacing). The force on the center of mass due to the internal degrees of freedom is given approximately by a dipole force [2]. The sign of this force depends on the sign of the detuning of the light frequency from resonance (of the electronic levels). The motion of the atoms is manipulated by fields with amplitudes which vary spatially, resulting in a force on the center of mass of the atoms. In many cases, the amplitude of the field can be assumed static. The atomic billiards described earlier consist of a time dependent field which results from the moving laser beam. Even at high frequencies of the motion of this beam, one might expect that this time dependence will have some dynamical consequences. The question is most interesting when the wavelength of atoms is of the order of the size of the billiards. In this work, the effect of a laser on the center-of-mass motion of the atoms will be modeled by a time dependent potential. For some situations of physical interest, this simpler model should still describe the dynamics in a high-frequency field without the need to specify the dynamics of the internal degrees of freedom or the quantum aspects of the light field. Therefore, in the present work the atoms are modeled by point particles moving in a rapidly oscillating potential that varies in space. This description is relevant for a wide class of light-atom interactions and is not confined to models of billiards, which motivated the present work.

The classical dynamics of particles influenced by a high-frequency field was studied in several contexts. Kapitza investigated a classical pendulum with a periodically moving point of suspension [6]. In this “Kapitza’s pendulum,” the motion can be separated into a slow part and a fast part which consists of a rapid motion around the slow part. The fast motion results in an effective potential for the slow motion. In some range of parameters this pendulum performs harmonic (slow) oscillations around the point where it points upwards. This point is unstable in the absence of the time dependent perturbation. This phenomenon is called “dynamical stabilization.” Later, Landau and Lifshitz general-

ized this result for motion in the presence of a rapid periodic force with a spatially dependent amplitude [7] (see also Ref. [8]), and calculated the leading term in an expansion in powers of the inverse frequency.

Dynamical stabilization is used to trap atoms in electromagnetic fields. The most notable example is the Paul trap [9]. In this trap, the time dependent electric fields are used to localize ions in the region where the field amplitude is minimal. The fields are well approximated by restoring forces which are linear in the distance from the equilibrium point. The resulting Hamiltonian is that of a time dependent oscillator. It is possible to find exact quantum-mechanical solutions for this problem, which are based on the corresponding classical system. That is, the states are simply related to the ones of the harmonic oscillator. Therefore the states of the motion in the Paul trap are known [10–12]. It is of interest to find some of the states of problems of a more general nature, even if only approximately.

The work of Kapitza was first extended to quantum-mechanical systems in a pioneering paper by Grozdanov and Raković [13]. They introduced a unitary gauge transformation resulting in an effective Hamiltonian that describes the slow motion and demonstrated that its eigenvalues are the quasienergies of the time dependent problem. The effective Hamiltonian was calculated as an expansion in powers of the inverse frequency. In that paper, the analysis is restricted to a driving potential that has a particularly simple time dependence. Moreover, the final results are restricted to forces that are uniform in space, a situation natural in standard spectroscopy, but too restrictive for the interesting problems in atom optics. These restrictions are avoided in the present work.

Other studies of quantum systems with periodic time dependent fields were also published. Gavrilă [14,15] developed a perturbation theory for the Floquet states and the quasienergies in terms of the states of the time-averaged problem. The scattering from a periodically driven barrier was studied by Vorobeichik *et al.* [16], by Bagwell and Lake [17], and by Wagner [18], while the quantum and classical dynamics of some one-dimensional systems were investigated by Hensler *et al.* [19]. In the limit of high frequencies, the systems behave as if the particles were subject to an effective potential which is the time average of the time dependent one. Fredholm theory was used by Georgeot and Prange [20] to study quasiclassical scattering from various systems, including a one-dimensional periodically kicked potential.

Another approach to time dependent systems (not necessarily periodic) is to use the Magnus expansion [21] in order to compute the propagator. Time periodic systems were used as examples in order to check the convergence of this expansion [22–24]. For these time periodic systems, the Magnus expansion is of similar nature to the method presented here, and the differences are discussed in Sec. III.

There are numerous other works regarding periodically driven systems. Here we mention few of them. Of special physical interest is the ionization of atoms by light (see Ref. [25] and references therein). Some toy models for ionization that consist of one-dimensional time dependent δ functions were treated rigorously [26]. In particular, it was shown that

typically there is full ionization, namely, that at long times the probability to be at the bound state (of the time-averaged problem) approaches zero. For some arrangements of the δ functions, stable bound Floquet states exist. The transport through driven mesoscopic devices [27], and in the presence of oscillating fields [28], attracted some interest.

In the present work, we study the dynamics of classical and quantum high-frequency driven systems. The classical problem is discussed in Sec. II, where the motion is separated into a “slow” and a “fast” part. A systematic perturbation theory is developed for the motion of the “slow” part. The equation of motion of the “slow” dynamics is then computed to the order ω^{-4} , which is an extension of the order ω^{-2} (presented in Ref. [7]). This slow motion is shown to result from an effective Hamiltonian. In Sec. III, an adaption of the Floquet theory to the problem is reviewed. An effective (time independent) Hamiltonian operator is defined following and generalizing [13]. The eigenvalues of this operator are the quasienergies of the system. This effective Hamiltonian is then computed perturbatively (to the order ω^{-4}) in Sec. IV. The restrictions introduced in Ref. [13] are avoided and consequently detailed expressions for the various terms of the effective Hamiltonian are calculated explicitly. The classical effective Hamiltonian of Sec. II is found to be the classical limit of this quantum effective Hamiltonian. A known exactly solvable simple example of the method is presented in Sec. V. In Sec. VI, the scattering from a time dependent potential is discussed. In particular, the resonances of the time dependent problem are found to agree with those of the effective (time independent) Hamiltonian of Sec. IV. Finally, the results, the implications, and some related open problems are discussed in Sec. VII.

II. CLASSICAL MOTION IN A HIGH-FREQUENCY POTENTIAL

In this section, the dynamics of a classical particle moving in one dimension under the influence of a force which is periodic in time is studied. Typically, solutions for time dependent problems can only be attained numerically. However, when the period of the force is small compared with the other time scales of the problem, it is possible to separate the motion of the particle into “slow” and “fast” parts. This simplification is due to the fact that the particle does not have the time to react to the periodic force before this force changes its sign, namely, the contribution of the periodic force to the acceleration in one period is negligible (compared to the contribution of the effective force, in a sense which will be specified in what follows). Thus we will consider the limit of small periods (or large frequencies) of the driving field.

The leading order (with respect to $1/\omega$) of the dynamics was computed by Kapitza [6] for the “Kapitza’s pendulum,” namely a pendulum where the point of suspension is moved periodically. It turns out to be very general [7]. Here the next order is computed, and it is demonstrated that the equation of motion of the slow part of the dynamics can be derived from a time independent Hamiltonian. This Hamiltonian will be computed explicitly to the order $1/\omega^4$. Later, this Hamil-

tonian will be compared with an effective Hamiltonian which will be derived for the corresponding quantum problem.

The existence of such a Hamiltonian might seem to contradict the fact that the time dependent dynamics do not possess a constant of motion. Moreover, the classical motion may be chaotic. The existence of this effective time independent Hamiltonian implies that a constant of motion exists for the slow dynamics (it is just the effective Hamiltonian), and for a one-dimensional system the slow dynamics is integrable. To avoid confusion, it should be emphasized that the effective Hamiltonian depends on a coordinate which describes the “slow” part of the motion. *This coordinate is not the location of the particle* (although they are almost identical at high frequencies). The actual motion consists of a rapid motion in the proximity of the trajectory of the slow dynamics. The relation between the slow coordinate and the coordinate of the particle is nonlinear and extremely complicated as will be demonstrated in what follows. We will demonstrate that an effective Hamiltonian for the “slow” motion may exist.

Newton’s equation for the motion in the periodic field is given by

$$m \frac{d^2 x}{dt^2} = -V'_0(x) - V'_1(x, \omega t), \quad (1)$$

where V_1 is a periodic function of ωt of period 2π and its average over a period vanishes. We denote derivatives with respect to coordinates by primes and with respect to time by dots. This separation of the potential to an average part $V_0(x)$ and a periodic part with vanishing average, $V_1(x, \omega t)$, is natural and will simplify the following calculations. We look for a solution of the form

$$x(t) = X(t) + \xi(X, \dot{X}, \omega t), \quad (2)$$

where $\dot{X} = dX/dt$ and

$$\bar{\xi} \equiv \frac{1}{2\pi} \int_0^{2\pi} d\tau \xi(X, \dot{X}, \tau) = 0. \quad (3)$$

The bar denotes in this paper the time average over one period. The fast part of the motion, which is nearly periodic in time, is denoted by ξ . It will be shown later that it can be chosen to depend only on X and \dot{X} , but not on the higher-order time derivatives. Since X and \dot{X} are slowly varying functions of time t , ξ is *not* periodic in time t , in spite of Eq. (3). The coordinate X describes the slow part of the motion and its equation of motion will be computed in the following. Our method of solution is to choose ξ so that Eq. (1) will lead to an equation for X which is time independent. An exact solution using Eq. (2) is too complicated to obtain. However, at high frequencies, one can determine ξ order by order in $1/\omega$. In order to separate terms in powers of the frequency, it is convenient to introduce the new time variable $\tau \equiv \omega t$. Using

$$\frac{d\xi}{dt} = \omega \frac{\partial \xi}{\partial \tau} + \frac{\partial \xi}{\partial X} \dot{X} + \frac{\partial \xi}{\partial \dot{X}} \ddot{X}, \quad (4)$$

Newton’s equation (1) is given by

$$\begin{aligned} m \left(\ddot{X} + \omega^2 \frac{\partial^2 \xi}{\partial \tau^2} + 2\omega \left[\frac{\partial^2 \xi}{\partial X \partial \tau} \dot{X} + \frac{\partial^2 \xi}{\partial \dot{X} \partial \tau} \dot{X} \right] + \frac{\partial \xi}{\partial X} \ddot{X} + \frac{\partial \xi}{\partial \dot{X}} \ddot{\dot{X}} \right. \\ \left. + \frac{\partial^2 \xi}{\partial X^2} \dot{X}^2 + 2 \frac{\partial^2 \xi}{\partial X \partial \dot{X}} \dot{X} \ddot{X} + \frac{\partial^2 \xi}{\partial \dot{X}^2} \ddot{X}^2 \right) \\ = -V'_0(X + \xi) - V'_1(X + \xi, \tau). \end{aligned} \quad (5)$$

The variables τ and t will be treated as independent variables. This calculation is similar to the ones performed within the method of multiple time scales. Indeed, the result of the following calculation is equivalent to the one obtained using the method of multiple time scales analysis, as demonstrated in the Appendix.

In the limit of high frequencies, ξ is going to be small (of the order ω^{-2}) and therefore it is convenient to expand $V_0(X + \xi)$ and $V_1(X + \xi, \tau)$ in powers of ξ (we assume that V_0 and V_1 are smooth functions of the coordinate). Then ξ is expanded in powers of $1/\omega$,

$$\xi = \sum_{i=1}^{\infty} \frac{1}{\omega^i} \xi_i. \quad (6)$$

The ξ_i are chosen so that the equation for X that results from Eq. (5) does not depend on τ .

