Quantum-correlated double-well tunneling of two-component Bose-Einstein condensates

H. T. Ng, C. K. Law, and P. T. Leung
Department of Physics, The Chinese University of Hong Kong, Shatin, NT, Hong Kong, China
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We investigate a double-well tunneling process in Bose condensates of two interacting species. We show that particles of different species can tunnel as pairs through the potential barrier between the two wells in opposite directions. Such a correlated motion of tunneling atoms leads to the generation of quantum entanglement between two macroscopically coherent systems. The strong correlations are manifested in the particle number sum and the phase difference variables.

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I. INTRODUCTION

Quantum tunneling of macroscopically coherent systems is an intriguing phenomena well known in the context of Josephson-junction effects in superconducting electronic systems. For superfluids consisting of neutral particles, detailed investigations of tunneling are aided by the recent realization Bose-Einstein condensation of atomic vapor in a well controllable environment. Indeed, recent experiments have successfully demonstrated quantum tunneling for condensates confined in an array of optical potentials [1,2]. One prominent feature of tunneling in Bose condensates is the nonlinear dynamics arising from the interaction between atoms. Quite remarkably, for single-component condensates trapped in double-well configurations, previous studies have indicated that a self-trapping mechanism can suppress the tunneling rate significantly due to the atom-atom interaction [3–6].

An interesting extension of the tunneling problem involves Bose condensates of two interacting species. The main question is how the interspecies interaction affects the tunneling process, and particularly the quantum coherence as the two condensates mix together. Previous studies of the general properties of two-component Bose condensates have emphasized the important role of the interspecies interaction, which leads to novel features, such as the components separation [7,8], cancellation of the mean-field energy shift [9], and the suppression of quantum phase diffusion [10]. However, the investigation of the influence of interspecies interaction on tunneling dynamics has only just begun [11–13].

In this paper, we examine the tunneling dynamics of a two-component condensate trapped in a double well. Initially, the atoms of the components *A* and *B* are separately prepared in the left and right potential well, respectively. We discover the condition under which the interspecies interaction can eliminate the self-trapping effect and thus enhances the tunneling significantly. Such an enhanced tunneling originates from the correlated quantized motion of the two condensates. We will show that atoms of different species tunnel through the barrier as correlated pairs in opposite directions, i.e., a form of *quantum entangled tunneling*. Therefore, tunneling serves as a mechanism to buildup a strong correlation among atoms of different species, and this leads to the generation of quantum entanglement between two multiparticle systems.

II. THE MODEL

The configuration of our double-well system is sketched in Fig. 1. Our focus in this paper is the quantum dynamics beyond the mean-field description. An exact many-body description is difficult even for single-component condensate problems. The usual method to capture the essential physics is based on the two-mode approximation in which the evolution is confined by the left and right localized mode functions associated with the respective potential wells [3–6,14,15]. Such an approximation is valid when each potential well is sufficiently deep so that higher modes of the wells essentially do not participate in the dynamics.

In the two-mode approximation, the system is modeled by the Hamiltonian ($\hbar=1$)

$$H=\frac{\Omega}{2}(a_L^\dagger a_R+a_R^\dagger a_L+b_L^\dagger b_R+b_R^\dagger b_L)+\kappa(a_L^\dagger a_L b_L^\dagger b_L + a_R^\dagger a_R b_R^\dagger b_R)+\frac{\kappa_a}{2}[(a_L^\dagger a_L)^2+(a_R^\dagger a_R)^2] + \frac{\kappa_b}{2}[(b_L^\dagger b_L)^2+(b_R^\dagger b_R)^2]. \quad (1)$$

Here, the subscripts *L* and *R*, respectively, denote the localized modes in the left and right potential wells. Since there are two modes available for each component, the model in fact consists of four-mode operators. We use $a_j^\dagger$ and $b_j^\dagger$ (j

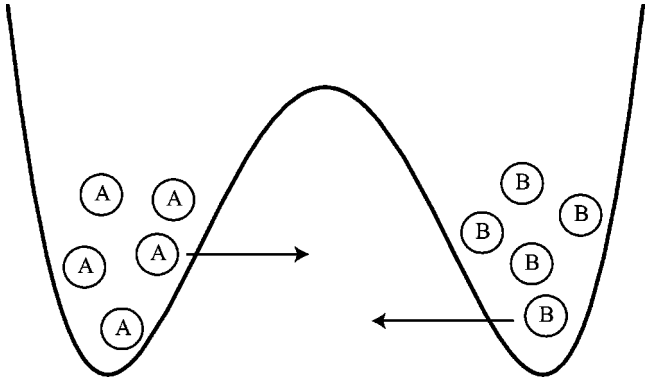

FIG. 1. A sketch of the double-well tunneling system. Initially, the atoms of the component *A* are trapped in the left side and the atoms of the component *B* are trapped in the right side.

