Theory of the forces exerted by Laguerre-Gaussian light beams on dielectrics

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The classical theory of the electromagnetic field associated with paraxial Laguerre-Gaussian light is generalized to apply to propagation in a bulk dielectric, and the theory is quantized to obtain expressions for the electric and magnetic field operators. The forms of the Poynting vector and angular momentum density operators are derived and their expectation values for a single-photon wave packet are obtained. The Lorentz force operator in the dielectric is resolved into longitudinal, radial, and azimuthal components. The theory is extended to apply to an interface between two semi-infinite dielectric media, one of which is transparent with an incident single-photon pulse, and the other of which is weakly attenuating. For a pulse that is much shorter than the attenuation length, the theory can separately identify the surface and bulk contributions to the Lorentz force on the attenuating dielectric. Particular attention is given to the transfer of longitudinal and angular momentum to the dielectric from light incident from free space. The resulting expressions for the shift and rotation of a transparent dielectric slab are shown to agree with those obtained from Einstein box theories.

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I. INTRODUCTION

The main purpose of this paper is a calculation of the Lorentz forces on a dielectric produced by a pulse of Laguerre-Gaussian light. The calculation generalizes earlier work on the radiation pressure associated with a plane-wave pulse of uniform cross-sectional intensity [1], which produces only a longitudinal force on the dielectric in the direction of propagation of the light. Laguerre-Gaussian light also exerts a longitudinal force but there are additional transverse forces in the radial and azimuthal directions. The longitudinal force causes a radiation pressure when the pulse impinges on a dielectric surface, with a corresponding transfer of linear momentum from light to dielectric material. The radial, or gradient, force acts towards the radius of maximum intensity in the pulse cross section and it has a confining, or trapping, effect on dielectric particles immersed in a liquid. The azimuthal force causes a torque on the dielectric, with a corresponding transfer of angular momentum from pulse to dielectric. The longitudinal and azimuthal forces find practical applications in the optical tweezers and spanners used to position and rotate biological specimens for examination under a microscope. The azimuthal force is of strong current interest [2–4].

The calculation presented here uses the simplest possible optical and dielectric systems that preserve the characteristic properties of the three force components and illustrate their natures and magnitudes. Thus the Laguerre-Gaussian mode function of zero radial index is assumed, with a beam waist much larger than the wavelength, and the evaluations are restricted to longitudinal positions close to the focus. The pulse is assumed to contain a single photon so that the various quantities are conveniently normalized per photon. The expectation values of interest are determined first for a lossless isotropic dielectric of infinite extent and secondly for a semi-infinite weakly attenuating dielectric with a flat surface. The propagation direction of the pulse in the latter system is assumed to be perpendicular to the surface and the weak attenuation ensures that the pulse within the dielectric never reaches any exit or reflecting surface. The pulse length is taken to be much smaller than the attenuation distance so that the calculation can separate the contributions to the forces from the passage of the pulse through the surface and from its subsequent attenuation in the bulk dielectric. Particular attenuation is given to the longitudinal and azimuthal forces, as they determine the transfers of linear and angular momentum to the dielectric. The magnitudes of these momenta have been a topic of controversy for some time [5] and, as in previous work [1], the calculations are restricted here to determinations of the Lorentz forces, without any assumptions of the photon momenta in dielectrics.

The classical properties of Laguerre-Gaussian light are summarized in Sec. II and the theory is extended to cover propagation within an infinite lossless dielectric medium. The theory is quantized in Sec. III to provide expressions for the electric and magnetic field operators in the infinite dielectric. The normalization of the field operators is verified by determination of the expectation value of the Poynting vector operator for a single-photon pulse. The angular momentum density operator is constructed in Sec. IV. The Lorentz force operator for an infinite isotropic dielectric is resolved into longitudinal, radial, and azimuthal components in Sec. V. The theory is generalized in Sec. VI to apply to an interface between two semi-infinite dielectric media, one of which is transparent, with an incident single-photon pulse, and the other of which is weakly attenuating. The surface and bulk contributions to the Lorentz force on the attenuating dielectric for a pulse incident from free space are calculated in Sec. VII. Particular attention is given to the transfers of longitudinal and angular momentum from the light to the dielectric. The conclusions of the work are discussed in Sec. VIII and the predicted shift and rotation of a dielectric slab are shown to agree with the results of simple Einstein box theories.

II. PARAXIAL MODE FUNCTIONS AND ELECTROMAGNETIC FIELDS IN DIELECTRICS

The positive-frequency part of the Lorentz-gauge vector potential in a material with real refractive index $\eta(\omega)$ for

Laguerre-Gaussian light in the form of a paraxial wave of frequency ω that travels in the positive z direction is

$$\mathbf{A}_L^+(\mathbf{r},t)=A_0(\alpha\tilde{\mathbf{x}}+\beta\tilde{\mathbf{y}})u_{k,l}(\mathbf{r})\exp(-i\omega t+ikz), \quad (2.1)$$

where A_0 is a complex amplitude, $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are unit vectors,

$$k=\eta(\omega)\omega/c \quad \text{and} \quad |\alpha|^2+|\beta|^2=1. \quad (2.2)$$

We consider only the simplest form of mode function, for a radial index $p=0$, given approximately by

$$u_{k,l}(\mathbf{r})=\frac{1}{\sqrt{\pi|l|!}}\left(\frac{\sqrt{2}}{w_0}\right)^{|l|+1}\rho^{|l|}\exp\left\{-\frac{\rho^2}{w_0^2}+\frac{ikz\rho^2}{2z_R^2}+il\phi-i(|l|+1)\frac{z}{z_R}\right\}, \quad l=0,\pm 1,\pm 2,..., \quad (2.3)$$

where $\mathbf{r}=(\rho,\phi,z)$ in cylindrical polar coordinates and z_R is the Rayleigh range, related to the beam waist w_0 by

$$z_R=\tfrac{1}{2}kw_0^2. \quad (2.4)$$

The z coordinate is measured from the position of the waist and the form (2.3) is valid when $z\ll z_R$ so that terms involving $(z/z_R)^2$ and higher powers can be neglected. The mode function is normalized,

$$\int_0^\infty d\rho\int_0^{2\pi}d\phi\,\rho|u_{k,l}(\mathbf{r})|^2=1, \quad (2.5)$$

and it satisfies the paraxial wave equation

$$\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+2ik\frac{\partial}{\partial z}\right\}u_{k,l}(\mathbf{r})=\left\{\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+2ik\frac{\partial}{\partial z}\right\}u_{k,l}(\mathbf{r})=0, \quad (2.6)$$

where the paraxial assumption

$$|\partial u_{k,l}(\mathbf{r})/\partial z|\ll ku_{k,l}(\mathbf{r}) \quad (2.7)$$

or

$$k\gg 1/w_0 \quad (2.8)$$

applies. In words, the beam waist is assumed to be much larger than its wavelength. In considering the paraxial equation after substitution of the approximate solution (2.3), the definition (2.4) and the limit $z\ll z_R$ are used. The above relations reduce to those for Laguerre-Gaussian modes in free space [6,7] for $\eta(\omega)=1$, except that several previous papers have the wrong signs for the second and fourth terms in the exponent in Eq. (2.3) when the form $\exp(-i\omega t)$ is used for the time dependence. These two terms in the exponent are needed for the mode function to satisfy the paraxial wave equation but they do not contribute to most of the quantities calculated below.

The strength of the mode function is given by

$$|u_{k,l}(\mathbf{r})|^2=\frac{2^{|l|+1}}{\pi|l|!w_0^{2(|l|+1)}}\rho^{2|l|}\exp(-2\rho^2/w_0^2), \quad (2.9)$$

independent of ϕ and z. The mode clearly has zero strength on the z axis at $\rho=0$, except for $l=0$, and its peak strength occurs at the radial coordinate ρ_0 given by

$$\rho_0^2=\tfrac{1}{2}|l|w_0^2. \quad (2.10)$$

The coefficients in the polarization factor in Eq. (2.1) can be used to form the spin angular-momentum quantum number of the beam [8],

$$\sigma=i(\alpha\beta^*-\alpha^*\beta), \quad (2.11)$$

which takes the values ± 1 for right and left circularly polarized light and 0 for linearly polarized light. The mode function (2.3) has the property

$$\frac{\partial u_{k,l}(\mathbf{r})}{\partial\phi}=ilu_{k,l}(\mathbf{r}) \quad (2.12)$$

and l is identified as the orbital angular-momentum quantum number of the beam [2,9]. These contributions to the angular momentum of the beam, defined relative to its direction of propagation, are considered in Sec. IV.