Before obtaining the slow equation of motion from Eq. (5), order by order, there are two points regarding our method of solution which should be discussed. First, we note that the fast part ξ is expanded in powers of $1/\omega$ while X is not expanded, which seems to be inconsistent. One may also expand X in powers of $1/\omega$ as $X = \sum_{i=0}^{\infty} (1/\omega^i) X_i$. When one does so the equation of motion for X is then replaced by a series of equations for X_i . In this series of equations, each X_i can be determined from the lower-order terms X_j , where $j < i$. This is the standard method of separation of time scales, and its application to the present problem is demonstrated in the Appendix. These equations are equivalent, in any order, to the equation of motion of (unexpanded) X , which will be obtained in what follows. At a given order ω^{-n} of the present calculation, *all* contributions that are found by the method of separation of time scales are included, but *some* of the higher-order terms are included as well. Second, we note that while we assumed that ξ depends only on X and \dot{X} , the higher-order derivatives of X with respect to time appear in Eq. (5). In the leading order in $1/\omega$, as will be demonstrated, one can replace \ddot{X} by $-(1/m)V'_0(X)$. The error is of higher order in $1/\omega$, leading to the correct contribution to ξ_i at the order where \ddot{X} appeared. Corrections of higher orders of $1/\omega$ to ξ_i result from the corrections of higher orders to \ddot{X} . These corrections will affect ξ_j with $j > i$, since these are chosen to cancel the τ dependence at any given order. The higher-order derivatives of X can be found by repeated differentiation of

\ddot{X} . This enables us to obtain an expression for ξ which depends on X and \dot{X} , but not on the higher-order derivatives of X .

To proceed we gather all the terms in Eq. (5), using Eq. (6), which are of the same order, say ω^{-n} , and choose ξ_{n+2} (which is still undetermined) so that the explicit τ dependence cancels. In the leading order (ω), the only contribution is

$$\frac{\partial^2 \xi_1}{\partial \tau^2} = 0. \tag{7}$$

Therefore we can choose

$$\xi_1 = 0. \tag{8}$$

In the next order (ω^0), we find the contributions

$$m \left(\ddot{X} + \frac{\partial^2 \xi_2}{\partial \tau^2} \right) = -V'_0(X) - V'_1(X, \tau). \tag{9}$$

Our goal is to balance the τ dependence. To do this we have to solve

$$\frac{\partial^2 \xi_2}{\partial \tau^2} = -\frac{1}{m} V'_1(X, \tau), \tag{10}$$

moreover, we also require that ξ_2 is periodic in τ . The integral over the right-hand side (RHS) of Eq. (10) can have terms which are time independent and thus ξ_2 can grow linearly in τ . To ensure that ξ_2 is small even at long times such secular terms must be avoided. This can be done by requiring that the time integral has a vanishing average over a period. Let $f(x, \tau)$ be any periodic function of τ with a vanishing average, $\bar{f} = 0$. Assume that the Fourier expansion of f is given by $f = \sum_{n \neq 0} f_n e^{in\tau}$, then we define the following integral:

$$\int_{\tau} [f] \equiv \sum_{n \neq 0} \frac{1}{in} f_n e^{in\tau}, \tag{11}$$

and its repeated application will be denoted by

$$\int^{(2)\tau} [f] = \int^{\tau} \left[\int^{\tau} [f] \right] \tag{12}$$

and j applications by

$$\int^{(j)\tau} [f] = \underbrace{\int^{\tau} \left[\dots \int^{\tau} [f] \dots \right]}_{j \text{ times}}. \tag{13}$$

This definition, which is actually a specific choice of the integration constant, is natural since it ensures that the result is periodic even after repeated integrations. It also helps to separate periodic terms (with vanishing average) and secular terms (which will be time independent in the current calculation). Integration of Eq. (10) implies

$$\xi_2 = -\frac{1}{m} \int^{(2)\tau} [V'_1(X, \tau)]. \tag{14}$$

Note that we did not really find the general solution of Eq. (9), but rather chose ξ so that it is satisfied. Substituting ξ_2 in Eq. (9) gives the leading-order equation for the slow coordinate X . The terms in this equation are just the time independent terms which were not canceled by ξ_2 ,

$$m\ddot{X} = -V'_0(X). \tag{15}$$

The contributions from Eq. (5) at the next order ω^{-1} are

$$m \left(\frac{\partial^2 \xi_3}{\partial \tau^2} + 2\dot{X} \frac{\partial^2 \xi_2}{\partial X \partial \tau} \right) = 0, \tag{16}$$

which, with the help of Eq. (14), is satisfied by

$$\xi_3 = \frac{2}{m} \dot{X} \int^{(3)\tau} [V''_1(X, \tau)]. \tag{17}$$

The terms of order ω^{-2} in Eq. (5) are given by

$$\begin{aligned} m \left[\frac{\partial^2 \xi_4}{\partial \tau^2} + 2 \left(\dot{X} \frac{\partial^2 \xi_3}{\partial X \partial \tau} + \ddot{X} \frac{\partial^2 \xi_3}{\partial \dot{X} \partial \tau} \right) + \ddot{X} \frac{\partial \xi_2}{\partial X} + \dot{X}^2 \frac{\partial^2 \xi_2}{\partial X^2} \right] \\ = -V''_0(X) \xi_2 - V''_1(X, \tau) \xi_2. \end{aligned} \tag{18}$$

Substituting Eqs. (14) and (17) leads to

$$\begin{aligned} \frac{\partial^2 \xi_4}{\partial \tau^2} = \frac{V''_0}{m^2} \int^{(2)\tau} [V'_1] + \frac{V''_1}{m^2} \int^{(2)\tau} [V'_1] \\ - \frac{3}{m} \dot{X}^2 \int^{(2)\tau} [V_1^{(3)}(X, \tau)] - \frac{3\ddot{X}}{m} \int^{(2)\tau} [V''_1]. \end{aligned} \tag{19}$$

Equation (19) cannot be solved, if ξ_4 is required to be periodic in τ , since the RHS has a nonvanishing average which will lead to solutions that grow like τ^2 (these are the secular solutions that one wishes to avoid when using the multiple time scales analysis). We will choose ξ_4 so that it will balance the τ dependent part of Eq. (19) and will be periodic in τ . The remaining τ independent terms in Eq. (19) will be included in the equation of motion of the slow coordinate X . Defining

$$f_1(X, \tau) \equiv \frac{1}{m^2} V''_1 \int^{(2)\tau} [V'_1] - \frac{1}{m^2} \overline{V''_1 \int^{(2)\tau} [V'_1]} \tag{20}$$

and choosing

$$\begin{aligned} \xi_4 = \frac{V''_0}{m^2} \int^{(4)\tau} [V'_1] + \int^{(2)\tau} [f_1] - \frac{3\dot{X}^2}{m} \int^{(4)\tau} [V_1^{(3)}] \\ - \frac{3\ddot{X}}{m} \int^{(4)\tau} [V''_1] \end{aligned} \tag{21}$$

balances all the τ dependent terms on the RHS of Eq. (19) but leaves an extra term

$$\frac{1}{m} \overline{V_1'' \int^{(2)\tau} [V_1']},$$

which is not balanced. This is actually a term of order ω^{-2} that is left on the RHS of Eq. (5) when we substitute ξ in Eq. (5). The resulting equation for the slow motion is then

$$m\ddot{X} = -V_0'(X) + \frac{1}{m\omega^2} \overline{V_1'' \int^{(2)\tau} [V_1']} + O(\omega^{-3}). \quad (22)$$

This is the leading order correction due to the periodic potential V_1 . It was calculated before in Refs. [6,7]. With the help of Eq. (15) or of the leading term in Eq. (22), \ddot{X} can be eliminated from expression (21) for ξ_4 . This method allows us to compute corrections order by order. We will continue the calculation up to the order ω^{-4} .

The next order is ω^{-3} . We do not need to compute ξ_5 explicitly since it can only change the slow equation in order ω^{-5} . To obtain the next correction to Eq. (22), one needs only the average over τ of the terms of order ω^{-3} . The reason is that ξ_5 will be chosen in such a way that it will cancel all the periodic terms with vanishing average. This further simplifies the calculation since all the terms (except $m\ddot{X}$) on the LHS of Eq. (5) have a vanishing average (over τ), thus only the terms from the RHS can contribute to the equation of the slow coordinate. In this order, the contributions to the equation of the slow coordinate X can result only

from

$$-\overline{V_0'' \xi_3} - \overline{V_1'' \xi_3}. \quad (23)$$

The first term will vanish since V_0 is τ independent and ξ_3 has a vanishing average. The second term can be computed using Eq. (17),

$$\overline{V_1'' \xi_3} = \frac{2\ddot{X}}{m} \overline{V_1'' \int^{(3)\tau} [V_1']} = -\frac{2\ddot{X}}{m} \int^\tau [V_1''] \int^{(2)\tau} [V_1''] = 0. \quad (24)$$

In the last calculation, we have used integration by parts and then the fact that the average of a derivative of a periodic function over a period must vanish. This leads to the conclusion that one can choose a periodic ξ_5 in such a way that all τ dependent terms of the order ω^{-3} in Eq. (5) are canceled.

We turn to the order ω^{-4} that is the last order that will be considered here. Again one can get the contributions to the equation of X by averaging terms of this order in Eq. (5). The average over τ of the LHS (excluding $m\ddot{X}$) vanishes and the contribution of the terms on the RHS is

$$-\overline{V_0'' \xi_4} - \frac{1}{2} \overline{V_0^{(3)} \xi_2^2} - \overline{V_1'' \xi_4} - \frac{1}{2} \overline{V_1^{(3)} \xi_2^2}. \quad (25)$$

The first term will vanish but the other terms have a nonvanishing average. Using Eqs. (14), (21) and integration by parts (in the averages) yields

$$\begin{aligned} -\frac{1}{2} \overline{V_0^{(3)} \xi_2^2} - \overline{V_1'' \xi_4} - \frac{1}{2} \overline{V_1^{(3)} \xi_2^2} = & -\frac{1}{2m^2} V_0^{(3)} \left(\int^{(2)\tau} [V_1'] \right)^2 - \frac{1}{m^2} V_0'' \int^{(2)\tau} [V_1''] \int^{(2)\tau} [V_1'] - \frac{1}{2m^2} V_1^{(3)} \left(\int^{(2)\tau} [V_1'] \right)^2 \\ & - \frac{1}{m^2} V_1'' \int^{(2)\tau} [V_1''] \int^{(2)\tau} [V_1'] + \frac{3\ddot{X}^2}{m} \int^{(2)\tau} [V_1''] \int^{(2)\tau} [V_1^{(3)}] - \frac{3V_0'}{m^2} \left(\int^{(2)\tau} [V_1''] \right)^2. \end{aligned} \quad (26)$$

In the last term, \ddot{X} in ξ_4 was replaced by $-V_0'/m$ resulting in errors that are of the order ω^{-6} in the final result. Equation (26) gives the ω^{-4} contribution to the equation for the slow coordinate X .