$=L,R$) to denote the creation operators of the components A and B, respectively. The parameters Ω, $\kappa_a(\kappa_b)$, and κ describe the tunneling rate, self-interaction strength of the component $A(B)$, and the interspecies interaction strength, respectively.

We consider the initial condition in which all atoms in the component $A(B)$ are localized in the left (right) potential well. The general form of the state vector at time t is given by

$$|\Psi(t)\rangle = e^{-i\kappa N^2 t}\sum_{n=0}^{N}\sum_{m=0}^{N} c_{n,m}(t)|n,N-n\rangle_A|m,N-m\rangle_B . \tag{2}$$

Here, $|p,q\rangle_S$ denotes the state with p atoms of species S ($S=A,B$) in the left well and q atoms of species S in the right well. The amplitudes $c_{n,m}(t)$ are governed by the Schrödinger equation according to Hamiltonian (1):

$$\begin{aligned} i\dot{c}_{n,m} = &\frac{\Omega}{2}[\sqrt{(n+1)(N-n)}c_{n+1,m} + \sqrt{n(N-n+1)}c_{n-1,m}] \\ &+\frac{\Omega}{2}[\sqrt{(m+1)(N-m)}c_{n,m+1} \\ &+\sqrt{m(N-m+1)}c_{n,m-1}] + \kappa[(n+m-N)^2]c_{n,m}, \end{aligned} \tag{3}$$

with the initial condition $c_{n,m}(0)=\delta_{n,N}\delta_{m,0}$. In this paper, we shall limit our study to the $4\kappa \gg N\Omega$ regime where the nonlinear interaction is dominant.

III. ENTANGLED TUNNELING DYNAMICS

A. Two-atom case

To gain insight of the quantum correlation developing in the tunneling process, we first consider the exactly solvable case with only one A atom in the left well and one B atom in the right well. In this case, the system is spanned by four basis vectors: $|1,0\rangle_A|1,0\rangle_B$, $|1,0\rangle_A|0,1\rangle_B$, $|0,1\rangle_A|1,0\rangle_B$, and $|0,1\rangle_A|0,1\rangle_B$. The eigenvalues and eigenvectors of H can be found explicitly (see Appendix A). In the regime where the interspecies interaction is sufficiently strong such that $\kappa \gg \Omega$, the system evolves as

$$\begin{aligned} |\Psi(t)\rangle = &e^{-i[(\kappa_a+\kappa_b)/2-\omega_0]t}[\cos\omega_0 t|1,0\rangle_A|0,1\rangle_B \\ &+ i\sin\omega_0 t|0,1\rangle_A|1,0\rangle_B] + O(\Omega/\kappa). \end{aligned} \tag{4}$$

In writing Eq. (4), we have defined $\omega_0=\Omega^2/2\kappa$ as an effective tunneling frequency. Because of the strong interaction between the atoms, the probability of finding both particles in the same well at any time t is negligible (of the order of Ω^2/κ^2). The tunneling motion of the two atoms are anticorrelated in the sense that the atom A and the atom B always move in opposite directions. Such an anticorrelated tunneling motion gives rise to quantum entanglement between the two atoms. At time $t=(n+1/4)\pi/\omega_0$, ($n=$ integers), the state is a form of Bell's state that is maximally entangled in the two-particle two-mode subspace.

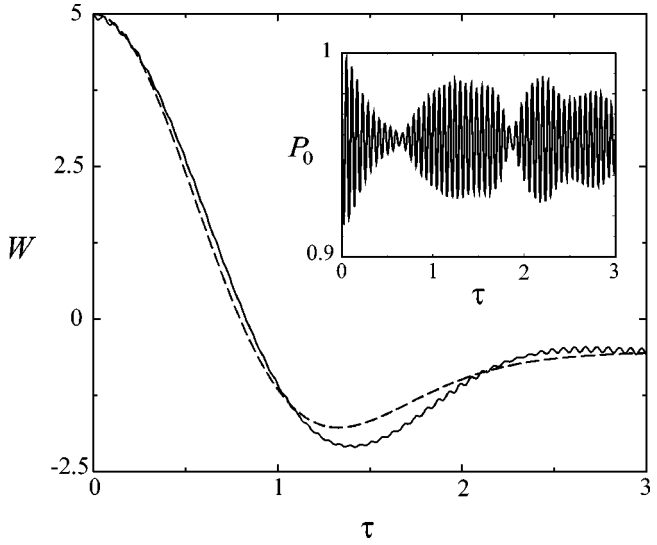

FIG. 2. Particle number difference of the component A between the two potential wells as a function of a dimensionless time $\tau \equiv (\Omega^2/\kappa)t$ with $N=5$ and $\kappa=\kappa_a=\kappa_b=10\ \Omega$. The dashed line is the solution based on the effective Hamiltonian in Eq. (6). The inset shows the overlap probability P_0 of finding the system belonging to the degenerate set of states $|\phi_q\rangle$.