Maxwell's equations for the positive-frequency fields in a homogeneous and isotropic dielectric are

$$\nabla\cdot\mathbf{E}^+=\mathbf{0}, \quad \nabla\times\mathbf{E}^+=i\omega\mathbf{B}^+ \quad (2.13)$$

and

$$\nabla\cdot\mathbf{B}^+=\mathbf{0}, \quad \nabla\times\mathbf{B}^+=-i\omega\frac{\eta^2}{c^2}\mathbf{E}^+. \quad (2.14)$$

The gauge condition on the vector and scalar potentials in a dielectric takes the form

$$\nabla\cdot\mathbf{A}_L(\mathbf{r},t)+\frac{\eta^2}{c^2}\frac{\partial\varphi_L(\mathbf{r},t)}{\partial t}=0, \tag{2.15}$$

and in free space, where $\eta=1$, this becomes the usual Lorentz gauge condition. The positive-frequency part of the scalar potential is [10]

$$\varphi_L^+(\mathbf{r},t)=-\frac{ic^2}{\eta^2\omega}\nabla\cdot\mathbf{A}_L^+(\mathbf{r},t)=-\frac{i\omega}{k^2}\nabla\cdot\mathbf{A}_L^+(\mathbf{r},t). \tag{2.16}$$

The electric and magnetic fields are obtained from the potentials as

$$\mathbf{E}^+(\mathbf{r},t)=-\frac{\partial\mathbf{A}_L^+(\mathbf{r},t)}{\partial t}-\nabla\varphi_L^+(\mathbf{r},t)=A_0\frac{c}{\eta}\bigg\{ik(\alpha\tilde{\mathbf{x}}+\beta\tilde{\mathbf{y}})u-\left(\alpha\frac{\partial u}{\partial x}+\beta\frac{\partial u}{\partial y}\right)\tilde{\mathbf{z}}\bigg\}\exp(-i\omega t+ikz) \tag{2.17}$$

and

$$\mathbf{B}^+(\mathbf{r},t)=\nabla\times\mathbf{A}_L^+(\mathbf{r},t)=A_0\bigg\{-ik(\beta\tilde{\mathbf{x}}-\alpha\tilde{\mathbf{y}})u+\left(\beta\frac{\partial u}{\partial x}-\alpha\frac{\partial u}{\partial y}\right)\tilde{\mathbf{z}}\bigg\}\exp(-i\omega t+ikz), \tag{2.18}$$

where u is shorthand for $u_{k,l}(\mathbf{r})$. These field expressions neglect terms in each component that are smaller than those retained in accordance with the paraxial assumption in Eq. (2.7) or Eq. (2.8). The z components are smaller than the x and y components by a factor of order $1/kw_0$. It is readily verified that the fields (2.17) and (2.18) satisfy Maxwell's equations (2.13) and (2.14). The Cartesian derivatives are converted to polar ρ and ϕ derivatives in the usual way.

III. QUANTIZED PARAXIAL FIELDS AND POYNTING VECTOR

The field operators corresponding to the classical fields in Eqs. (2.17) and (2.18) are obtained by the usual quantization procedure [11] as

$$\hat{\mathbf{E}}^+(\mathbf{r},t)=\int_0^\infty d\omega\left(\frac{\hbar}{4\pi\varepsilon_0 c\eta^3\omega}\right)^{1/2}\hat{a}(\omega)\exp\left[-i\omega\left(t-\frac{\eta z}{c}\right)\right]\times\left\{i\eta\omega(\alpha\tilde{\mathbf{x}}+\beta\tilde{\mathbf{y}})u-c\left(\alpha\frac{\partial u}{\partial x}+\beta\frac{\partial u}{\partial y}\right)\tilde{\mathbf{z}}\right\} \tag{3.1}$$

and

$$\hat{\mathbf{B}}^+(\mathbf{r},t)=\int_0^\infty d\omega\left(\frac{\hbar}{4\pi\varepsilon_0 c^3\eta\omega}\right)^{1/2}\hat{a}(\omega)\exp\left[-i\omega\left(t-\frac{\eta z}{c}\right)\right]\times\left\{-i\eta\omega(\beta\tilde{\mathbf{x}}-\alpha\tilde{\mathbf{y}})u+c\left(\beta\frac{\partial u}{\partial x}-\alpha\frac{\partial u}{\partial y}\right)\tilde{\mathbf{z}}\right\}, \tag{3.2}$$

where $\hat{a}(\omega)$ is the photon destruction operator, with commutation relation

$$[\hat{a}(\omega),\hat{a}^\dagger(\omega')]=\delta(\omega-\omega'). \tag{3.3}$$

These quantized field expressions reduce to those for a uniform plane wave of transverse cross section A in a dielectric [12] when the mode function u is replaced by $1/\sqrt{A}$. They satisfy the operator Maxwell equations similar to Eqs. (2.13) and (2.14) and their normalization is shown below to give the correct representation for the energy in a single-photon wave packet.

The expressions (3.1) and (3.2) generalize the classical fields (2.17) and (2.18) to excitations that include a spread of frequencies, as is needed to represent a wave packet. Thus, a single-photon pulse is represented by the state vector [11,12]

$$|1\rangle=\int d\omega\,\xi(\omega)\hat{a}^\dagger(\omega)|0\rangle, \tag{3.4}$$

where $|0\rangle$ is the vacuum state. Use of the photon operator commutation relation shows that the single-photon state vector satisfies

$$\hat{a}(\omega)|1\rangle=\xi(\omega)|0\rangle. \tag{3.5}$$

The function $\xi(\omega)$, which describes the spectrum of the photon wave packet, is normalized,

$$\int d\omega|\xi(\omega)|^2=1, \tag{3.6}$$

and a simple choice is the narrowband Gaussian function of spatial length L,

$$\xi(\omega)=\left(\frac{L^2}{2\pi c^2}\right)^{1/4}\exp\left\{-\frac{L^2(\omega-\omega_0)^2}{4c^2}\right\},\quad c/L\ll\omega_0. \tag{3.7}$$

The narrow spectrum ensures that ω can be set equal to the central frequency ω_0 of the wave packet in such quantities as the Rayleigh range (2.4).

The normal-order Poynting vector operator is

$$:\hat{\mathbf{S}}(\mathbf{r},t):=\varepsilon_0 c^2\{\hat{\mathbf{E}}^-(\mathbf{r},t)\times\hat{\mathbf{B}}^+(\mathbf{r},t)-\hat{\mathbf{B}}^-(\mathbf{r},t)\times\hat{\mathbf{E}}^+(\mathbf{r},t)\}, \tag{3.8}$$

with z component

$$:\hat{S}_z(\mathbf{r},t):=\frac{\hbar}{4\pi}\int_0^\infty d\omega\int_0^\infty d\omega'(\omega\omega')^{1/2}\frac{\eta+\eta'}{(\eta\eta')^{1/2}}\hat{a}^\dagger(\omega)\hat{a}(\omega')\times\exp\left[i(\omega-\omega')t-i(\eta\omega-\eta'\omega')\frac{z}{c}\right]u^*u', \quad (3.9)$$

where u' is shorthand for $u_{k',l}(\mathbf{r})$ with $k'=\eta(\omega')\omega'/c$ and it is assumed that the coefficients α and β are independent of ω for frequencies within the excitation spectrum $\xi(\omega)$. There are also contributions to the Poynting vector operator, omitted from Eq. (3.9), that contain integrands with terms in $\hat{a}(\omega)\hat{a}(\omega')$ and $\hat{a}^\dagger(\omega)\hat{a}^\dagger(\omega')$. The expectation values of these terms vanish for the single-photon states defined in Eqs. (3.4)–(3.6). For more general states in which these expectation values do not vanish, their contributions oscillate at optical frequencies and they can be neglected.

The expression (3.9) simplifies, when integrated over all time, to give

$$\int_{-\infty}^\infty dt:\hat{S}_z(\mathbf{r},t):=\int_0^\infty d\omega\,\hbar\omega\hat{a}^\dagger(\omega)\hat{a}(\omega)|u|^2. \quad (3.10)$$

Use of the normalization condition (2.5) gives the integrated energy flow over the complete mode cross section as

$$\int_0^\infty d\rho\int_0^{2\pi}d\phi\,\rho\int_{-\infty}^\infty dt:\hat{S}_z(\mathbf{r},t):=\int_0^\infty d\omega\,\hbar\omega\hat{a}^\dagger(\omega)\hat{a}(\omega). \quad (3.11)$$

This expression shows the expected form of the total energy flow past each point on the z axis as the quantum $\hbar\omega$ weighted by the photon-number operator at frequency ω, and it justifies the normalization of the field operators (3.1) and (3.2). Similarly, the integration of the Poynting vector operator over the entire z axis gives

$$\int_{-\infty}^\infty dz:\hat{S}_z(\mathbf{r},t):=\int_0^\infty d\omega\,\hbar\omega\nu_G(\omega)\hat{a}^\dagger(\omega)\hat{a}(\omega)|u|^2 \quad (3.12)$$

and integration over the mode cross section gives the integrated energy flow over all space as

$$\int d\mathbf{r}:\hat{S}_z(\mathbf{r},t):=\int_0^\infty d\omega\,\hbar\omega\nu_G(\omega)\hat{a}^\dagger(\omega)\hat{a}(\omega). \quad (3.13)$$

The contribution of each frequency component to the flow is thus weighted by the appropriate group velocity $\nu_G(\omega)$, defined by

$$\frac{1}{\nu_G(\omega)}=\frac{\partial(\eta(\omega)\omega/c)}{\partial\omega}. \quad (3.14)$$

The Poynting vector x component operator is

$$:\hat{S}_x(\mathbf{r},t):=\frac{i\hbar c}{4\pi}\int_0^\infty d\omega\int_0^\infty d\omega'\frac{\hat{a}^\dagger(\omega)\hat{a}(\omega')}{(\eta\eta'\omega\omega')^{1/2}}\exp\left[i(\omega-\omega')t-i(\eta\omega-\eta'\omega')\frac{z}{c}\right]\left\{-\omega\left(\frac{\eta}{\eta'}|\alpha|^2+|\beta|^2\right)u^*\frac{\partial u'}{\partial x}+\omega\left(\alpha\beta^*-\frac{\eta}{\eta'}\alpha^*\beta\right)u^*\frac{\partial u'}{\partial y}+\omega'\left(\frac{\eta'}{\eta}|\alpha|^2+|\beta|^2\right)\frac{\partial u^*}{\partial x}u'+\omega'\left(\frac{\eta'}{\eta}\alpha\beta^*-\alpha^*\beta\right)\frac{\partial u^*}{\partial y}u'\right\} \quad (3.15)$$

and the expression for the y component is the same but with the exchanges of symbols $x\leftrightarrow y$ and $\alpha\leftrightarrow\beta$. The z component exceeds the x and y components by a factor of order kw_0. The resulting time-integrated energy flows in the x and y directions are

$$\int_{-\infty}^\infty dt:\hat{S}_x(\mathbf{r},t):=\int_0^\infty d\omega\frac{\hbar c}{2\eta}\hat{a}^\dagger(\omega)\hat{a}(\omega)\left\{i\left(\frac{\partial u^*}{\partial x}u-u^*\frac{\partial u}{\partial x}\right)+\sigma\frac{\partial|u|^2}{\partial y}\right\} \quad (3.16)$$

and

$$\int_{-\infty}^\infty dt:\hat{S}_y(\mathbf{r},t):=\int_0^\infty d\omega\frac{\hbar c}{2\eta}\hat{a}^\dagger(\omega)\hat{a}(\omega)\left\{i\left(\frac{\partial u^*}{\partial y}u-u^*\frac{\partial u}{\partial y}\right)-\sigma\frac{\partial|u|^2}{\partial x}\right\}. \quad (3.17)$$