The equation for X to the order ω^{-4} is obtained when ξ is substituted into Eq. (5) and the remaining terms are averaged over τ resulting in

$$\begin{aligned} m\ddot{X} = & -V_0' - \frac{1}{m\omega^2} \int^\tau [V_1'] \int^\tau [V_1''] - \frac{1}{2m^2\omega^4} V_0^{(3)} \left(\int^{(2)\tau} [V_1'] \right)^2 - \frac{1}{m^2\omega^4} V_0'' \int^{(2)\tau} [V_1''] \int^{(2)\tau} [V_1'] - \frac{1}{2m^2\omega^4} V_1^{(3)} \left(\int^{(2)\tau} [V_1'] \right)^2 \\ & - \frac{1}{m^2\omega^4} V_1'' \int^{(2)\tau} [V_1''] \int^{(2)\tau} [V_1'] + \frac{3\ddot{X}^2}{m\omega^4} \int^{(2)\tau} [V_1''] \int^{(2)\tau} [V_1^{(3)}] - \frac{3V_0'}{m^2\omega^4} \left(\int^{(2)\tau} [V_1''] \right)^2 + O(\omega^{-5}). \end{aligned} \quad (27)$$

It is instructive to introduce the effective potential

$$V_{eff}(X) \equiv V_0 + \frac{1}{2m\omega^2} \left(\int^\tau [V_1'] \right)^2 + \frac{1}{2m^2\omega^4} V_1'' \left(\int^{(2)\tau} [V_1'] \right)^2 + \frac{1}{2m^2\omega^4} V_0'' \left(\int^{(2)\tau} [V_1''] \right)^2. \quad (28)$$

Substituting Eq. (28) in Eq. (27) results in the equation of the slow motion

$$m\ddot{X} = -V'_{eff} + \frac{3\dot{X}^2}{m\omega^4} \overline{\int^{(2)\tau} [V''_1] \int^{(2)\tau} [V_1^{(3)}]} - \frac{3V'_0}{m^2\omega^4} \left(\overline{\int^{(2)\tau} [V''_1]} \right)^2 + O(\omega^{-5}). \quad (29)$$

Given a solution of this, $X(t)$, the solution for the original problem can be easily obtained (to the appropriate order of $1/\omega$) since ξ is known in terms of X [see Eqs. (14), (17), and (21)]. From these equations, one sees that in the case where the oscillating force V'_1 is independent of position X the fast coordinate ξ is independent of X and \dot{X} to the order ω^{-4} [note that in Eq. (21) only the order ω^0 of X is required see also Eq. (A16)]. The final result of Ref. [13] is confined to the case where V'_1 is independent of X . Equation (29) can be derived from the Hamiltonian

$$H_{eff} = \frac{P^2}{2m} + V_{eff}(X) + \frac{3}{2m^3\omega^4} \left(\overline{\int^{(2)\tau} [V''_1]} \right)^2 P^2 + O(\omega^{-5}), \quad (30)$$

where P is the momentum conjugate to X .

We have shown that using the natural separation of time scales, it is possible to separate the motion of a particle in a high-frequency periodic field into “slow” and “fast” parts. The slow dynamics can be derived from an effective Hamiltonian which is time independent. We turn to discuss the corresponding quantum problem.

III. FLOQUET THEORY AND THE EFFECTIVE HAMILTONIAN

Consider a quantum system with a Hamiltonian that is periodic in time, $\hat{H}(t+T) = \hat{H}(t)$. Such systems can be treated using the Floquet theory [29–33]. The symmetry with respect to discrete time translations implies that the solutions of the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi = \hat{H} \psi \quad (31)$$

are linear combinations of functions of the form

$$\psi_\lambda = e^{-i(\lambda t/\hbar)} u_\lambda(x, \omega t), \quad (32)$$

where u_λ are periodic with respect to ωt with period 2π , that is $u_\lambda(x, \omega(t+T)) = u_\lambda(x, \omega t)$ with $\omega = 2\pi/T$. The states u_λ are called the quasienergy or the Floquet states and λ is referred to as the quasienergy (we will also call the states ψ_λ as the quasienergy states). This is the content of the Bloch-Floquet theorem in time. The states u_λ are the eigenstates of the Floquet Hamiltonian

$$\hat{\mathcal{H}}_F = -i\hbar \frac{\partial}{\partial t} + \hat{H}. \quad (33)$$

The quasienergy (or Floquet) states have a natural separation into a “slow” part $e^{-i(\lambda t/\hbar)}$ (with the natural choice $0 \leq \lambda/\hbar \leq \omega$), which includes the information about the quasienergies, and to a fast part $u_\lambda(x, \omega t)$ that depends only on the “fast” time ωt . It is expected that one will be able to find an equation of motion for the slow part of the dynamics as was done for the classical systems in Sec. II. Such an equation will include information regarding the quasienergies of the quantum system, and will be developed in what follows. It establishes a natural link between the separation into the fast and the slow motion in classical mechanics, which can be formalized by the theory of separation of time scales, and Bloch-Floquet theory in quantum mechanics.

It is known that one may write the propagator in the form [33]

$$\hat{U}(t) = \hat{\mathcal{P}}(t) e^{-it\hat{\mathcal{G}}/\hbar}, \quad (34)$$

where $\hat{\mathcal{G}}$ is self-adjoint and $\hat{\mathcal{P}}$ is unitary and periodic with the period of the Hamiltonian. The eigenvalues of $\hat{\mathcal{G}}$ are the quasienergies of the system provided the eigenstates of $\hat{\mathcal{G}}$ are in the domain of $\hat{H}(0)$. Sometimes $\hat{\mathcal{G}}$ is called the quasienergy or Floquet operator. The actual calculation of $\hat{\mathcal{G}}$ might be complicated. Such an operator was calculated in Refs. [22,13] by introducing expansions for $\hat{\mathcal{P}}$ and $\hat{\mathcal{G}}$. The result turns out to depend on the phase of the periodic part of the Hamiltonian or on the initial time. [See for example Ref. [22], Eqs. (25) and (26) and Ref. [13], Eq. (16).] Inspired by Eqs. (31)–(34) an approach of a somewhat similar spirit is used.

The goal is to find a unitary gauge transformation $e^{i\hat{F}(t)}$, where $\hat{F}(t)$ is a Hermitian operator (function of \hat{x} and \hat{p}) defined at a certain time t , which is a periodic function of time with the same period as \hat{H} , such that in the new gauge the Hamiltonian in the Schrödinger equation is *time independent*. Such a Hamiltonian was found by Grozdanov and Rasković [13] if the time dependent part is of the restricted form $V_{GR} = \tilde{V}(x) \sin(\omega t + \theta)$. It was analyzed with the further strong restriction that for one dimension it takes the form $d\tilde{V}/dx = \text{const}$ (uniform force). In what follows, a general analysis that is free of these restrictions is presented. Applying $e^{i\hat{F}}$ to both sides of Eq. (31) and adding $i\hbar[(\partial/\partial t)e^{i\hat{F}}]\psi$ to both sides leads to

$$i\hbar \frac{\partial}{\partial t} (e^{i\hat{F}} \psi) = e^{i\hat{F}} \hat{H} \psi + i\hbar \left(\frac{\partial}{\partial t} e^{i\hat{F}} \right) \psi. \quad (35)$$

In terms of the functions in the new gauge, $\phi = e^{i\hat{F}} \psi$, this equation is

$$i\hbar \frac{\partial}{\partial t} \phi = \hat{G} \phi, \quad (36)$$

where the Hamiltonian is

$$\hat{G} = e^{i\hat{F}} \hat{H} e^{-i\hat{F}} + i\hbar \left(\frac{\partial e^{i\hat{F}}}{\partial t} \right) e^{-i\hat{F}}. \quad (37)$$

In the classical limit, it reduces to

$$G = H - \hbar \frac{\partial F}{\partial t}. \quad (38)$$

Therefore in the classical limit $-\hbar F$ is the generating function of the canonical transformation corresponding to the unitary transformation $e^{-i\hat{F}}$ [34].

Let us assume that such an operator \hat{F} exists so that \hat{G} is time independent. Then the eigenfunctions of \hat{G} are $v_\lambda(x)$, and their evolution takes the form

$$\phi_\lambda(t, x) = e^{-i(\lambda t/\hbar)} v_\lambda(x). \quad (39)$$

These states, in the original gauge, correspond to

$$\psi_\lambda(t, x) = e^{-i\hat{F}} \phi_\lambda = e^{-i(\lambda t/\hbar)} e^{-i\hat{F}} v_\lambda(x), \quad (40)$$

since \hat{F} does not include any time derivative. The function $e^{-i\hat{F}} v_\lambda$ is periodic in time with the period of \hat{H} and therefore ψ_λ is a Floquet state with quasienergy $\lambda \pmod{\hbar\omega}$. It should be compared with Eq. (32) with the identification $u_\lambda = e^{-i\hat{F}} v_\lambda$. It is assumed that $e^{i\hat{F}}$ (and $e^{-i\hat{F}}$) are such that they map the domain of $\hat{H}(t)$ into that of \hat{G} and vice versa. This may not be true in general, and one cannot exclude the possibility that examples, where only some of the quasienergies can be found using this method, exist. For example, problems of this nature may occur if for a function ψ in the Hilbert space of \hat{H} , the function $e^{i\hat{F}} \psi$ is not in this space. The limitations on the validity of the method should be subject to further mathematical studies.

To emphasize the difference between the effective Hamiltonian \hat{G} and \hat{G} given by Eq. (34), let us write the propagator in terms of \hat{F} , and \hat{G} . To propagate any state in time using \hat{G} , it has to be transformed to the time independent gauge, then propagated, and finally transformed back. This results in the propagator

$$\hat{U}(t) = e^{-i\hat{F}(t)} e^{-i(t\hat{G}/\hbar)} e^{i\hat{F}(0)}. \quad (41)$$

Since \hat{G} and \hat{F} do not commute, \hat{G} generally differs from \hat{G} of Eq. (34).

We note that an approximate solution of the time dependent problem in terms of an expansion of \hat{F} and \hat{G} has some superior properties compared to the more customary expansion of \hat{P} and \hat{G} of Eq. (34). For instance, if \hat{F} is Hermitian at any order then $e^{i\hat{F}}$ is manifestly unitary, while some care is needed to obtain unitary approximations for \hat{P} . In addition, \hat{G} does not depend on the phase of the time dependent field, while \hat{G} does depend on this phase (see Refs. [13,22]). Therefore in the present work, a description in terms of \hat{G} and \hat{F} is used rather than the one in terms of \hat{P} and \hat{G} .

In the following section, the derivation of \hat{G} and \hat{F} will be presented explicitly as an expansion in powers of $1/\omega$. It will be shown that at high frequencies \hat{F} can be chosen to be

small, of the order of $1/\omega$. In this limit, one can easily calculate the matrix elements of an observable \hat{O} between the quasienergy (Floquet) states using the eigenvalues and eigenstates of the effective Hamiltonian \hat{G} :

$$\begin{aligned} \langle \psi_{\lambda_1} | \hat{O} | \psi_{\lambda_2} \rangle &= \langle \phi_{\lambda_1} | e^{i\hat{F}} \hat{O} e^{-i\hat{F}} | \phi_{\lambda_2} \rangle \\ &= \langle \phi_{\lambda_1} | \hat{O} | \phi_{\lambda_2} \rangle + i \langle \phi_{\lambda_1} | [\hat{F}, \hat{O}] | \phi_{\lambda_2} \rangle \\ &\quad - \frac{1}{2} \langle \phi_{\lambda_1} | [\hat{F}, [\hat{F}, \hat{O}]] | \phi_{\lambda_2} \rangle + \dots \end{aligned} \quad (42)$$

The result is an effective expansion in powers of $1/\omega$. Since observables have a meaningful classical limit $\hbar \rightarrow 0$, their expectation should reduce to the expansion in powers of $1/\omega$ for the corresponding classical quantity as calculated in Sec. II and the Appendix. The expansion of \hat{G} presented in the following section can be considered an extension of the multiple time scales analysis to quantum mechanics. The effective Hamiltonian, which will be obtained, will be compared with the classical Hamiltonian for the slow motion that was computed in Sec. II.

IV. THE EFFECTIVE HAMILTONIAN OF QUANTUM SYSTEMS WITH A HIGH-FREQUENCY POTENTIAL

In Sec. III, we demonstrated that the quasienergies and the Floquet states of a quantum system can be determined if one can find a gauge transformation so that the Hamiltonian is time independent. The transformation and the resulting effective Hamiltonian are obtained here. Typically \hat{F} and \hat{G} cannot be computed exactly. For high frequencies, one can determine \hat{F} and \hat{G} order by order in $1/\omega$. In the following, we present a derivation of \hat{F} and \hat{G} accurate to the order $1/\omega^4$.

We consider the Hamiltonian (which is more general than the one studied in Ref. [13])

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}_0(x) + \hat{V}_1(x, \omega t). \quad (43)$$

This is the quantum system which corresponds to the classical system that was discussed in Sec. II. It should be noted that the method which is described in the present section also applies to the Hamiltonians that differ from Eq. (43), for example in the presence of magnetic fields and for spins (see Sec. V). We choose to examine the Hamiltonian (43) since it is of interest to compare the resulting effective Hamiltonian with its classical counterpart (30). As mentioned in Sec. III we are looking for a unitary transformation $e^{i\hat{F}}$ so that the resulting Hamiltonian (37) is time independent. It is convenient to define $\tau = \omega t$, since the Hamiltonian depends on time only through τ . Using this definition, Eq. (37) is given by

$$\hat{G} = e^{i\hat{F}} \hat{H} e^{-i\hat{F}} + i\hbar \omega \left(\frac{\partial e^{i\hat{F}}}{\partial \tau} \right) e^{-i\hat{F}}. \quad (44)$$

At high frequencies, \hat{F} is assumed to be small, of the order of $1/\omega$, an assumption that will be explicitly satisfied by the following calculation. This enables us to expand \hat{G} and \hat{F} in powers of $1/\omega$ and to choose \hat{F} so that \hat{G} is time independent in any given order. The expansions are given by

$$\hat{G} = \sum_{n=0}^{\infty} \frac{1}{\omega^n} \hat{G}_n \quad (45)$$

and

$$\hat{F} = \sum_{n=1}^{\infty} \frac{1}{\omega^n} \hat{F}_n. \quad (46)$$

The periodicity $\hat{F}(\tau + 2\pi) = \hat{F}(\tau)$ is assumed. The calculation is performed by computing \hat{G}_l in terms of $\hat{F}_1, \dots, \hat{F}_{l+1}$ and then choosing \hat{F}_{l+1} so that \hat{G}_l is time independent. The terms in Eq. (44) are calculated with the help of the operator expansions (that can be obtained with the help of Refs. [35,36]),

$$e^{i\hat{F}} \hat{H} e^{-i\hat{F}} = \hat{H} + i[\hat{F}, \hat{H}] - \frac{1}{2!} [\hat{F}, [\hat{F}, \hat{H}]] - \frac{i}{3!} [\hat{F}, [\hat{F}, [\hat{F}, \hat{H}]]] + \dots \quad (47)$$

and

$$\left(\frac{\partial e^{i\hat{F}}}{\partial \tau} \right) e^{-i\hat{F}} = i \frac{\partial \hat{F}}{\partial \tau} - \frac{1}{2!} \left[\hat{F}, \frac{\partial \hat{F}}{\partial \tau} \right] - \frac{i}{3!} \left[\hat{F}, \left[\hat{F}, \frac{\partial \hat{F}}{\partial \tau} \right] \right] + \dots \quad (48)$$

In the leading order, $O(\omega^0)$, \hat{G}_0 is given by

$$\hat{G}_0 = \frac{\hat{p}^2}{2m} + \hat{V}_0(x) + \hat{V}_1(x, \tau) - \hbar \frac{\partial \hat{F}_1}{\partial \tau}. \quad (49)$$

The potentials \hat{V}_0 and \hat{V}_1 do not depend on \hat{p} . To cancel any time dependence, we choose

$$\hat{F}_1 = \frac{1}{\hbar} \int_{\tau}^{\tau} [\hat{V}_1(x, \tau)]. \quad (50)$$

It is easily computed in the coordinate representation. Note that \hat{F}_1 is determined only up to a Hermitian time independent operator. It was assumed to vanish here. Substituting Eq. (50) in Eq. (49) leads to

$$\hat{G}_0 = \frac{\hat{p}^2}{2m} + \hat{V}_0(x). \quad (51)$$

This is the leading order of the effective Hamiltonian. The dynamics do not depend on the fast time dependent potential \hat{V}_1 as expected. The corrections due to \hat{V}_1 will appear at higher orders in $1/\omega$.

At the order $1/\omega$, the effective Hamiltonian obtained from Eqs. (44)–(48) is

$$\hat{G}_1 = i[\hat{F}_1, \hat{H}] - \hbar \frac{\partial \hat{F}_2}{\partial \tau} - \frac{i\hbar}{2} \left[\hat{F}_1, \frac{\partial \hat{F}_1}{\partial \tau} \right]. \quad (52)$$

Note that \hat{F}_1 , given by Eq. (50), depends only on the coordinate and therefore it commutes with its time derivative and also with \hat{V}_0 . If a periodic \hat{F}_2 can be chosen so that

$$\frac{\partial \hat{F}_2}{\partial \tau} = \frac{i}{\hbar} [\hat{F}_1, \hat{H}] = \frac{i}{\hbar} \left[\hat{F}_1, \frac{\hat{p}^2}{2m} \right], \quad (53)$$

then \hat{G}_1 vanishes. Indeed, by choosing

$$\hat{F}_2 = \frac{i}{2m} \int^{(2)\tau} [V_1''] + \frac{i}{m} \int^{(2)\tau} [V_1'] \frac{\partial}{\partial x}, \quad (54)$$

we obtain

$$\hat{G}_1 = 0. \quad (55)$$

We have presented \hat{F}_2 in the coordinate representation since the simple dependence of \hat{H} on the momentum makes it the most natural representation. We will use it also when calculating higher orders.

At the next order, ω^{-2} , \hat{G}_2 found from Eqs. (44)–(48) is

$$\begin{aligned} \hat{G}_2 = & i[\hat{F}_2, \hat{H}] - \frac{1}{2} [\hat{F}_1 [\hat{F}_1, \hat{H}]] - \hbar \frac{\partial \hat{F}_3}{\partial \tau} - \frac{i\hbar}{2} \left[\hat{F}_1, \frac{\partial \hat{F}_2}{\partial \tau} \right] \\ & - \frac{i\hbar}{2} \left[\hat{F}_2, \frac{\partial \hat{F}_1}{\partial \tau} \right] + \frac{\hbar}{6} \left[\hat{F}_1 \left[\hat{F}_1, \frac{\partial \hat{F}_1}{\partial \tau} \right] \right]. \end{aligned} \quad (56)$$

Substituting $\hat{H} = \hat{G}_0 + \hbar(\partial \hat{F}_1 / \partial \tau)$ and using Eq. (53) to eliminate the commutation relation $[\hat{F}_1, \hat{H}]$ results in

$$\hat{G}_2 = i[\hat{F}_2, \hat{G}_0] - \hbar \frac{\partial \hat{F}_3}{\partial \tau} + \frac{i\hbar}{2} \left[\hat{F}_2, \frac{\partial \hat{F}_1}{\partial \tau} \right]. \quad (57)$$

We can choose a periodic \hat{F}_3 in order to balance the time dependence of \hat{G}_2 . Note that \hat{G}_2 has some time independent part that cannot be canceled by a periodic \hat{F}_3 . Therefore we separate \hat{G}_2 into a τ independent part and a part that is periodic with vanishing average and choose \hat{F}_3 so that the latter vanishes [in Eq. (57)]. For this purpose, \hat{F}_3 must satisfy

$$\frac{\partial \hat{F}_3}{\partial \tau} = \frac{i}{\hbar} [\hat{F}_2, \hat{G}_0] + \frac{i}{2} \left(\left[\hat{F}_2, \frac{\partial \hat{F}_1}{\partial \tau} \right] - \overline{\left[\hat{F}_2, \frac{\partial \hat{F}_1}{\partial \tau} \right]} \right), \quad (58)$$

where \hat{G}_0 is given by Eq. (51) and an average over a period is denoted by bar. After some algebraic manipulations, \hat{F}_3 is found to be

$$\begin{aligned}\hat{F}_3 = & -\frac{\hbar}{m^2} \int^{(3)\tau} [V_1''] \frac{\partial^2}{\partial x^2} - \frac{\hbar}{m^2} \int^{(3)\tau} [V_1^{(3)}] \frac{\partial}{\partial x} \\ & - \frac{\hbar}{4m^2} \int^{(3)\tau} [V_1^{(4)}] - \frac{1}{m\hbar} V_0' \int^{(3)\tau} [V_1'] \\ & + \frac{1}{2m\hbar} \int^\tau [\mathcal{P}_1] + \hat{f}_3(\hat{x}, \hat{p}),\end{aligned}\quad (59)$$

where

$$\begin{aligned}\mathcal{P}_1(x, \tau) \equiv & im\hbar \left(\left[\hat{F}_2, \frac{\partial \hat{F}_1}{\partial \tau} \right] - \overline{\left[\hat{F}_2, \frac{\partial \hat{F}_1}{\partial \tau} \right]} \right) \\ = & V_1' \int^{(2)\tau} [V_1'] - V_1' \int^{(2)\tau} [V_1''].\end{aligned}\quad (60)$$

The constant of the integration over τ is the Hermitian operator \hat{f}_3 that depends only on \hat{x} and \hat{p} , and will be determined at the next order. We will use the freedom to choose

\hat{f}_3 to cause \hat{G}_3 to have a simple form. Using Eq. (59) in Eq. (57) will cancel the time dependent terms, resulting in

$$\hat{G}_2 = \frac{i\hbar}{2} \overline{\left[\hat{F}_2, \frac{\partial \hat{F}_1}{\partial \tau} \right]} = -\frac{1}{2m} V_1' \int^{(2)\tau} [V_1'] = \frac{1}{2m} \left(\int^\tau [V_1'] \right)^2, \quad (61)$$

where we have used integration by parts.