It is straightforward to perform the differentiations in these expressions by conversion to polar coordinates and use of the mode function (2.3). The terms all contain either $\sin\phi$ or $\cos\phi$, so that integration over the mode cross section as in Eq. (3.11) gives

$$\int_0^\infty d\rho\int_0^{2\pi}d\phi\,\rho\int_{-\infty}^\infty dt:\hat{S}_x(\mathbf{r},t):=\int_0^\infty d\rho\int_0^{2\pi}d\phi\,\rho\int_{-\infty}^\infty dt:\hat{S}_y(\mathbf{r},t):=0. \quad (3.18)$$

The results (3.16) and (3.17) are expressed more compactly in terms of radial and azimuthal components of the Poynting vector, defined by

$$\hat{S}_\rho=\hat{S}_x\cos\phi+\hat{S}_y\sin\phi \quad \text{and} \quad \hat{S}_\phi=-\hat{S}_x\sin\phi+\hat{S}_y\cos\phi. \quad (3.19)$$

The time-integrated energy flows in the radial and azimuthal directions are thus found to be

$$\int_{-\infty}^{\infty} dt{:}\hat{S}_\rho(\mathbf{r},t){:} = i\int_0^\infty d\omega \frac{\hbar c}{2\eta}\hat{a}^\dagger(\omega)\hat{a}(\omega)\left(\frac{\partial u^*}{\partial\rho}u - u^*\frac{\partial u}{\partial\rho}\right) = \int_0^\infty d\omega \frac{\hbar c}{\eta}\hat{a}^\dagger(\omega)\hat{a}(\omega)\frac{4\rho z}{kw_0^4}|u|^2 \tag{3.20}$$

and

$$\int_{-\infty}^{\infty} dt{:}\hat{S}_\phi(\mathbf{r},t){:} = \int_0^\infty d\omega \frac{\hbar c}{\eta}\hat{a}^\dagger(\omega)\hat{a}(\omega)\left(\frac{l}{\rho}|u|^2 - \frac{\sigma}{2}\frac{\partial|u|^2}{\partial\rho}\right) = \int_0^\infty d\omega \frac{\hbar c}{\eta}\hat{a}^\dagger(\omega)\hat{a}(\omega)\left(\frac{l}{\rho} - \frac{\sigma|l|}{\rho} + \frac{2\sigma\rho}{w_0^2}\right)|u|^2. \tag{3.21}$$

This azimuthal component is smaller than the time-integrated z component in Eq. (3.10) by a factor of order $1/kw_0$, while the radial component has an additional reduction factor z/z_R. The spin-dependent terms in the azimuthal component cancel at $\rho=\rho_0$, given by Eq. (2.10).

The relative magnitudes of the three time-integrated Poynting-vector components in Eqs. (3.10), (3.20), and (3.21) agree with the results of a classical calculation [13]. It is seen that the effect of the dielectric with $\eta>1$ is to reduce the transverse components of the Poynting vector by the factor $1/\eta$ relative to the longitudinal component. The energy in a Laguerre-Gaussian beam in a dielectric therefore flows more closely parallel to the z axis than in a similar beam in free space.

The expectation values of all of the operators in the present section for the single-photon pulse defined in Eq. (3.4) are given by the same expressions as the operators themselves except for the replacements $\hat{a}^\dagger(\omega)\rightarrow\xi^*(\omega)$ and $\hat{a}(\omega')\rightarrow\xi(\omega')$. The integrals in the expectation values are difficult to evaluate because of the dispersion in $\eta(\omega)$, whose main role in integrations over z is the provision of an additional factor $\nu_G(\omega)$ relative to integrations over the time, as in Eqs. (3.10) and (3.12). If the dispersion is ignored, the expectation value of the z-component Poynting vector operator (3.9), for example, can be written

$$\langle 1|{:}\hat{S}_z(\mathbf{r},t){:}|1\rangle = \frac{\hbar}{2\pi}\left|\int_0^\infty d\omega\, \omega^{1/2}\xi(\omega) \times\exp\left[-i\omega\left(t-\frac{\eta z}{c}\right)\right]u\right|^2. \tag{3.22}$$

In the expression (2.3) for u, ω appears only in the k factor in the second term of the exponent and this term is negligible compared to the kz term in the exponent of Eq. (3.22). Thus u can be taken outside the integral and insertion of the Gaussian spectrum from Eq. (3.7) then gives

$$\langle 1|{:}\hat{S}_z(\mathbf{r},t){:}|1\rangle = \frac{\hbar\omega_0 c}{L}\left(\frac{2}{\pi}\right)^{1/2}\exp\left[-\frac{2c^2}{L^2}\left(t-\frac{\eta z}{c}\right)^2\right]|u|^2, \tag{3.23}$$

where the narrow spectrum justifies the approximation of replacing $\omega^{1/2}$ in the integrand by $\omega_0^{1/2}$. The peak of the wave packet at time t lies at position $z=ct/\eta$. The time integral of Eq. (3.23) is

$$\int_{-\infty}^{\infty} dt\langle 1|{:}\hat{S}_z(\mathbf{r},t){:}|1\rangle = \hbar\omega_0|u|^2 \tag{3.24}$$

and the normalization in Eq. (2.5) confirms the total energy content $\hbar\omega_0$ of the photon wave packet.

IV. ANGULAR MOMENTUM DENSITY

The form of the linear momentum of the electromagnetic field in a dielectric is a matter of debate and controversy [5], and this uncertainty applies equally to the form of the angular momentum. The effective values of these momenta are discussed in Secs. VII and VIII in terms of their transfers to an attenuating dielectric. We follow here the conventional approach [2,9], with the angular momentum of the electromagnetic field per unit volume defined as

$$\mathbf{j} = \varepsilon_0\mathbf{r}\times(\mathbf{E}\times\mathbf{B}) = \mathbf{r}\times\mathbf{S}/c^2, \tag{4.1}$$

where $\mathbf{S}$ is the Poynting vector. Thus the z component of the normally ordered angular-momentum density operator is

$$:\hat{j}_z: = (x{:}\hat{S}_y{:} - y{:}\hat{S}_x{:})/c^2 = \rho{:}\hat{S}_\phi{:}/c^2, \tag{4.2}$$

where Eq. (3.19) has been used. Substitution into the first form in Eq. (4.2) from Eq. (3.15) and the corresponding expression for the y component gives a complicated expression for $:\hat{j}_z:$.

If the dispersion in the refractive index is again ignored, and with a transformation to cylindrical polar coordinates, the expression simplifies greatly to

$$:\hat{j}_z(\mathbf{r},t): = \frac{\hbar}{4\pi c\eta}\int_0^\infty d\omega\int_0^\infty d\omega'(\omega\omega')^{1/2}\hat{a}^\dagger(\omega)\hat{a}(\omega')\exp\left[i(\omega-\omega')\left(t-\frac{\eta z}{c}\right)\right] \times\left\{l\frac{\omega+\omega'}{\omega\omega'}u^*u' - \sigma\rho\left(\frac{1}{\omega}\frac{\partial u^*}{\partial\rho}u' + \frac{1}{\omega'}u^*\frac{\partial u'}{\partial\rho}\right)\right\}, \tag{4.3}$$

where the property (2.12) has been used. The time-integrated angular-momentum density operator is accordingly

$$\int_{-\infty}^{\infty} dt : \hat{j}_z(\mathbf{r},t) := \int_0^{\infty} d\omega \frac{\hbar}{2c\,\eta(\omega)} \hat{a}^{\dagger}(\omega)\hat{a}(\omega) \left\{ 2l|u|^2 - \sigma\rho \frac{\partial |u|^2}{\partial \rho} \right\}, \tag{4.4}$$

where the dispersion in the refractive index can be restored in the integrated expression. This is the quantum-mechanical version of the classical expression for the angular-momentum density [9]. Note the different spatial variations of the orbital and spin contributions, with the former proportional to the mode strength and the latter to its radial gradient. The spin thus contributes only in the presence of a transverse variation in $|u|^2$ and it vanishes at $\rho=\rho_0$ [14].

The square modulus (2.9) of the mode function (2.3) has the property

$$\rho \frac{\partial |u_{k,l}(\mathbf{r})|^2}{\partial \rho} = 2\left(|l| - \frac{2\rho^2}{w_0^2} \right) |u_{k,l}(\mathbf{r})|^2 = 2|l||u_{k,l}(\mathbf{r})|^2 - 2(|l|+1)|u_{k,l+1}(\mathbf{r})|^2, \tag{4.5}$$

so that Eq. (4.4) can be written

$$\int_{-\infty}^{\infty} dt : \hat{j}_z(\mathbf{r},t) := \int_0^{\infty} d\omega \frac{\hbar}{c\,\eta(\omega)} \hat{a}^{\dagger}(\omega)\hat{a}(\omega) \left(l - \sigma|l| + \frac{2\sigma\rho^2}{w_0^2} \right) |u|^2. \tag{4.6}$$

This result agrees with the form of time-integrated azimuthal Poynting vector in Eq. (3.21) and its relation (4.2) to the angular-momentum density. Integration over the complete mode cross section with use of the second expression on the right of Eq. (4.5) gives

$$\int_0^{\infty} d\rho \int_0^{2\pi} d\phi\, \rho \int_{-\infty}^{\infty} dt : \hat{j}_z(\mathbf{r},t) := \frac{\hbar(l+\sigma)}{c} \int_0^{\infty} d\omega \frac{1}{\eta(\omega)} \hat{a}^{\dagger}(\omega)\hat{a}(\omega). \tag{4.7}$$

The analogous spatial integral of Eq. (4.3) gives

$$\int d\mathbf{r} : \hat{j}_z(\mathbf{r},t) := \hbar(l+\sigma) \int_0^{\infty} d\omega \frac{v_G(\omega)}{c\,\eta(\omega)} \hat{a}^{\dagger}(\omega)\hat{a}(\omega), \tag{4.8}$$

where the group velocity is defined in Eq. (3.14). The free-space orbital angular momentum $\hbar l$ and spin angular momentum $\hbar\sigma$ are thus affected in the same way by immersion in a dielectric medium.

V. FORCE OPERATOR

The force exerted on the dielectric by the light beam at position $\mathbf{r}$ is determined by the Lorentz force-density operator, defined as a sum of electric and magnetic contributions [15]

$$\hat{\mathbf{f}}(\mathbf{r},t) = \hat{\mathbf{f}}^{E}(\mathbf{r},t) + \hat{\mathbf{f}}^{B}(\mathbf{r},t) = [\hat{\mathbf{P}}(\mathbf{r},t)\cdot\nabla]\hat{\mathbf{E}}(\mathbf{r},t) + \frac{\partial \hat{\mathbf{P}}(\mathbf{r},t)}{\partial t} \times \hat{\mathbf{B}}(\mathbf{r},t). \tag{5.1}$$

The polarization $\hat{\mathbf{P}}(\mathbf{r},t)$ is expressed in terms of the electric field operator via the dielectric function, equal to the square of the refractive index, and its form is obtained from Eq. (3.1) as

$$\hat{\mathbf{P}}^{+}(\mathbf{r},t) = \int_0^{\infty} d\omega \left(\frac{\varepsilon_0 \hbar}{4\pi c\,\eta^3 \omega} \right)^{1/2} (\eta^2 - 1)\hat{a}(\omega) \exp\left[-i\omega\left(t - \frac{\eta z}{c} \right) \right] \left\{ i\eta\omega(\alpha\tilde{\mathbf{x}} + \beta\tilde{\mathbf{y}})u - c\left(\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} \right)\tilde{\mathbf{z}} \right\}. \tag{5.2}$$

The total force on the dielectric at time t is represented by the force operator

$$\hat{\mathbf{F}}(t) = \int d\mathbf{r}\, \hat{\mathbf{f}}(\mathbf{r},t). \tag{5.3}$$

It is convenient to separate the force density into longitudinal, radial, and azimuthal components.

A. Longitudinal component

The x, y, and z components of the force-density operator are straightforwardly but tediously obtained by substitution of Eqs. (3.1), (3.2), and (5.2) into Eq. (5.1). For the z component, it is found that the electric contribution in Eq. (5.1) is smaller than the magnetic contribution by a factor of order $(kw_0)^{-2}$ and the former can be neglected in view of Eq. (2.8). The normally ordered magnetic contribution alone gives

$$:\hat{f}_z(\mathbf{r},t):=\frac{i\hbar}{4\pi c^2}\int_0^\infty d\omega\int_0^\infty d\omega'\left(\frac{\omega\omega'}{\eta\eta'}\right)^{1/2}\hat{a}^\dagger(\omega)\hat{a}(\omega')\exp\left[i(\omega-\omega')t-i(\eta\omega-\eta'\omega')\frac{z}{c}\right]$$
$$\times[\omega\eta'(\eta^2-1)-\omega'\eta(\eta'^2-1)]u^*u'. \tag{5.4}$$

This expression reduces to that for the force-density operator for a uniform plane wave in a dispersionless dielectric [1] if η is assumed independent of frequency and u is replaced by $1/\sqrt{A}$. In general, the normally ordered part of the total force operator (5.3) vanishes,

$$:\hat{F}_z(t):=\int d\mathbf{r}:\hat{f}_z(\mathbf{r},t):=0, \tag{5.5}$$

and the time integral of Eq. (5.4) also vanishes,

$$\int_{-\infty}^{\infty}dt:\hat{f}_z(\mathbf{r},t):=0. \tag{5.6}$$

With dispersion neglected, Eq. (5.4) can be written in the form

$$:\hat{f}_z(\mathbf{r},t):=\frac{\hbar(\eta^2-1)}{4\pi c^2}\frac{\partial}{\partial t}\int_0^\infty d\omega\int_0^\infty d\omega'(\omega\omega')^{1/2}\hat{a}^\dagger(\omega)\hat{a}(\omega')\exp\left[i(\omega-\omega')\left(t-\frac{\eta z}{c}\right)\right]u^*u'. \tag{5.7}$$

The expectation value of this operator for the single-photon wave packet is obtained by the same method as used for the Poynting vector in Eqs. (3.22) and (3.23) with the result

$$\langle 1|:\hat{f}_z(\mathbf{r},t):|1\rangle=-\frac{2\hbar\omega_0 c(\eta^2-1)}{L^3}\left(\frac{2}{\pi}\right)^{1/2}\left(t-\frac{\eta z}{c}\right)\exp\left[-\frac{2c^2}{L^2}\left(t-\frac{\eta z}{c}\right)^2\right]|u|^2. \tag{5.8}$$

The distribution of the force density in the xy plane is determined by the mode strength $|u|^2$. Integration of Eq. (5.8) over the plane gives the same total z force as derived previously [1] for a uniform plane wave. The force density is antisymmetric around the peak of the pulse, with positive values in front and negative values behind [1,15]. The dielectric experiences a local stretching force that travels with the pulse, centered on its peak, but the spatially integrated force vanishes, in accordance with Eq. (5.5).

B. Radial component

The electric and magnetic contributions to the x and y components of the force-density operator (5.1) have the same order of magnitude, being smaller than the z component calculated above by a factor of order $(kw_0)^{-1}$, and both must be retained. The resulting expressions are quite complicated and it is acceptable to simplify them by neglect of dispersion in the refractive index. It is convenient to give results, not for the x and y components, but for the radial and azimuthal components defined by

$$\hat{f}_\rho=\hat{f}_x\cos\phi+\hat{f}_y\sin\phi \quad \text{and} \quad \hat{f}_\phi=-\hat{f}_x\sin\phi+\hat{f}_y\cos\phi, \tag{5.9}$$

similar to the Poynting vector decomposition in Eq. (3.19). The radial component is

$$:\hat{f}_\rho(\mathbf{r},t):=\frac{\hbar(\eta^2-1)}{4\pi c\eta}\int_0^\infty d\omega\int_0^\infty d\omega'(\omega\omega')^{1/2}\hat{a}^\dagger(\omega)\hat{a}(\omega')\exp\left[i(\omega-\omega')\left(t-\frac{\eta z}{c}\right)\right]$$
$$\times\left\{\frac{\omega'}{\omega}\frac{\partial u^*}{\partial\rho}u'+\frac{\omega}{\omega'}u^*\frac{\partial u'}{\partial\rho}-\sigma l\frac{(\omega-\omega')^2}{\omega\omega'}\frac{u^*u'}{\rho}\right\} \tag{5.10}$$

and its time integral is

$$\int_{-\infty}^{\infty}dt:\hat{f}_\rho(\mathbf{r},t):=\frac{\hbar(\eta^2-1)}{c\eta}\left(\frac{|l|}{\rho}-\frac{2\rho}{w_0^2}\right)|u|^2\int_0^\infty d\omega\,\omega\hat{a}^\dagger(\omega)\hat{a}(\omega'), \tag{5.11}$$

where the ρ derivative of $|u|^2$ is taken from Eq. (4.5).

The expectation value of Eq. (5.10) for the single-photon wave packet is

$$\langle 1|:\hat{f}_\rho(\mathbf{r},t):|1\rangle=\frac{\hbar\omega_0(\eta^2-1)}{\sqrt{2\pi}\eta L}\left(\frac{\partial}{\partial\rho}+\frac{\sigma l}{\omega_0^2\rho}\frac{\partial^2}{\partial t^2}\right)\left\{\exp\left[-\frac{2c^2}{L^2}\left(t-\frac{\eta z}{c}\right)^2\right]|u|^2\right\}. \tag{5.12}$$

The time-derivative contribution is very much smaller than that of the radial derivative on account of the inequality in Eq. (3.7) and it can safely be neglected to give

$$\langle 1|:\hat{f}_{\rho}(\mathbf{r},t):|1\rangle = \frac{2\hbar\omega_0(\eta^2-1)}{\sqrt{2\pi}\,\eta L}\left(\frac{|l|}{\rho}-\frac{2\rho}{w_0^2}\right)\exp\left[-\frac{2c^2}{L^2}\left(t-\frac{\eta z}{c}\right)^2\right]|u|^2. \tag{5.13}$$

The radial force density is independent of the spin quantum number σ and it is localized within the pulse. It vanishes at the radius ρ_0 defined in Eq. (2.10), it is positive for $\rho<\rho_0$, and negative for $\rho>\rho_0$. The radial force thus compresses the dielectric towards the ring of radius ρ_0.