The calculation of \hat{G}_3 and \hat{G}_4 can be performed along similar lines. This calculation is tedious but straightforward, and only the main results are presented. Using the freedom in the choice of \hat{f}_3 , we choose it to satisfy

$$f_3'(x) = -\frac{1}{m\hbar} \overline{\int^\tau [V_1'] \int^{(2)\tau} [V_1'']}. \quad (62)$$

This choice for \hat{f}_3 leads to

$$\hat{G}_3 = 0. \quad (63)$$

Then \hat{F}_4 is found to satisfy

$$\begin{aligned}\hat{F}_4 = & -\frac{\hbar^2 i}{m^3} \int^{(4)\tau} [V_1^{(3)}] \frac{\partial^3}{\partial x^3} - \frac{3\hbar^2 i}{2m^3} \int^{(4)\tau} [V_1^{(4)}] \frac{\partial^2}{\partial x^2} - \frac{3\hbar^2 i}{4m^3} \int^{(4)\tau} [V_1^{(5)}] \frac{\partial}{\partial x} - \frac{\hbar^2 i}{8m^3} \int^{(4)\tau} [V_1^{(6)}] \\ & - \frac{i}{2m^2} \left(\int^\tau [\mathcal{P}_3'] + \int^\tau [\mathcal{P}_2'] \right) \\ & + 2 \int^\tau [\mathcal{P}_3] \frac{\partial}{\partial x} + 2 \int^\tau [\mathcal{P}_2] \frac{\partial}{\partial x} + \frac{i}{4m^2} \left(\int^{(2)\tau} [\mathcal{P}_1''] + 2 \int^{(2)\tau} [\mathcal{P}_1'] \frac{\partial}{\partial x} \right) + \hat{f}_4(\hat{x}, \hat{p}),\end{aligned}\quad (64)$$

where

$$\mathcal{P}_2(x, \tau) \equiv V_1' \int^{(3)\tau} [V_1''] - V_1' \int^{(3)\tau} [V_1'''], \quad \mathcal{P}_3(x, \tau) \equiv V_0'' \int^{(3)\tau} [V_1'] + 3V_0' \int^{(3)\tau} [V_1'']. \quad (65)$$

The time independent part of \hat{F}_4 is denoted by \hat{f}_4 . Using the freedom in the choice of gauge, we choose \hat{f}_4 so that in the classical limit the effective Hamiltonian \hat{G} reduces to its classical counterpart (30). To achieve this \hat{f}_4 is chosen to satisfy

$$\hat{f}_4 = \tilde{g}(x) \hat{p} + \hat{p} \tilde{g}(x) = 2 \frac{\hbar}{i} \tilde{g}(x) \frac{\partial}{\partial x} + \frac{\hbar}{i} \tilde{g}'(x), \quad (66)$$

where

$$\tilde{g}(x) = -\frac{3}{4m^2 \hbar} \overline{\int^{(2)\tau} [V_1'] \int^{(2)\tau} [V_1'']}. \quad (67)$$

This results in

$$\begin{aligned}\hat{G}_4 = & \frac{1}{2m^2} V_0'' \left(\int^{(2)\tau} [V_1'] \right)^2 + \frac{1}{2m^2} V_1'' \left(\int^{(2)\tau} [V_1'] \right)^2 - \frac{\hbar^2}{2m^3} \left\{ 3 \left(\int^{(2)\tau} [V_1''] \right)^2 \frac{\partial^2}{\partial x^2} + 6 \int^{(2)\tau} [V_1''] \int^{(2)\tau} [V_1^{(3)}] \frac{\partial}{\partial x} \right. \\ & \left. + \frac{3}{2} \int^{(2)\tau} [V_1''] \int^{(2)\tau} [V_1^{(4)}] + \frac{5}{4} \left(\int^{(2)\tau} [V_1^{(3)}] \right)^2 \right\}.\end{aligned}\quad (68)$$

This is the highest order of \hat{G} that is computed here.

The freedom in the choice of gauge was used here and the time independent parts of the \hat{F}_i were chosen in a specific way. Generally, this choice is arbitrary. In the present work, a choice was made so that in the classical limit the effective Hamiltonian reduces to the specific classical counterpart (30), which resulted in a natural way within the derivation of Sec. II. In Ref. [13], on the other hand, the choice of the time independent parts of the \hat{F}_i is made so that the average of the fast variables over a period reduces to the slow variables within an order ω^{-4} calculation. It is found there that with this choice, the requirement can be satisfied only if the oscillating force is independent of the position (in one dimension).

We have used a perturbation theory (in $1/\omega$) to obtain a periodic gauge transformation $e^{i\hat{F}}$ and an effective Hamiltonian \hat{G} so that the quasienergies are the eigenvalues of \hat{G} . Its eigenstates are related to the quasienergy states by Eq. (40). For a Hamiltonian of form Eq. (43), this effective Hamiltonian is given by Eqs. (44), (45), (51), (55), (61), (63) and (68). Collecting all contributions, one finds

$$\begin{aligned} \hat{G} = & \frac{\hat{p}^2}{2m} + \hat{V}_{eff} + \frac{1}{4\omega^4} (\hat{p}^2 g(x) + 2\hat{p}g(x)\hat{p} + g(x)\hat{p}^2) \\ & + \frac{\hbar^2}{\omega^4} \hat{V}_q + O(\omega^{-5}), \end{aligned} \quad (69)$$

where

$$\begin{aligned} V_{eff}(x) = & V_0(x) + \frac{1}{2m\omega^2} \left(\int_{-\tau}^{\tau} [V_1'] \right)^2 \\ & + \frac{1}{2m^2\omega^4} V_0'' \left(\int_{-\tau}^{\tau} [V_1'] \right)^2 \\ & + \frac{1}{2m^2\omega^4} V_1'' \left(\int_{-\tau}^{\tau} [V_1'] \right)^2 \end{aligned} \quad (70)$$

is the effective potential corresponding to Eq. (28),

$$g(x) = \frac{3}{2m^3} \left(\int_{-\tau}^{\tau} [V_1''] \right)^2 \quad (71)$$

is the coefficient of P^2 in Eq. (30), while

$$\hat{V}_q = \frac{1}{8m^3} \left(\int_{-\tau}^{\tau} [V_1^{(3)}] \right)^2 \quad (72)$$

is a quantum correction to the classical Hamiltonian [its form obviously depends on the ordering of operators in Eq. (69)].

The effective Hamiltonian is the main result of this section. The classical limit of Eq. (69) is the classical effective Hamiltonian (30). The freedom in the choice of gauge in the quantum problem was used, and \hat{f}_3 and \hat{f}_4 were chosen specifically to achieve this. We did not use the freedom of a

canonical transformation in the classical calculation. The specific canonical transformation from the Hamiltonian (43) to the Hamiltonian (30) is generated by the classical limit of $-\hbar\hat{F}$ with the specific choice of the time independent parts, which was made in the present work.

The perturbation theory that was developed here enables one to calculate not only the quasienergies but also the corresponding Floquet states. If the eigenfunctions of \hat{G} are known, then the quasienergy (or Floquet) states can be computed up to the order ω^{-4} using Eq. (40) with

$$\begin{aligned} \hat{F} = & \frac{1}{\hbar\omega} \int_{-\tau}^{\tau} [V_1] + \frac{i}{2m\omega^2} \int_{-\tau}^{(2)\tau} [V_1''] + \frac{i}{m\omega^2} \int_{-\tau}^{(2)\tau} [V_1'] \frac{\partial}{\partial x} \\ & + \frac{1}{\omega^3} \hat{F}_3 + \frac{1}{\omega^4} \hat{F}_4 + O(\omega^{-5}), \end{aligned} \quad (73)$$

where \hat{F}_3 is given by Eqs. (59), (60), and (62) while \hat{F}_4 is given by Eqs. (64), (65), (66), and (67).

For the driven harmonic oscillator where $V_0(x) = \frac{1}{2}m\omega_0^2x^2$ and $V_1(x, \tau) = \mathcal{E}x \cos(\tau)$ (69)–(72) yield

$$\hat{G} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_0^2x^2 + \frac{\mathcal{E}^2}{4m\omega^2} + \frac{\mathcal{E}^2}{4m\omega^2} \frac{\omega_0^2}{\omega^2} + O(\omega^{-5}). \quad (74)$$

This simple model is exactly solvable. The effective Hamiltonian of this model was calculated in Ref. [13]. It is given by

$$\hat{G} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2x^2 + \frac{\mathcal{E}^2}{4m(\omega^2 - \omega_0^2)}. \quad (75)$$

The expansion of the exact result (75) to the fourth order (in $1/\omega$) leads to Eq. (74) as expected.

V. DYNAMICS OF SPINS IN TIME DEPENDENT MAGNETIC FIELDS

A simple example that demonstrates the methods presented in this work is a spin in a field, which is a combination of a static and a periodic time dependent magnetic field. This example demonstrates that also Hamiltonians that are not of the form $\hat{p}^2/2m + \hat{V}(x)$ can be treated in the way presented in Secs. III and IV. The systems that are considered here consist of a spin in a constant magnetic field combined with a perpendicular periodic field linear or with circular polarization. The Hamiltonian for the linearly polarized field is given by

$$\hat{H}_l = -\omega_0 \hat{I}_z + \omega_1 \cos(\omega t) \hat{I}_x, \quad (76)$$

while for the circularly polarized field it is

$$\hat{H}_c = -\omega_0 \hat{I}_z + \omega_1 (\cos(\omega t) \hat{I}_x + \sin(\omega t) \hat{I}_y). \quad (77)$$

For a spin in a circularly polarized field, the problem was solved exactly by Rabi [37]. This system also appears in

textbooks as a paradigm of time dependent two-level systems [38]. Our goal is to demonstrate that for this simple system, the quasienergies can be computed exactly using the method presented in Secs. III and IV. First, we derive some results that are valid for any Hamiltonian linear in the spin operators.

These spin problems turn out to be simple since the spin operators have a closed algebra,

$$[\hat{I}_x, \hat{I}_y] = i\hbar \hat{I}_z, \quad [\hat{I}_y, \hat{I}_z] = i\hbar \hat{I}_x, \quad [\hat{I}_z, \hat{I}_x] = i\hbar \hat{I}_y. \quad (78)$$

The effective Hamiltonian (44) is obtained with the help of the expansions in commutation relations (47) and (48). For a Hamiltonian and \hat{F} that are linear in the spin operators, these expansions can be summed. Consider a transformation generated by

$$\hat{F} = A(\tau)\hat{I}_x + B(\tau)\hat{I}_y + C(\tau)\hat{I}_z, \quad (79)$$

where $\tau = \omega t$ and A , B , and C are the real functions of time. Let \hat{Q} be an arbitrary operator which is linear in \hat{I}_i . A straightforward calculation shows that

$$[\hat{F}, [\hat{F}, [\hat{F}, \hat{Q}]]] = \alpha^2 [\hat{F}, \hat{Q}] \quad (80)$$

with

$$\alpha = \hbar \sqrt{A^2 + B^2 + C^2}. \quad (81)$$

Therefore for any Hamiltonian linear in the spin operators any commutation relation in Eq. (47) can be reduced to $[\hat{F}, \hat{H}]$ or to $[\hat{F}, [\hat{F}, \hat{H}]]$, and the series is given by

$$\begin{aligned} e^{i\hat{F}} \hat{H} e^{-i\hat{F}} &= \hat{H} - \frac{1}{2!} [\hat{F}, [\hat{F}, \hat{H}]] + \frac{1}{4!} \alpha^2 [\hat{F}, [\hat{F}, \hat{H}]] \\ &\quad - \frac{1}{6!} \alpha^4 [\hat{F}, [\hat{F}, \hat{H}]] + \dots + i [\hat{F}, \hat{H}] \\ &\quad - \frac{i}{3!} \alpha^2 [\hat{F}, \hat{H}] + \frac{i}{5!} \alpha^4 [\hat{F}, \hat{H}] + \dots \\ &= \hat{H} + \frac{\cos \alpha - 1}{\alpha^2} [\hat{F}, [\hat{F}, \hat{H}]] + \frac{\sin \alpha}{\alpha} i [\hat{F}, \hat{H}]. \end{aligned} \quad (82)$$