C. Azimuthal component

Use of the definition in Eq. (5.9) gives

$$:\hat{f}_{\phi}(\mathbf{r},t):=\frac{\hbar(\eta^2-1)}{4\pi c\,\eta}\int_0^\infty d\omega\int_0^\infty d\omega'(\omega\omega')^{1/2}\hat{a}^\dagger(\omega)\hat{a}(\omega')\exp\left[i(\omega-\omega')\left(t-\frac{\eta z}{c}\right)\right]i(\omega-\omega')$$
$$\times\left\{l\,\frac{\omega+\omega'}{\omega\omega'}\,\frac{u^*u'}{\rho}-\sigma\left(\frac{1}{\omega}\frac{\partial u^*}{\partial\rho}u'+\frac{1}{\omega'}u^*\frac{\partial u'}{\partial\rho}\right)\right\}. \tag{5.14}$$

The angular-momentum density operator in Eq. (4.3) and the azimuthal force density satisfy a form of continuity equation

$$\rho:\hat{f}_{\phi}(\mathbf{r},t):+(\eta^2-1)\frac{\partial}{\partial t}:\hat{j}_z(\mathbf{r},t):=0. \tag{5.15}$$

The time integral of the azimuthal force density vanishes,

$$\int_{-\infty}^{\infty}dt:\hat{f}_{\phi}(\mathbf{r},t):=0. \tag{5.16}$$

The expectation value of the azimuthal force density for the single-photon wave packet is

$$\langle 1|:\hat{f}_{\phi}(\mathbf{r},t):|1\rangle = -\frac{4\hbar c^2(\eta^2-1)}{\eta L^3}\left(\frac{2}{\pi}\right)^{1/2}\left(t-\frac{\eta z}{c}\right)\exp\left[-\frac{2c^2}{L^2}\left(t-\frac{\eta z}{c}\right)^2\right]\left(\frac{l}{\rho}-\frac{\sigma|l|}{\rho}+\frac{2\sigma\rho}{w_0^2}\right)|u|^2. \tag{5.17}$$

The azimuthal force density thus shares the property of the longitudinal component (5.8) of having balanced positive and negative values in the front and rear of the pulse. The pulse thus carries a localized twisting force but there is zero total azimuthal force on the bulk dielectric.

VI. REFLECTION FROM DIELECTRIC SURFACE

Suppose now that space is divided into two regions with a dielectric of real refractive index $\eta_0(\omega)$ at $z<0$ and a dielectric of complex refractive index

$$n(\omega)=\eta(\omega)+i\kappa(\omega) \tag{6.1}$$

at $z>0$. The theory outlined so far applies to the dielectric at $z<0$ with η replaced by η_0 and the real wave vector k replaced by

$$k_0=\eta_0(\omega)\omega/c. \tag{6.2}$$

The complex wave vector at $z>0$ is denoted

$$k=n(\omega)\omega/c=[\eta(\omega)+i\kappa(\omega)]\omega/c. \tag{6.3}$$

The dielectric at $z>0$ cannot in reality be of infinite extent but its thickness is assumed to be much larger than the characteristic attenuation distance $c/\omega\kappa(\omega)$, so that there is no need to allow for waves reflected from its right-hand boundary. In addition, the attenuation distance is assumed to be much longer than the pulse length L, so that surface and bulk effects resulting from the entry of the pulse into the dielectric can be separated.

Suppose that an incident Laguerre-Gaussian beam, whose electric and magnetic field operators are given by Eqs. (3.1) and (3.2) with the above replacements, impinges normally on the dielectric interface from the left. The amplitudes of the reflected and transmitted beams are determined by the boundary conditions at $z=0$ in the usual way. Solutions for the same problem but with an incident transverse plane wave have been given previously [1,16,17]. The incident beam considered here has field components in all three coordinate directions and it is necessary to apply both tangential and normal boundary conditions. The x and y components of $\hat{\mathbf{E}}^{+}(\mathbf{r},t)$ retain the same sign in the reflected beam but the z component changes sign, while the x and y components of $\hat{\mathbf{B}}^{+}(\mathbf{r},t)$ change sign but the z component retains the same sign in the reflected beam. The field operators at $z<0$ are thus found to be

$$\hat{\mathbf{E}}^{+}(\mathbf{r},t)=\int_0^\infty d\omega\left(\frac{\hbar}{4\pi\varepsilon_0 c\,\eta_0^3\omega}\right)^{1/2}\hat{a}(\omega)\exp(-i\omega t)\left\{\left[i\eta_0\omega(\alpha\tilde{\mathbf{x}}+\beta\tilde{\mathbf{y}})u_{k_0}-c\left(\alpha\frac{\partial u_{k_0}}{\partial x}+\beta\frac{\partial u_{k_0}}{\partial y}\right)\tilde{\mathbf{z}}\right]\exp\left(i\frac{\eta_0\omega z}{c}\right)\right.$$
$$\left.+R(\omega)\left[i\eta_0\omega(\alpha\tilde{\mathbf{x}}+\beta\tilde{\mathbf{y}})u_{-k_0}+c\left(\alpha\frac{\partial u_{-k_0}}{\partial x}+\beta\frac{\partial u_{-k_0}}{\partial y}\right)\tilde{\mathbf{z}}\right]\exp\left(-i\frac{\eta_0\omega z}{c}\right)\right\} \tag{6.4}$$

and

$$\hat{\mathbf{B}}^{+}(\mathbf{r},t)=\int_0^\infty d\omega\left(\frac{\hbar}{4\pi\varepsilon_0 c^3\eta_0\omega}\right)^{1/2}\hat{a}(\omega)\exp(-i\omega t)\left\{\left[-i\eta_0\omega(\beta\tilde{\mathbf{x}}-\alpha\tilde{\mathbf{y}})u_{k_0}+c\left(\beta\frac{\partial u_{k_0}}{\partial x}-\alpha\frac{\partial u_{k_0}}{\partial y}\right)\tilde{\mathbf{z}}\right]\exp\left(i\frac{\eta_0\omega z}{c}\right)\right.$$
$$\left.+R(\omega)\left[i\eta_0\omega(\beta\tilde{\mathbf{x}}-\alpha\tilde{\mathbf{y}})u_{-k_0}+c\left(\beta\frac{\partial u_{-k_0}}{\partial x}-\alpha\frac{\partial u_{-k_0}}{\partial y}\right)\tilde{\mathbf{z}}\right]\exp\left(-i\frac{\eta_0\omega z}{c}\right)\right\}, \tag{6.5}$$

where the subscripts on the mode function u defined in Eq. (2.3) distinguish the incident and reflected beams and

$$R(\omega)=\frac{\eta_0(\omega)-n(\omega)}{\eta_0(\omega)+n(\omega)} \tag{6.6}$$

is the usual reflection coefficient at the interface.

The corresponding field operators at $z>0$ are

$$\hat{\mathbf{E}}^{+}(\mathbf{r},t)=\int_0^\infty d\omega\left(\frac{\hbar}{4\pi\varepsilon_0 c\,\eta_0\omega}\right)^{1/2}\hat{a}(\omega)\exp\left[-i\omega\left(t-\frac{n(\omega)z}{c}\right)\right]T(\omega)\left\{i\omega(\alpha\tilde{\mathbf{x}}+\beta\tilde{\mathbf{y}})u_k-\frac{c}{n(\omega)}\left(\alpha\frac{\partial u_k}{\partial x}+\beta\frac{\partial u_k}{\partial y}\right)\tilde{\mathbf{z}}\right\} \tag{6.7}$$

and

$$\hat{\mathbf{B}}^{+}(\mathbf{r},t)=\int_0^\infty d\omega\left(\frac{\hbar}{4\pi\varepsilon_0 c^3\eta_0\omega}\right)^{1/2}\hat{a}(\omega)\exp\left[-i\omega\left(t-\frac{n(\omega)z}{c}\right)\right]T(\omega)\left\{-in(\omega)\omega(\beta\tilde{\mathbf{x}}-\alpha\tilde{\mathbf{y}})u_k+c\left(\beta\frac{\partial u_k}{\partial x}-\alpha\frac{\partial u_k}{\partial y}\right)\tilde{\mathbf{z}}\right\}, \tag{6.8}$$

where the occurrence of η_0 in the square-root factors is a consequence of the boundary conditions and

$$T(\omega)=\frac{2\eta_0(\omega)}{\eta_0(\omega)+n(\omega)} \tag{6.9}$$

is the usual transmission coefficient at the interface. Continuity of energy flow at the interface is ensured by the relation

$$\eta_0[1-|R(\omega)|^2]=\eta|T(\omega)|^2. \tag{6.10}$$

It is not difficult to verify that the standard tangential and normal boundary conditions are all satisfied by the field operators in Eqs. (6.4), (6.5), (6.7), and (6.8). The forms of the field operators reduce to those given in [1] when the incident light beam is a uniform plane wave. The complete field operators also contain noise contributions associated with the material loss [16,17], but these are omitted from the above expressions as they do not contribute to the quantities calculated here.