The operator \hat{F} of Eq. (79) is linear in the spin operators and, therefore, such is also $\partial \hat{F} / \partial \tau$. In a similar manner, Eq. (48) can be summed to

$$\begin{aligned} \left(\frac{\partial \hat{F}}{\partial \tau} \right) e^{-i\hat{F}} &= i \frac{\partial \hat{F}}{\partial \tau} + \frac{\sin \alpha - \alpha}{\alpha^3} i \left[\hat{F}, \left[\hat{F}, \frac{\partial \hat{F}}{\partial \tau} \right] \right] \\ &\quad + \frac{\cos \alpha - 1}{\alpha^2} \left[\hat{F}, \frac{\partial \hat{F}}{\partial \tau} \right]. \end{aligned} \quad (83)$$

The Hamiltonian in the new gauge is thus given by

$$\begin{aligned} \hat{G} &= \hat{H} + \frac{\cos \alpha - 1}{\alpha^2} [\hat{F}, [\hat{F}, \hat{H}]] + \frac{\sin \alpha}{\alpha} i [\hat{F}, \hat{H}] - \hbar \omega \frac{\partial \hat{F}}{\partial \tau} \\ &\quad + \hbar \omega \frac{\cos \alpha - 1}{\alpha^2} i \left[\hat{F}, \frac{\partial \hat{F}}{\partial \tau} \right] - \hbar \omega \frac{\sin \alpha - \alpha}{\alpha^3} \left[\hat{F}, \left[\hat{F}, \frac{\partial \hat{F}}{\partial \tau} \right] \right]. \end{aligned} \quad (84)$$

The problem of finding the effective Hamiltonian is thus reduced to finding three functions of time, i.e., $A(\tau)$, $B(\tau)$, and $C(\tau)$ so that Eq. (84) is time independent. Equation (84) is valid for any Hamiltonian which is linear in the spin operators. Therefore, the problem is reduced to the solution of three coupled nonlinear differential equations, that is a well-defined mathematical problem. Generally, this may be hard to do since \hat{G} is not linear in terms of these functions. We turn now to examine the simplest case, i.e., of a circularly polarized field (77).

For the spin in a circularly polarized field, a perturbative solution in powers of $1/\omega$ for \hat{F} and \hat{G} can be found. The computation is done exactly as the one in Sec. IV. Thus only a brief outline of the calculation is presented. At the order ω^0 ,

$$\hat{G}_0 = \hat{H} - \hbar \frac{\partial \hat{F}_1}{\partial \tau} = -\omega_0 \hat{I}_z + \omega_1 (\cos \tau \hat{I}_x + \sin \tau \hat{I}_y) - \hbar \frac{\partial \hat{F}_1}{\partial \tau} \quad (85)$$

and therefore

$$\hat{F}_1 = \frac{\omega_1}{\hbar} (\sin \tau \hat{I}_x - \cos \tau \hat{I}_y) \quad (86)$$

and

$$\hat{G}_0 = -\omega_0 \hat{I}_z. \quad (87)$$

Note that here $[\hat{F}_1, (\partial \hat{F}_1 / \partial \tau)] \neq 0$, which changes some of the expressions obtained in Sec. IV.

At order ω^{-1} a straightforward calculation leads to

$$\hat{G}_1 = -\hbar \omega_0 \frac{\partial \hat{F}_1}{\partial \tau} - \hbar \frac{\partial \hat{F}_2}{\partial \tau} - \frac{\omega_1^2}{2} \hat{I}_z, \quad (88)$$

which results in

$$\hat{F}_2 = -\omega_0 \hat{F}_1 = -\frac{\omega_0 \omega_1}{\hbar} (\sin \tau \hat{I}_x - \cos \tau \hat{I}_y), \quad (89)$$

while

$$\hat{G}_1 = -\frac{\omega_1^2}{2} \hat{I}_z. \quad (90)$$

A similar calculation at the next order leads to

$$\hat{F}_3 = \left(\omega_0^2 - \frac{\omega_1^2}{3} \right) \hat{F}_1 \quad (91)$$

and

$$\hat{G}_2 = \frac{1}{2} \omega_0 \omega_1^2 \hat{I}_z. \quad (92)$$

The expansions for \hat{F} and \hat{G} are obtained by collecting all the terms from Eqs. (85)–(92). These expansions are given by

$$\hat{G}_c = \left(-\omega_0 - \frac{\omega_1^2}{2\omega} + \frac{\omega_0 \omega_1^2}{2\omega^2} + O(\omega^{-3}) \right) \hat{I}_z \quad (93)$$

and by

$$\begin{aligned} \hat{F}_c &= \frac{\omega_1}{\hbar \omega} \left(1 - \frac{\omega_0}{\omega} + \frac{\omega_0^2}{\omega^2} - \frac{\omega_1^2}{3\omega^2} + O(\omega^{-3}) \right) \\ &\times (\sin \tau \hat{I}_x - \cos \tau \hat{I}_y), \end{aligned} \quad (94)$$

where the subscript c denotes that this result is obtained for the circularly polarized field.

An examination of Eq. (94) suggests that \hat{F}_c may have the exact form

$$\hat{F}_c = \frac{\alpha(\omega)}{\hbar} (\sin \tau \hat{I}_x - \cos \tau \hat{I}_y). \quad (95)$$

It turns out that \hat{F}_c of this form leads to a time independent Hamiltonian if α is chosen appropriately. Substituting \hat{F}_c and $\hat{H} = \hat{H}_c$ of Eq. (77) in Eq. (84) leads to

$$\begin{aligned} \hat{G}_c &= (\omega_1 \cos \alpha - \omega_0 \sin \alpha - \omega \sin \alpha) (\cos \tau \hat{I}_x + \sin \tau \hat{I}_y) \\ &+ (-\omega_0 \cos \alpha - \omega_1 \sin \alpha + \omega \cos \alpha + \omega) \hat{I}_z. \end{aligned} \quad (96)$$

The Hamiltonian \hat{G}_c is time independent if

$$\omega_1 \cos \alpha - \omega_0 \sin \alpha - \omega \sin \alpha = 0. \quad (97)$$

Solving for α the Hamiltonian in the new gauge, Eq. (96) reduces to

$$\hat{G}_c = (\omega - \sqrt{\omega_1^2 + (\omega_0 + \omega)^2}) \hat{I}_z, \quad (98)$$

and is time independent. The quasienergies of the spin in a circularly polarized field are the eigenvalues of Eq. (98). They are given by

$$E_s = (\omega - \sqrt{\omega_1^2 + (\omega_0 + \omega)^2}) \hbar s, \quad (99)$$

where $s = -S, -S+1, \dots, +S$ (S is the magnitude of the spin).

For spin $S=1/2$, not only the quasienergies but also the quasienergy states can be computed rather easily. The spin operator can be represented by the Pauli matrices

$$\hat{I}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{I}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{I}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (100)$$

The Hamiltonian is then given by

$$\hat{H}_c = \frac{\hbar}{2} \begin{pmatrix} -\omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & \omega_0 \end{pmatrix}. \quad (101)$$

The unitary transformation $\hat{U}_c = e^{-i\hat{F}_c}$, which transforms the eigenstates of the effective Hamiltonian (98) to the quasienergy states of Eq. (101), can be obtained by calculating the various powers of \hat{F}_c . For $S=1/2$,

$$\hat{F}_c = \frac{\alpha}{2i} \begin{pmatrix} 0 & -e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix}. \quad (102)$$

Since \hat{G}_c is proportional to \hat{I}_z its eigenstates are the eigenstates of \hat{I}_z . Thus, the quasienergy states of the Hamiltonian (101), corresponding to the quasienergies

$$E_{\pm} = \pm \frac{\hbar}{2} (\omega - \sqrt{\omega_1^2 + (\omega_0 + \omega)^2}), \quad (103)$$

are

$$u_{\pm} = \hat{U}_c \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \frac{\alpha}{2} \\ -e^{i\omega t} \sin \frac{\alpha}{2} \end{pmatrix} \quad (104)$$

and

$$u_{\pm} = \hat{U}_c \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-i\omega t} \sin \frac{\alpha}{2} \\ \cos \frac{\alpha}{2} \end{pmatrix}. \quad (105)$$

This is exactly the problem that was solved by Rabi [37], and is discussed in Ref. [38]. The physical quantity of interest is typically the number of spins that flip (if all spins are polarized initially), rather than the Floquet states. We note that the term $\sqrt{\omega_1^2 + (\omega_0 + \omega)^2}$ in the expression for the quasienergies is the Rabi frequency. It is the frequency of oscillations of these ‘‘spin flips.’’

For the spin in a linearly polarized field, a perturbative computation of \hat{G} leads to

$$\hat{G}_l = \left(-\omega_0 + \frac{\omega_0 \omega_1^2}{4\omega^2} + \frac{\omega_0^3 \omega_1^2}{4\omega^4} - \frac{\omega_0 \omega_1^4}{64\omega^4} + O(\omega^{-5}) \right) \hat{I}_z. \quad (106)$$

To the best of our knowledge, an exact expression for the quasienergies of this system is not known. If one substitutes \hat{F} of the form (79) in Eq. (84), the problem of finding the effective time independent Hamiltonian reduces to the problem of finding the three functions of time, $A(\tau)$, $B(\tau)$, and $C(\tau)$, so that the new Hamiltonian is time independent. These satisfy first-order nonlinear differential equations. Typically solutions to such equations exist but it is not easy to find them explicitly. It is possible to choose parameters so that also the exact \hat{G}_c is proportional to \hat{I}_z . The approximate effective Hamiltonian (106), which was obtained in a similar way as Eq. (93), can be compared to the previously pub-

lished results. While we are not aware of any $1/\omega$ expansion for the quasienergies, some expansions in the strength of the time dependent field have been published. If one examines, for instance, the expansion given by Eq. (2.10) and Appendix A of Ref. [39] (which is valid for spin $S=1/2$) and expands it in powers of $1/\omega$, one obtains the quasienergies of Eq. (106).

In this section, we have studied some problems involving spins in crossed constant and time dependent magnetic fields. We have shown that the perturbation theory presented in Sec. IV can be used for such systems. For a circularly polarized field, we were able to compute the quasienergies exactly, in agreement with the previously published results.

VI. SCATTERING FROM AN OSCILLATING GAUSSIAN POTENTIAL

The system considered in Sec. V is simple, in the sense that the spectrum of the effective Hamiltonian \hat{G} is discrete and simply related to the one of the time independent part of the original Hamiltonian \hat{H}_0 . Moreover, for this example also the eigenfunctions of these Hamiltonians are simply related. It is of interest to examine examples that are more complicated and where such simple relations cannot be found. In this section, we examine such a system, the oscillating Gaussian, where an additional difference is that the spectrum is continuous and one is interested in the scattering states.

Consider a system which consists of a particle that interacts with an oscillatory Gaussian potential. The Hamiltonian is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \gamma e^{-\beta x^2} \cos(\omega t). \quad (107)$$

The system is of interest for two reasons. First, when $x \rightarrow \infty$ the potential vanishes and therefore one expects to find scattering quasienergy states. Second, the average of the potential vanishes, namely $V_0(x)=0$, consequently any interesting effect is due to the rapidly oscillating potential. This system describes trapping by an oscillating field, a phenomenon that is of physical interest. The physical properties of this system and the numerical methods used to analyze it are discussed elsewhere [40]. Here we only state briefly the results that are related to the properties of the effective Hamiltonian.

The effective Hamiltonian (69), which corresponds to Eq. (107), is

$$\begin{aligned} \hat{G} = & \frac{\hat{p}^2}{2m} + \frac{\beta^2 \gamma^2 x^2}{m \omega^2} e^{-2\beta x^2} + \frac{3\beta^2 \gamma^2}{m^3 \omega^4} (1 - 2\beta x^2)^2 e^{-2\beta x^2} \hat{p}^2 \\ & - \frac{12i\hbar \gamma^2 \beta^2 x}{m^3 \omega^4} (2\beta x^2 - 1)(3 - 2\beta x^2) e^{-2\beta x^2} \hat{p} \\ & - \frac{\hbar^2 \beta^3 \gamma^2}{m^3 \omega^4} (-9 + 99\beta x^2 - 114\beta^2 x^4 + 44\beta^3 x^6) e^{-2\beta x^2} \\ & + O(\omega^{-5}). \end{aligned} \quad (108)$$

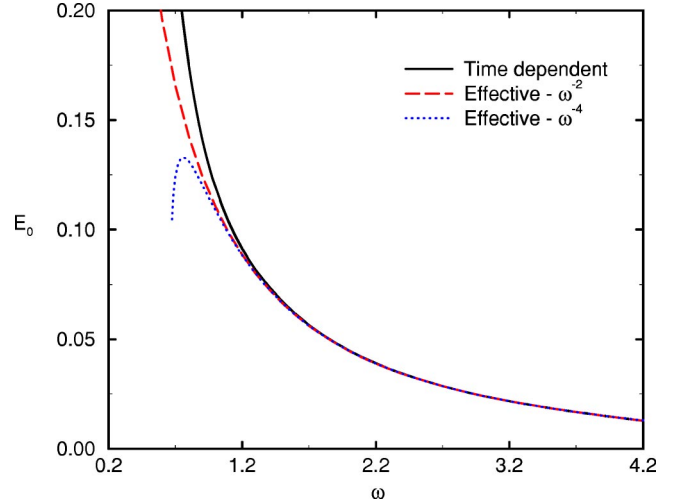


FIG. 1. The energy E_0 of the lowest quasienergy resonance of the Hamiltonian (107) as a function of the driving frequency (solid line), compared to the lowest resonance of the effective Hamiltonians (109)—dashed line, and (108)—dotted line, for $\gamma=9$ and $\beta=0.02$. The “atomic units” $m=\hbar=e=1$ are used here.

We examine separately the leading correction due to the oscillating field, which is given by the Hamiltonian

$$\hat{G}^{(2)} \equiv \hat{G}_0 + \hat{G}_1 + \hat{G}_2 = \frac{\hat{p}^2}{2m} + \frac{\beta^2 \gamma^2 x^2}{m \omega^2} e^{-2\beta x^2}, \quad (109)$$

where the error is of the order ω^{-4} . It has a simple physical meaning as the potential consists of a double barrier and the spectrum is continuous.

Since the effective potential of Eq. (109) is a double barrier one expects to find that this system exhibits resonances. These resonances describe long-lived unstable states. Each resonance is characterized by a complex energy $E - i\Gamma/2$. The real part E is the location of the resonance, while Γ is the width which is inversely proportional to the lifetime. For a review on the relevant properties of resonances and useful methods to compute them, see Ref. [41].

For any resonance of Eq. (108) and (109), it is natural to look for the corresponding resonance of the time dependent original Hamiltonian (107). More precisely, one looks for the resonances of the Floquet Hamiltonian (33) with \hat{H} of Eq. (107). This is done numerically by using a combination of the (t, t') method and complex scaling [41].

The energy E_0 and the width Γ_0 of the lowest (corresponding to the smallest real part E_0) quasienergy resonance of Eq. (107) are compared with the lowest resonance of the effective Hamiltonians (108) and (109) in Figs. 1 and 2. It is clear that for large frequencies there is an excellent agreement between the resonance of the time dependent Hamiltonian (107) and the ones of the effective Hamiltonians (108) and (109). At low frequencies, the location and width of the exact resonance differ from those of the effective Hamiltonian. The deviation for the order ω^{-4} Hamiltonian (108) is large indicating that the expansion is asymptotic. This is ex-

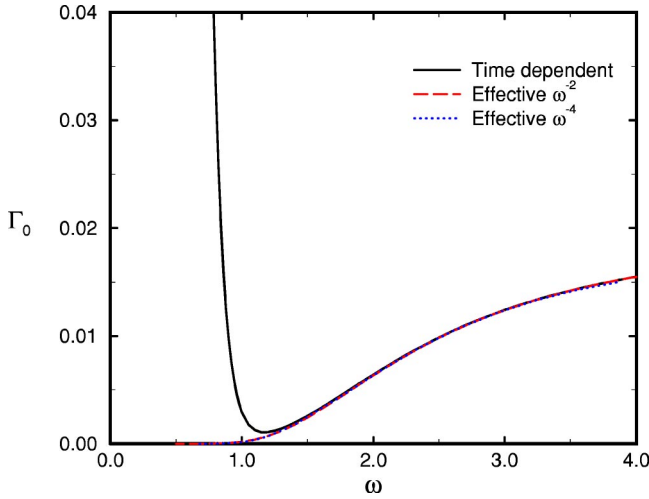


FIG. 2. Same as Fig. 1 for Γ_0 , the width of the lowest resonance (“atomic units” $m = \hbar = e = 1$).

pected since the perturbation theory developed in Sec. IV assumes high frequencies. A more complete study of this specific system and a discussion regarding the physical implications of this resonance are given in Ref. [40].

In this section, we demonstrated that the effective Hamiltonian \hat{G} can be used to obtain some physical properties of systems that are more complicated than those presented in Sec. V. In particular, the resonances of a periodic time dependent system were found to be given by the resonances of the corresponding time independent effective Hamiltonian. Resonances for oscillating barriers were computed numerically in Refs. [17,18]. The calculation of the present section demonstrates the physical origin of the results.

VII. SUMMARY AND DISCUSSION

In this paper we investigated the classical and quantum motion in high-frequency fields. The classical motion can be treated by separation of time scales. In Sec. II, this motion is separated into a slow part and a fast part, which consists of rapid oscillations around the slow part. The fast part and the resulting equation for the slow motion are solved perturbatively to the order ω^{-4} . This perturbation series is a generalization of the calculation presented by Landau and Lifshitz [7]. We also demonstrated that this perturbation theory is equivalent (to the order ω^{-4}) to the standard mathematical method of multiple time scales analysis [42], which is more complicated. The resulting equation for the slow motion is found to result from a time independent Hamiltonian.

Following a review of the Floquet theory in Sec. III, the corresponding effective quantum Hamiltonian is computed explicitly, using a high-frequency perturbation theory up to the order ω^{-4} , in Sec. IV. The resulting Hamiltonian (69) is rather simple. Its classical limit is the classical effective Hamiltonian (30). This effective Hamiltonian is therefore a generalization of the classical results of Kapitza [6] and of Landau and Lifshitz [7] to quantum mechanics. In the classical limit, the unitary gauge transformation $e^{-i\hat{F}}$ reduces to a canonical transformation generated by the classical limit of

$-\hbar\hat{F}$ [see Eq. (38) and Ref. [13]]. The Hamiltonian (30) for the slow variables was obtained in a natural way in Sec. II. The freedom in the choice of gauge was used to choose \hat{F} so that in the classical limit the effective quantum Hamiltonian (69) reduces to Eq. (30). Consequently, the classical limit of $-\hbar\hat{F}$ is the generating function of the canonical transformation from the original time dependent Hamiltonian to the time independent Hamiltonian (30). This limit explains the fact that the classical dynamics of the slow coordinate X is generated by a Hamiltonian. Using the freedom of choice of gauge, one can generate Hamiltonians that differ from Eq. (30) and (69), but are related to them by canonical and gauge transformations. The present work extends [13] to general driving potentials and is not restricted to the driving given by Eq. (6a) of Ref. [13]. The perturbation theory which was developed can, in principle, be used to compute it to any given order in $1/\omega$. This is a significant extension beyond [13] in the spirit of separation of time scales [42] that enables a systematic expansion in powers of ω^{-1} . For this, the requirement (23) of Ref. [13] is avoided and the expansion can be performed for any driving potential V_1 and is not restricted to driving forces that are uniform in space. It should be emphasized that this perturbation theory is an expansion in $1/\omega$ and *not* in powers of the time dependent potential. The potential V_1 does not have to be small in order to obtain a good approximation of the original system.

Several examples were discussed. The spin in a rotating magnetic field is a simple, exactly solvable example, which was used as a demonstration for the method. For another system, the oscillating Gaussian, we showed numerically that its lowest resonance is given by the resonance of the corresponding effective Hamiltonian (69) of Sec. VI. Thus, for time dependent traps, such as the atomic billiards discussed earlier, the time independent effective Hamiltonian can be used to compute the resonances and the lifetimes of particles in these traps.

While the examples presented in this work indicate that this effective potential is a meaningful concept and is also useful for calculations, there are points that require further research. The convergence properties of the $1/\omega$ expansions for \hat{F} and \hat{G} are not clear. There may be situations in which the perturbation theory fails to converge or where smaller than any power corrections are of physical importance. Consider for example a system similar to the oscillating Gaussian where V_1 vanishes outside some finite domain. In this case, one may expect to find scattering states as the eigenstates of \hat{G} . In the limit of high energies, such a state may be roughly approximated by the plane wave $e^{ipx/\hbar}$ (with large p). The leading momentum contributions to \hat{F} are of the form

$$\begin{aligned} & \frac{1}{\hbar\omega} \int^\tau [V_1] - \frac{1}{\hbar m \omega^2} \int^{(2)\tau} [V_1'] \hat{p} + \frac{1}{\hbar m^2 \omega^3} \int^{(3)\tau} [V_1''] \hat{p}^2 \\ & - \frac{1}{\hbar m^3 \omega^4} \int^{(4)\tau} [V_1^{(3)}] \hat{p}^3, \end{aligned}$$

as one can see from Eq. (73) with Eq. (59) and (64). While

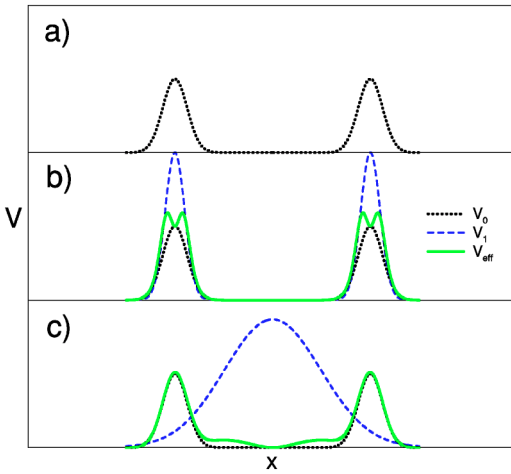


FIG. 3. Qualitative effect of driving on trapping times: (a) the average potential V_0 ; (b) the position dependence of the driving potential V_1 (dashed line) results in the effective potential V_{eff} (solid line); (c) as in (b) with driving mainly inside the trap. The units are arbitrary.

the general structure of the series for \hat{F} is unknown, it seems that for states with momentum which scales as ω^α with $\alpha > 1$ the series do not converge.