The Poynting vector operator in the dielectric at $z>0$ is obtained in the same way as that in Sec. III but with the field operators in Eqs. (3.1) and (3.2) replaced by those in Eqs. (6.7) and (6.8). Thus the z component operator in Eq. (3.9) is replaced by

$$:\hat{S}_z(\mathbf{r},t):=\frac{\hbar}{4\pi\eta_0}\int_0^\infty d\omega\int_0^\infty d\omega'(\omega\omega')^{1/2}\hat{a}^\dagger(\omega)\hat{a}(\omega')\exp\left[i(\omega-\omega')t-i(n^*\omega-n'\omega')\frac{z}{c}\right]T^*(\omega)T(\omega')(n'+n^*)u^*u'. \tag{6.11}$$

The time-integrated Poynting vector is

$$\int_{-\infty}^{\infty} dt{:}\hat{S}_z(\mathbf{r},t){:} = \int_0^{\infty} d\omega\, \hbar\omega \hat{a}^{\dagger}(\omega)\hat{a}(\omega)\exp\left(-\frac{-2\omega\kappa z}{c}\right)\frac{4\eta_0\eta}{(\eta_0+\eta)^2+\kappa^2}|u|^2 \tag{6.12}$$

and the expectation value of this expression for the narrowband single-photon pulse is

$$\int_{-\infty}^{\infty} dt\langle 1|{:}\hat{S}_z(\mathbf{r},t){:}|1\rangle = \hbar\omega_0 \exp\left(-\frac{2\omega_0\kappa z}{c}\right)\frac{4\eta_0\eta}{(\eta_0+\eta)^2+\kappa^2}|u|^2, \tag{6.13}$$

where η_0, η, and κ are evaluated at frequency ω_0. The average total energy of the pulse that enters the dielectric at $z=0$ is therefore

$$\int_0^{\infty} d\rho \int_0^{2\pi} d\phi\, \rho \int_{-\infty}^{\infty} dt\langle 1|{:}\hat{S}_z(\rho,\phi,0;t){:}|1\rangle = \hbar\omega_0 \frac{4\eta_0\eta}{(\eta_0+\eta)^2+\kappa^2}. \tag{6.14}$$

This radiative energy is all transferred to the dielectric as the light beam is totally attenuated in accordance with the exponential decay factor in Eq. (6.13). Note that no energy enters the dielectric at $z>0$ via the x and y components of the Poynting vector on account of a property analogous to that given for the infinite homogeneous dielectric in Eq. (3.18).

VII. RADIATION FORCE AND TORQUE ON FREE-SPACE SURFACE

It is a simple matter to calculate the total transfers of linear and angular momentum to a lossy dielectric by a single-photon wave packet incident normally on its surface from free space. The free-space photon has a well-defined linear momentum $\hbar\omega_0/c$ and its reflection from the dielectric surface with coefficient R given by Eq. (6.6) with $\eta_0=1$ produces a momentum transfer

$$\frac{\hbar\omega_0}{c}[1+|R(\omega_0)|^2] = \frac{2\hbar\omega_0}{c}\,\frac{\eta^2+1+\kappa^2}{(\eta+1)^2+\kappa^2}, \tag{7.1}$$

where η and κ are again evaluated at frequency ω_0. This expression is valid for the assumed conditions in which there is no reemergence of any light transmitted into the dielectric. The linear momentum transfer has the expected value of $2\hbar\omega_0/c$ for reflection from the perfect mirror described by the limit $\kappa\rightarrow\infty$.

For the angular momentum, the reflected beam has quantum numbers $-\sigma$ and $-l$ with respect to its $-z$ propagation direction. The total transfer of angular momentum to the dielectric is therefore

$$\hbar(l+\sigma)[1-|R(\omega_0)|^2] = \hbar(l+\sigma)\frac{4\eta}{(\eta+1)^2+\kappa^2}. \tag{7.2}$$

There is thus no transfer of angular momentum to a perfect mirror.

The transfer of the energy in Eq. (6.14) from the light to the dielectric is distributed over the bulk material in accordance with the exponential attenuation. By contrast, the total transfers of momentum given by Eqs. (7.1) and (7.2) occur partly at the dielectric surface and partly in the bulk material. The separation into surface and bulk contributions requires a more detailed study of the radiation forces. In order to separate the surface and bulk contributions, it is assumed that κ is sufficiently small that the attenuation length greatly exceeds the pulse length. However, κ should also be sufficiently large that the attenuation length is much smaller than the Rayleigh range, so that z can be assumed much smaller than z_R. Thus, the various lengths satisfy the inequalities

$$z_R \gg c/2\omega_0\kappa \gg L \gg c/\omega_0, \tag{7.3}$$

where the inequality from Eq. (3.7) is included. It is easily verified that these inequalities are satisfied for visible light with a 10^{-1} m pulse length and a 10^{-3} m beam waist in a dielectric with $\eta=1.5$ and $\kappa=5\times10^{-8}$. The use of a short pulse ensures that the variations in force as the pulse passes through the surface occur on a much shorter time scale than the subsequent fall-off in force as the transmitted light is attenuated.

The three components of the force-density operator (5.1) are calculated by the same methods as used in Sec. V, where the field operators are now taken from Eqs. (6.7) and (6.8). The polarization operator (5.2) is replaced by

$$\hat{\mathbf{P}}^{+}(\mathbf{r},t) = \int_0^{\infty} d\omega\left(\frac{\varepsilon_0\hbar}{4\pi c\omega}\right)^{1/2}(n^2-1)\hat{a}(\omega)\exp\left[-i\omega\left(t-\frac{nz}{c}\right)\right]T(\omega)\left\{i\omega(\alpha\tilde{\mathbf{x}}+\beta\tilde{\mathbf{y}})u-\frac{c}{n}\left(\alpha\frac{\partial u}{\partial x}+\beta\frac{\partial u}{\partial y}\right)\tilde{\mathbf{z}}\right\}, \tag{7.4}$$

where n is the complex refractive index from Eq. (6.1). The expressions for the components of the force operator are more complicated than before and it is convenient to neglect dispersion in the complex refractive index at the outset.

A. Longitudinal component

The z component of the force-density operator can be written in the form

$$:\hat{f}_z(\mathbf{r},t):=\frac{\hbar}{\pi c^2}\frac{1}{|n+1|^2}\int_0^\infty d\omega\int_0^\infty d\omega'(\omega\omega')^{1/2}\hat{a}^\dagger(\omega)\hat{a}(\omega')\left\{\eta(|n|^2-1)\frac{\partial}{\partial t}+\kappa(|n|^2+1)(\omega+\omega')\right\}$$
$$\times\exp\left[i(\omega-\omega')t-i(n^*\omega-n\omega')\frac{z}{c}\right]u^*u', \tag{7.5}$$

which generalizes Eq. (5.7) to the dielectric interface system. The time integral of this operator is

$$\int_{-\infty}^\infty dt:\hat{f}_z(\mathbf{r},t):=\frac{2\hbar}{c^2}\kappa\frac{\eta^2+\kappa^2+1}{(\eta+1)^2+\kappa^2}|u|^2\int_0^\infty d\omega\,2\omega^2\hat{a}^\dagger(\omega)\hat{a}(\omega)\exp(-2\omega\kappa z/c) \tag{7.6}$$

and further integration over all space gives the total transfer of linear momentum to the dielectric as

$$\int d\mathbf{r}\int_{-\infty}^\infty dt:\hat{f}_z(\mathbf{r},t):=\frac{2\hbar}{c}\frac{\eta^2+\kappa^2+1}{(\eta+1)^2+\kappa^2}\int_0^\infty d\omega\,\omega\hat{a}^\dagger(\omega)\hat{a}(\omega). \tag{7.7}$$

The expectation value of the frequency integral is ω_0 for any form of narrowband single-photon pulse, so this expression agrees with Eq. (7.1).

The expectation value of the longitudinal force-density operator for the single-photon wave packet is obtained by the same method as before, with the result

$$\langle 1|:\hat{f}_z(\mathbf{r};t):|1\rangle=\frac{4\hbar\omega_0}{cL|n+1|^2}\left\{-\eta(|n|^2-1)\frac{2c^2}{L^2}\left(t-\frac{\eta z}{c}\right)+\kappa(|n|^2+1)\omega_0\right\}\left(\frac{2}{\pi}\right)^{1/2}\exp\left[-\frac{2\omega_0\kappa z}{c}-\frac{2c^2}{L^2}\left(t-\frac{\eta z}{c}\right)^2\right]|u|^2. \tag{7.8}$$

For a lossless dielectric with $\kappa=0$, this expression reduces to the force-density expectation value in Eq. (5.8), apart from an additional factor ηT^2 caused by the changes in the field operators from those of an infinite medium to those for the half-space dielectric.

The total force operator defined in Eq. (5.3) is now obtained by integration of Eq. (7.8) only over positive z. With use of the standard Gaussian integral, the definition of the complementary error function, and the inequalities (7.3), the total force exerted on the dielectric by the single-photon wave packet is

$$\langle 1|:\hat{F}_z(t):|1\rangle=\frac{4\hbar\omega_0}{c[(\eta+1)^2+\kappa^2]}\left\{(\eta^2+\kappa^2-1)\frac{c}{2L}\left(\frac{2}{\pi}\right)^{1/2}\exp\left[-\frac{2c^2t^2}{L^2}\right]+\frac{\omega_0\kappa}{\eta}\exp\left[-\frac{2\omega_0\kappa t}{\eta}\right]\mathrm{erfc}\left[-\frac{2^{1/2}ct}{L}\right]\right\}. \tag{7.9}$$

This total force is the same as that found for an incident plane wave [1], although its distribution in the xy plane is given by the function $|u|^2$ instead of a uniform distribution over area A. The time dependence of the total force is plotted in Fig. 2 of this reference. The first term in the large bracket of Eq. (7.9) is the surface contribution; it acts only for times t close to 0 as the pulse passes through the surface and the positive force from the front of the pulse is not wholly canceled by the negative force in the rear. The second term is the bulk contribution; it vanishes in the absence of dielectric attenuation but, when $\kappa \neq 0$, it shows a rapid rise as the transmitted pulse enters the medium followed by a slow exponential fall-off with decay time $\eta/2\omega_0\kappa$.

The time-integrated force, or the total linear momentum transfer to the dielectric, is

$$\int_{-\infty}^\infty dt\langle 1|:\hat{F}_z(t):|1\rangle=\frac{2\hbar\omega_0}{c}\left\{\underbrace{\frac{\eta^2+\kappa^2-1}{(\eta+1)^2+\kappa^2}}_{\text{surface}}+\underbrace{\frac{2}{(\eta+1)^2+\kappa^2}}_{\text{bulk}}\right\}=\underbrace{\frac{2\hbar\omega_0}{c}\frac{\eta^2+1+\kappa^2}{(\eta+1)^2+\kappa^2}}_{\text{total}}. \tag{7.10}$$

The total transfer again agrees with that given in Eq. (7.1). The energy transfer is given by Eq. (6.14) with $\eta_0=1$, so that the transfer of momentum per transmitted $\hbar\omega_0$ photon is

$$\frac{\hbar\omega_0}{c}\left\{\underbrace{\frac{\eta^2+\kappa^2-1}{2\eta}}_{\text{surface}}+\underbrace{\frac{1}{\eta}}_{\text{bulk}}\right\}=\underbrace{\frac{\hbar\omega_0}{c}\frac{\eta^2+1+\kappa^2}{2\eta}}_{\text{total}}. \tag{7.11}$$

The implications of these results are discussed in [1] and in Sec. VIII.