Even taking into account all the questions concerning the validity and the convergence of the perturbation expansions presented in this work, we find the effective potential to be useful for the following reasons. The perturbation theory leads to a time independent effective Hamiltonian. Physicists, who are used to working with the time independent systems, have developed an intuition for such systems, and thus the effective Hamiltonian may give physical insight that is absent when examining the corresponding time dependent problem. In addition, all the calculations used to obtain the effective Hamiltonian are straightforward. There are no differential equations or any complicated iterative schemes that may appear in more sophisticated perturbation theories. For comparison see Ref. [43]. Finally, one may use all the well-developed techniques for the time independent quantum systems to compute the eigenvalues of \hat{G} . In particular, one can use the time independent perturbation theory in the case where the eigenvalues and eigenstates of \hat{G}_0 are known.

The effective Hamiltonian (69) can be useful to predict the qualitative behavior without complicated numerical calculations. Assume first that the frequency is sufficiently high so that only terms of order up to $1/\omega^2$ should be included and denote $V_2(x) \equiv (1/2m\omega^2)(\int^x V_1')^2$. Let V_0 take the form depicted in Fig. 3(a). It exhibits resonances. A natural question is whether the linewidth increases or decreases as a result of the time dependent potential. It is clear that for a situation as in Fig. 3(b) the linewidth decreases, since the particle has to tunnel through the effective barriers which are higher than those of V_0 because the effective potential is $V_{eff} = V_0 + V_2$, while for Fig. 3(c) the linewidth (typically) increases since the energy of the resonances is shifted upwards by the time dependent perturbation. Numerical calculations of the type presented in Sec. VI and in Refs. [17,18] should confirm

these results. If the terms of the order ω^{-4} are important, more subtle considerations concerning the kinetic energy $(1/2m)\langle \hat{p}^2 \rangle$ and the semiclassical limit are required.

Since there are some physical situations where one expects that the perturbation theory fails, it is of importance to find ways to improve it. The goal is to be able to describe terms that are smaller than any power in the perturbations. This might be possible by using a superconvergent perturbation theory that was recently applied to the time dependent quantum systems [44]. In this perturbation theory, the small parameter is the size of the time dependent potential. It is of interest to modify it to a perturbation theory in $1/\omega$. Note that while these superconvergent perturbation schemes have superior convergence properties, they may turn out to be complicated for explicit calculations. Thus, one may lose the main advantage of the perturbation theory presented in this work, its simplicity. In addition, it is of interest to generalize the results of this work to systems of higher dimension. Such a generalization should be straightforward.

In conclusion, we have investigated the dynamics of high-frequency driven classical and quantum systems. A high-frequency perturbation theory was used to obtain an effective time independent Hamiltonian for the slow part of the classical and quantum motion. For quantum systems, the spectrum of this Hamiltonian is the quasienergy spectrum of the time dependent system. This effective Hamiltonian is computed in a high-frequency systematic perturbation theory. It is demonstrated that the effective Hamiltonian gives the exact quasienergies and quasienergy states of some simple examples as well as the lowest resonance (including the lifetime) for a time dependent atom trap.

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APPENDIX: MULTIPLE TIME SCALES ANALYSIS

The derivation given in Sec. II was, in some sense, not explicitly consistent. For example the slow motion was not solved by expanding its coordinates in orders of $1/\omega$. The terms of the order $1/\omega^n$ included all contributions up to that order, and also some contributions of higher order. It is of interest to show that the same result can be obtained using a standard method, namely the method of multiple time scales analysis [42]. In this appendix, we show how to derive the equations of motion of the slow dynamics using multiple time scales analysis. We will present only the first few orders in $1/\omega$ since this method turns out to be more complicated than the method used in Sec. II. While the two methods differ in details, they are equivalent and lead to the same results, when consistently expanded order by order in $1/\omega$.

In order to use the multiple time scales analysis, it is convenient to transform equation (1) to a standard form [42]. This can be done by defining $\tau \equiv \omega t$, $\epsilon \equiv 1/\omega$, and $y \equiv dx/dt = \omega dx/d\tau$. The first-order equations of motion are then given by

$$\frac{d}{d\tau} \vec{z} = \epsilon \vec{f}(\tau, \vec{z}), \quad (\text{A1})$$

where

$$\vec{z} = \begin{pmatrix} x \\ y \end{pmatrix} \quad (\text{A2})$$

and

$$\vec{f} = \begin{pmatrix} y \\ -\frac{1}{m}(V'_0(x) + V'_1(x, \tau)) \end{pmatrix}. \quad (\text{A3})$$

Then one introduces the following expansion of the solution:

$$\vec{z} = \sum_{n=0}^{\infty} \epsilon^n \vec{z}_n(\tau, t), \quad (\text{A4})$$

where $t = \epsilon\tau$ and τ are treated as independent variables. Using this expansion together with

$$\frac{d}{d\tau} = \frac{\partial}{\partial\tau} + \epsilon \frac{\partial}{\partial t}, \quad (\text{A5})$$

results in

$$\frac{\partial \vec{z}_0}{\partial\tau} + \epsilon \frac{\partial \vec{z}_0}{\partial t} + \epsilon \frac{\partial \vec{z}_1}{\partial\tau} + \dots = \epsilon \vec{f}(\tau, \vec{z}_0 + \epsilon \vec{z}_1 + \dots). \quad (\text{A6})$$

The solution is obtained by expanding \vec{f} , matching powers of ϵ , and solving for \vec{z} order by order. This is the standard multiple time scales analysis, see Ref. [42] for a detailed description of the method. (Note that in Ref. [42] the role of t and τ is opposite to the one in the present work.) We proceed to solve the first few orders in ϵ .

At order ϵ^0 , the leading order of Eq. (A3) results in

$$\begin{aligned} \frac{\partial x_0}{\partial\tau} &= 0, \\ \frac{\partial y_0}{\partial\tau} &= 0. \end{aligned} \quad (\text{A7})$$

Therefore x_0 and y_0 can be any functions of the slow time t :

$$\begin{aligned} x_0 &= \bar{x}_0(t), \\ y_0 &= \bar{y}_0(t). \end{aligned} \quad (\text{A8})$$

Note that at the leading order the solution is found to depend only on the slow time scale, as expected. We will denote the τ independent slow part of the solution by \bar{x}_i and \bar{y}_i at any

order. Additional conditions (A10) on \bar{x}_0 and \bar{y}_0 will be obtained from the requirement that the solution is not secular at the next order.

At order ϵ , Eq. (A3) results in

$$\begin{aligned} \frac{\partial x_1}{\partial\tau} &= -\frac{\partial \bar{x}_0}{\partial t} + \bar{y}_0(t), \\ \frac{\partial y_1}{\partial\tau} &= -\frac{\partial \bar{y}_0}{\partial t} - \frac{1}{m} V'_0(\bar{x}_0) - \frac{1}{m} V'_1(\bar{x}_0, \tau), \end{aligned} \quad (\text{A9})$$

where we have used Eq. (A8). We are interested in the solutions of Eq. (A9) which are not secular, namely, they do not grow with the fast time scale τ . The RHS of Eq. (A9) can be decomposed into periodic functions of τ (with vanishing average) and τ independent terms. Any τ independent term will result in a secular contribution. Therefore, to avoid such terms we demand

$$\begin{aligned} \frac{\partial \bar{x}_0}{\partial t} &= \bar{y}_0(t), \\ \frac{\partial \bar{y}_0}{\partial t} &= -\frac{1}{m} V'_0(\bar{x}_0(t)). \end{aligned} \quad (\text{A10})$$

This is the leading order of the equation which governs the slow time scale. The nonsecular equation of order ϵ is now

$$\begin{aligned} \frac{\partial x_1}{\partial\tau} &= 0, \\ \frac{\partial y_1}{\partial\tau} &= -\frac{1}{m} V'_1(\bar{x}_0, \tau) \end{aligned} \quad (\text{A11})$$

and its solution is

$$\begin{aligned} x_1(\tau, t) &= \bar{x}_1(t), \\ y_1(\tau, t) &= -\frac{1}{m} \int^{\tau} [V'_1(\bar{x}_0(t), \tau)] + \bar{y}_1(t). \end{aligned} \quad (\text{A12})$$

This process can be repeated order after order. At each order, all the terms with the same power of ϵ are gathered. The resulting equation may be secular and therefore an additional condition on the slow part of the solution is enforced. Then, one can solve for x and y at that order. The calculation is rather tedious and will not be presented here. We only give the secularity conditions that result when the next two orders are computed:

$$\begin{aligned} \frac{\partial \bar{x}_1}{\partial t} &= \bar{y}_1(t), \\ \frac{\partial \bar{y}_1}{\partial t} &= -\frac{1}{m} \bar{x}_1(t) V''_0(\bar{x}_0(t)), \end{aligned} \quad (\text{A13})$$

and

$$\frac{\partial \bar{x}_2}{\partial t} = \bar{y}_2(t),$$

$$\frac{\partial \bar{y}_2}{\partial t} = \frac{1}{m^2} \overline{V_1''(\bar{x}_0(t), \tau) \int^{(2)\tau} [V_1'(\bar{x}_0(t), \tau)] - \frac{1}{m} \bar{x}_2(t) V_0''(\bar{x}_0(t)) - \frac{\bar{x}_1^2(t)}{2m} V_0^{(3)}(\bar{x}_0(t))}. \quad (\text{A14})$$

Equations (A10), (A13), and (A14) are the first orders of an expansion of the following system of equations:

$$\frac{\partial \bar{X}}{\partial t} = \bar{Y}(t),$$

$$\frac{\partial \bar{Y}}{\partial t} = -\frac{1}{m} V_0'(\bar{X}(t)) + \frac{\epsilon^2}{m^2} \overline{V_1''(\bar{X}(t), \tau) \int^{(2)\tau} [V_1'(\bar{X}(t), \tau)]} \quad (\text{A15})$$

in powers of ϵ , where

$$\bar{X}(t) \equiv \bar{x}_0(t) + \epsilon \bar{x}_1(t) + \epsilon^2 \bar{x}_2(t) + \dots,$$

$$\bar{Y}(t) \equiv \bar{y}_0(t) + \epsilon \bar{y}_1(t) + \epsilon^2 \bar{y}_2(t) + \dots \quad (\text{A16})$$

Equations (A15) are accurate to the order ϵ^2 and are identical to the slow equation (22). We have used the multiple time scales analysis up to the order ϵ^4 and found that the resulting slow equations analogous to Eq. (A15) are equivalent to Eq. (27) in Sec. II, with the identification $\bar{X} = X$ and $\bar{Y} = \dot{X}$.

The method presented in Sec. II and the multiple time scales analysis used in this appendix have a similar structure. Both separate the motion into τ dependent and independent parts, and both lead to equations for the slow motion which have to be satisfied to avoid solutions that grow with τ . The main difference between the methods is that when using multiple time scales the slow coordinate is expanded in powers of $\epsilon = 1/\omega$. This leads to a large number of terms that result from the fact that the functions in the slow equation are evaluated at $\bar{x}_0(t)$ rather than at $\bar{X}(t)$. Consequently, the derivation in Sec. II is much simpler than the one presented in this appendix. This is also the reason that here we did not present explicitly the calculation using the multiple time scales analysis up to the order ϵ^4 .

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