B. Transverse components

The x component of the force operator obtained from Eq. (5.1) upon substitution of the field operators from Eqs. (6.7), (6.8), and (7.4) is

$$
\begin{aligned}
:\hat{f}_x(\mathbf{r},t):=&\frac{\hbar}{\pi c}\frac{1}{|n+1|^2}\int_0^\infty d\omega\int_0^\infty d\omega'(\omega\omega')^{1/2}\hat{a}^\dagger(\omega)\hat{a}(\omega')\exp\left[i(\omega-\omega')t-i(n^*\omega-n\omega')\frac{z}{c}\right]\Bigg\{(n^{*2}-1)\Bigg[\Bigg(|\alpha|^2\\
&+\frac{\omega}{\omega'}|\beta|^2\Bigg)u^*\frac{\partial u'}{\partial x}-\frac{n}{n^*}\left(1-\frac{\omega'}{\omega}\right)|\alpha|^2\frac{\partial u^*}{\partial x}u'-\left(\frac{\omega}{\omega'}-1\right)\alpha\beta^*u^*\frac{\partial u'}{\partial y}-\frac{n}{n^*}\left(1-\frac{\omega'}{\omega}\right)\alpha\beta^*\frac{\partial u^*}{\partial y}u'\Bigg]+(n^2-1)\\
&\times\left[\frac{n^*}{n}\left(\frac{\omega}{\omega'}-1\right)|\alpha|^2u^*\frac{\partial u'}{\partial x}+\left(|\alpha|^2+\frac{\omega'}{\omega}|\beta|^2\right)\frac{\partial u^*}{\partial x}u'+\frac{n^*}{n}\left(\frac{\omega}{\omega'}-1\right)\alpha^*\beta u^*\frac{\partial u'}{\partial y}+\left(1-\frac{\omega'}{\omega}\right)\alpha^*\beta\frac{\partial u^*}{\partial y}u'\right]\Bigg\},
\end{aligned}
\tag{7.12}
$$

and the y component is given by the same expression with the exchanges $x\leftrightarrow y$ and $\alpha\leftrightarrow\beta$. The time-integrated operators are accordingly

$$
\int_{-\infty}^\infty dt{:}\hat{f}_x(\mathbf{r},t){:}=\frac{2\hbar}{c}\frac{1}{(\eta+1)^2+\kappa^2}\left\{(n^{*2}-1)u^*\frac{\partial u}{\partial x}+(n^2-1)\frac{\partial u^*}{\partial x}u\right\}\int_0^\infty d\omega\,\omega\hat{a}^\dagger(\omega)\hat{a}(\omega)\exp(-2\omega\kappa z/c) \tag{7.13}
$$

and

$$
\int_{-\infty}^\infty dt{:}\hat{f}_y(\mathbf{r},t){:}=\frac{2\hbar}{c}\frac{1}{(\eta+1)^2+\kappa^2}\left\{(n^{*2}-1)u^*\frac{\partial u}{\partial y}+(n^2-1)\frac{\partial u^*}{\partial y}u\right\}\int_0^\infty d\omega\,\omega\hat{a}^\dagger(\omega)\hat{a}(\omega)\exp(-2\omega\kappa z/c). \tag{7.14}
$$

The radial and azimuthal components are defined by the relations (5.9) as before and the time-integrated value of the former is obtained from Eqs. (7.13) and (7.14) as

$$
\int_{-\infty}^\infty dt{:}\hat{f}_\rho(\mathbf{r},t){:}=\frac{4\hbar\omega_0}{c}\frac{\eta^2-\kappa^2-1}{(\eta+1)^2+\kappa^2}\exp(-2\omega_0\kappa z/c)\left\{\frac{|l|}{\rho}-\frac{2\rho}{w_0^2}\right\}|u|^2\int_0^\infty d\omega\,\hat{a}^\dagger(\omega)\hat{a}(\omega), \tag{7.15}
$$

where the limit $z\ll z_R$ is assumed and the derivative expression from Eq. (4.5) is used. For a lossless dielectric with $\kappa=0$, this expression reduces to the integrated force density for an infinite medium given in Eq. (5.11), apart from the additional factor ηT^2. The time-integrated azimuthal force density is similarly obtained as

$$
\int_{-\infty}^\infty dt{:}\hat{f}_\phi(\mathbf{r},t){:}=\frac{8\hbar\omega_0}{c}\frac{\eta\kappa l}{(\eta+1)^2+\kappa^2}\exp(-2\omega_0\kappa z/c)\frac{|u|^2}{\rho}\int_0^\infty d\omega\,\hat{a}^\dagger(\omega)\hat{a}(\omega). \tag{7.16}
$$

This vanishes for a lossless dielectric with $\kappa=0$ in agreement with Eq. (5.16).

The main aim of the calculation of the transverse force components is the identification of the surface and bulk contributions to the transfer of angular momentum from the optical pulse to the dielectric, similar to the above results for the transfer of linear momentum via the longitudinal force component. For this, it is necessary to obtain the space and time dependence of the azimuthal force density and not merely its time-integrated value given in Eq. (7.16). The required force density is derived straightforwardly from Eq. (7.12) and the corresponding y component with the use of Eq. (5.9) but the resulting expression is extremely complicated. Considerable simplifications result if only those terms are retained that survive in the integration over ϕ that will be made in the calculation of the total torque defined in Eq. (7.22) below. Thus terms in $\sin\phi\cos\phi$ are neglected and, furthermore, $\sin^2\phi$ and $\cos^2\phi$ are set equal to their integrated average values of $\frac{1}{2}$. In addition, it is convenient to assume that the forces will be evaluated only for narrowband wave packets so that the integrated frequencies ω and ω' can be set equal to ω_0, except where they appear in the combination $\omega-\omega'$. With these manipulations, the azimuthal force-density operator reduces to

$$
\begin{aligned}
:\hat{f}_\phi(\mathbf{r},t):=&\frac{\hbar}{\pi c}\frac{1}{(\eta+1)^2+\kappa^2}\left\{\frac{\eta^2(\eta^2+\kappa^2-1)}{\eta^2+\kappa^2}\left(\frac{2l}{\rho}-\sigma\frac{\partial}{\partial\rho}\right)\frac{\partial}{\partial t}+\frac{4l\eta\kappa\omega_0}{\rho}\right\}|u|^2\int_0^\infty d\omega\int_0^\infty d\omega'\hat{a}^\dagger(\omega)\hat{a}(\omega')\\
&\times\exp\left[i(\omega-\omega')t-i(n^*\omega-n\omega')\frac{z}{c}\right].
\end{aligned}
\tag{7.17}
$$

The time integral of this operator agrees with the expression given in Eq. (7.16).

C. Torque

The operator that represents the density of the z component of the torque on the dielectric is

$$\hat{g}_z(\mathbf{r},t)=\rho\hat{f}_\phi(\mathbf{r},t)+(\hat{\mathbf{E}}\times\hat{\mathbf{P}})_z, \tag{7.18}$$

where the second term on the right is the ordinary expression for the torque on an electric dipole [18]. This term provides an additional torque proportional to the spin quantum number σ and, in combination with the azimuthal force from Eq. (7.17), it gives a torque density operator

$$:\hat{g}_z(\mathbf{r},t):=\frac{\hbar}{\pi c}\frac{1}{(\eta+1)^2+\kappa^2}\left\{\frac{\eta^2(\eta^2+\kappa^2-1)}{\eta^2+\kappa^2}\left(2l-\sigma\rho\frac{\partial}{\partial\rho}\right)\frac{\partial}{\partial t}+4(l+\sigma)\eta\kappa\omega_0\right\}|u|^2\int_0^\infty d\omega\int_0^\infty d\omega'\hat{a}^\dagger(\omega)\hat{a}(\omega')$$
$$\times\exp\left[i(\omega-\omega')t-i(n^*\omega-n\omega')\frac{z}{c}\right] \tag{7.19}$$

valid for application to a narrowband wave packet. The time-integrated torque is

$$\int_{-\infty}^{\infty}dt:\hat{g}_z(\mathbf{r},t):=\frac{8\hbar\omega_0}{c}\frac{\eta\kappa(l+\sigma)}{(\eta+1)^2+\kappa^2}\exp(-2\omega_0\kappa z/c)|u|^2\int_0^\infty d\omega\,\hat{a}^\dagger(\omega)\hat{a}(\omega) \tag{7.20}$$

and a further integration over all space gives

$$\int d\mathbf{r}\int_{-\infty}^{\infty}dt:\hat{g}_z(\mathbf{r},t):=\hbar(l+\sigma)\frac{4\eta}{(\eta+1)^2+\kappa^2}\int_0^\infty d\omega\,\hat{a}^\dagger(\omega)\hat{a}(\omega). \tag{7.21}$$

The expectation value of the integral is unity for a single-photon pulse and the total torque on the dielectric agrees with the expected transfer of angular momentum in Eq. (7.2).

The total torque on the dielectric at time t is represented by the torque operator

$$\hat{G}_z(t)=\int d\mathbf{r}\,\hat{g}_z(\mathbf{r},t), \tag{7.22}$$

analogous to the force operator defined in Eq. (5.3). The expectation value of this operator for the single-photon wave packet is evaluated in the same way as that in Eq. (7.9) with the result

$$\langle 1|:\hat{G}_z(t):|1\rangle=\frac{4\hbar(l+\sigma)\eta}{[(\eta+1)^2+\kappa^2](\eta^2+\kappa^2)}\left\{(\eta^2+\kappa^2-1)\frac{c}{L}\left(\frac{2}{\pi}\right)^{1/2}\exp\left[-\frac{2c^2t^2}{L^2}\right]+\frac{\omega_0\kappa}{\eta}\exp\left[-\frac{2\omega_0\kappa t}{\eta}\right]\mathrm{erfc}\left[-\frac{2^{1/2}ct}{L}\right]\right\}. \tag{7.23}$$

The large bracketed term in this expression is similar to that in the total force expectation value (7.9), except that the latter has one-half the ratio of surface to bulk contributions. With allowance for this change, the time dependences of the two contributions to the total torque can be seen from Fig. 2 of [1].

The time-integrated torque, or total angular-momentum transfer to the dielectric, is

$$\int_{-\infty}^{\infty}dt\langle 1|:\hat{G}_z(t):|1\rangle=\frac{4\hbar(l+\sigma)\eta}{(\eta+1)^2+\kappa^2}\left\{\underbrace{\frac{\eta^2+\kappa^2-1}{\eta^2+\kappa^2}}_{\text{surface}}+\underbrace{\frac{1}{\eta^2+\kappa^2}}_{\text{bulk}}\right\}=\underbrace{\frac{4\hbar(l+\sigma)\eta}{(\eta+1)^2+\kappa^2}}_{\text{total}}. \tag{7.24}$$

The total transfer again agrees with that given in Eq. (7.2). The energy transfer is given by Eq. (6.14) with $\eta_0=1$, so that the transfer of angular momentum per transmitted $\hbar\omega_0$ photon is

$$\hbar(l+\sigma)\left\{\underbrace{\frac{\eta^2+\kappa^2-1}{\eta^2+\kappa^2}}_{\text{surface}}+\underbrace{\frac{1}{\eta^2+\kappa^2}}_{\text{bulk}}\right\}=\underbrace{\hbar(l+\sigma)}_{\text{total}}. \tag{7.25}$$

The bulk component of the angular momentum in a lossless dielectric is thus reduced by a factor η^2 relative to its free-space value. However, the torque on an object immersed in the dielectric cannot be deduced from these results but requires an additional calculation for a three-component system, as in the corresponding problem of the transfer of linear momentum [1].

VIII. DISCUSSION

The first stage in the force calculations reported here is the quantization of the radiation field associated with Laguerre-Gaussian modes in dielectrics. The field operators so derived extend the usual results for uniform plane-wave modes to a more complicated mode structure that corresponds reasonably closely to the light beams used in many practical observations of radiation pressure and torque effects. The calculations could in principle be performed with the use of classical fields but the quantum version has no additional complication or difficulty and it provides results in terms of forces per single-photon wave packet, which is very convenient for interpretation of the overall effects. The results reported here are restricted to the mean values of the forces, which scale linearly with the mean photon numbers for more intense light beams. However, the quantum theory also allows for future calculations of the force fluctuations in, for example, the nonclassical light beams considered for use in some schemes of gravitational wave detection [19].

The calculations are all based on the standard expression for the Lorentz force and there is no reliance on any assumptions about the magnitudes of the linear and angular momenta of light in a dielectric. Their *free-space* magnitudes are used to derive expressions for the transfer of momenta to the dielectric in Eqs. (7.1) and (7.2), and these are confirmed by subsequent calculations of the Lorentz force components. The main aim of the work is the understanding of the forces caused by the passage of light from free space into a dielectric and, in particular, the division of the integrated force into surface and bulk contributions. This separation is made possible by the time resolution of the longitudinal and azimuthal forces as a single-photon pulse passes through the surface and is subsequently totally attenuated in the bulk. The forces exerted by light on dielectric samples are measurable quantities, whereas the momenta often assigned to the light beam or to individual photons in dielectrics are not directly observable. Indeed, it has been shown [1] that different measurements of the longitudinal force can produce quite different values of the apparent linear photon momentum in a dielectric.

The nature of the longitudinal force exerted by the single-photon Laguerre-Gaussian pulse and the corresponding transfer of linear momentum is similar to that for uniform plane-wave modes discussed previously [1], but some additional comments can be made. Consider again the total momentum transfer derived in Eqs. (7.1) and (7.10). This can be divided into contributions from the reflected and transmitted components of the single-photon wave packet, as the reflected pulse component transfers a momentum $2\hbar\omega_0/c$, and the remainder is the transfer from the transmitted photon component. The reflected photon component is entirely a surface contribution, so we can combine Eqs. (7.1) and (7.10) as

$$\underbrace{\frac{2\hbar\omega_0}{c}|R(\omega_0)|^2}_{\text{reflected}}+\underbrace{\frac{\hbar\omega_0}{c}(1-|R(\omega_0)|^2)}_{\text{transmitted}}=\frac{2\hbar\omega_0}{c}\left\{\underbrace{\frac{(\eta-1)^2+\kappa^2}{(\eta+1)^2+\kappa^2}+\frac{2(\eta-1)}{(\eta+1)^2+\kappa^2}}_{\text{surface}}+\underbrace{\frac{2}{(\eta+1)^2+\kappa^2}}_{\text{bulk}}\right\}. \tag{8.1}$$

The left-hand side is a rearrangement of the left of Eq. (7.1) to display the reflected contribution and the right-hand side is a rearrangement of the right of Eq. (7.10) to show the reflected surface contribution (first term) and the transmitted surface and bulk contributions (second and third terms, respectively). The transmitted energy is given by Eq. (6.14) with $\eta_0=1$ and the transmitted contributions to the linear momentum transfer to the dielectric per transmitted photon are therefore

$$\frac{\hbar\omega_0}{c}\left\{\underbrace{\frac{\eta-1}{\eta}}_{\text{surface}}+\underbrace{\frac{1}{\eta}}_{\text{bulk}}\right\}=\underbrace{\frac{\hbar\omega_0}{c}}_{\text{total}}, \tag{8.2}$$

independent of κ.

The surface momentum transfer from Eq. (8.2) can be used to obtain the shift ΔZ in position of a transparent slab of refractive index η, mass M, and thickness D with antireflection coatings as a normally incident single-photon wave packet passes through. Thus the velocity imparted to the slab by the above surface momentum is

$$V=\frac{\hbar\omega_0}{Mc}\frac{\eta-1}{\eta} \tag{8.3}$$

and the time of flight through the slab is

$$T=\eta D/c. \tag{8.4}$$

The slab comes to rest again after the wave packet emerges from the rear surface and the final shift in slab position is

$$\Delta Z=VT=\hbar\omega_0(\eta-1)D/Mc^2, \tag{8.5}$$

in exact agreement with a calculation of the shift by an Einstein box theory [20].

The Laguerre-Gaussian pulse also exerts an azimuthal force and there is a corresponding transfer of angular momentum given in Eq. (7.2) or Eq. (7.24). A remarkable feature of the transfer is the proportionality to $l+\sigma$, despite the distinct provenances of the orbital angular momentum in Eq. (2.12) and the spin angular momentum in Eq. (2.11) from the spatial and polarization parts of the vector potential (2.1), respectively. The angular momentum transfer has no contribution from the reflected component of the single-photon wave packet and the total transfer of angular momentum to the dielectric comes entirely from the transmitted component. Thus the separation of the latter into surface and bulk transfers, analogous to Eq. (8.2), is obtained directly from Eq. (7.25) as

$$\hbar(l+\sigma)\Bigg\{\underbrace{\frac{\eta^2-1}{\eta^2}}_{\text{surface}}+\underbrace{\frac{1}{\eta^2}}_{\text{bulk}}\Bigg\}=\underbrace{\hbar(l+\sigma)}_{\text{total}}, \tag{8.6}$$

where κ is set equal to zero for the transparent slab.

The surface angular momentum transfer can be used to obtain the angular rotation $\Delta\Phi$ of the dielectric slab, whose moment of inertia around the z axis is denoted I, as the wave packet passes through. Thus the angular velocity imparted by the above surface angular momentum is

$$\Omega=\frac{\hbar(l+\sigma)}{I}\,\frac{\eta^2-1}{\eta^2} \tag{8.7}$$

and the total rotation after the slab comes to rest again is

$$\Delta\Phi=\Omega T=\frac{\hbar(l+\sigma)D}{Ic}\,\frac{\eta^2-1}{\eta}. \tag{8.8}$$

This expression also agrees exactly with the result of an Einstein box theory [20].

The magnitudes of the linear and angular momenta carried by light in dielectric media have been topics of debate and controversy over many years [5]. Different values of the momenta appear to follow from different formulations of the energy-momentum properties of the electromagnetic fields in dielectrics by Abraham, Minkowski, and others. All these authors derive equivalent momentum-force conservation relations but with different identifications of the momentum density, momentum current density, and force [21]. The different formulations validly describe experiments on the transfer of momentum between field and dielectric when properly applied. The Lorentz force components for single-photon pulses of light in dielectrics calculated here are independent of any specific formulation of the energy-momentum tensor, as are the results for the shift and rotation of a transparent slab derived above. For the transparent dielectric, it is seen from Eqs. (8.2) and (8.6) that the bulk components of the linear and angular momenta are equal, respectively, to $\hbar\omega_0/c\eta$ and $\hbar(l+\sigma)/\eta^2$. However, it is emphasized that these momenta are deduced from results derived here for the radiation pressure and rotational forces on an effectively semi-infinite attenuating specimen and for the shift and rotation of a transparent dielectric slab. Any attempt to measure them within the bulk dielectric would require the detection of the forces exerted by the light on some additional physical component not included in the present theory.

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