B. Multiatom case

Now we examine the multiatom case. In order to facilitate the discussion, we assume the number of particles are the same for the two components, i.e., $N_a=N_b=N$, and the condensates share the same interaction strength, i.e., $\kappa_a=\kappa_b=\kappa$. The latter condition is a good approximation to ^{87}Rb condensate of atoms in hyperfine spin states $|F=2,m_f=1\rangle$ and $|F=1,m_f=-1\rangle$, which share similar scattering lengths [8]. However, we emphasize that these assumptions are not crucial, we shall discuss more general situations later in the paper.

1. Numerical results

Let us first present the typical results obtained from the numerical solutions of Eq. (3). In Fig. 2, we show the particle number difference $W \equiv \langle a_L^\dagger a_L - a_R^\dagger a_R\rangle$ of species A between the two wells as a function of time. The occurrence of tunneling is revealed by the decrease of W. At longer times W approaches zero, therefore the numbers of A atoms in the two potential wells are roughly equalized. We emphasize that the nonzero interspecies interaction is responsible for the tunneling to occur. If the two species do not interact with each other (i.e., $\kappa=0$), then a sufficiently strong self-interaction $\kappa_j > N\Omega$ ($j=a,b$) can suppress the tunneling almost completely by the self-trapping effect [3].

We note that the state vector $|\Psi(t)\rangle$ is a superposition of $(N+1)^2$ states of the form $|n,N-n\rangle_A|m,N-m\rangle_B$. However, a small number of them suffice to describe the process. This can be understood intuitively because $|n,N-n\rangle_A|m,N-m\rangle_B$ have different energies for different values of n and

m, and those with energies closer to that of the initial state are more accessible. The difference in energies is significant in the regime $4\kappa \gg N\Omega$ considered here. We find that the states $|\phi_q\rangle = |q, N-q\rangle_A |N-q, q\rangle_B$ with $q = 0,1,2,\ldots,N$ are approximately degenerate energy eigenvectors of H (to zero order in Ω), and the energies of states other than $|\phi_q\rangle$ are higher than those of $|\phi_q\rangle$ by an amount of the order of κ or higher. Since the initial state $|\Psi(0)\rangle$ is $|\phi_0\rangle$, $|\Psi(t)\rangle$ is mainly a superposition of $|\phi_q\rangle$ according to the energy argument above. We find that this is indeed the case. To provide a numerical evidence of our finding, we show in the inset of Fig. 2, the overlap probability defined by $P_0(t) = \sum_{q=0}^{N} |\langle \phi_q | \Psi(t) \rangle|^2$. For the parameters used in this figure, the set of $|\phi_q\rangle$ contributes more than 90% of $|\Psi(t)\rangle$. Our further numerical tests suggest that $P_0 \to 1$ in the limit $N\Omega/\kappa \to 0$.

We remark that all $|\phi_q\rangle$ have the same number of particles (component A plus component B) in the left potential well, and the same also holds for the right well. Therefore, $P_0(t) \approx 1$ implies small fluctuations of total particle number in the each potential well. In the case of Fig. 2, we find that the fluctuation of total particle number in the left well $\langle \Delta N_L \rangle$ is much smaller than $\langle N_L \rangle^{1/2}$, i.e., a sub-Poissonian distribution. Since, $N_L = a_L^\dagger a_L + b_L^\dagger b_L$, the particle numbers of the two condensates in the left well is strongly correlated. The same is also true in the right potential well.

2. Effective Hamiltonian

The time-dependent problem is now significantly simplified because the evolution of the condensates mainly involves the set of degenerate states $|\phi_q\rangle$. For those states that do not have the same energy as $|\phi_q\rangle$, they act as intermediate states that are rarely populated. We may eliminate such intermediate states to obtain the effective Hamiltonian (see Appendix B)

$$H_{\rm eff} = -\frac{\Omega^2}{2\kappa}(a_L^\dagger a_R b_L b_R^\dagger + a_L a_R^\dagger b_L^\dagger b_R + a_L^\dagger a_L a_R^\dagger a_R + b_L^\dagger b_L b_R^\dagger b_R), \tag{5}$$

where a constant term proportional to the total number of particles is dropped. The effective Hamiltonian $H_{\rm eff}$ is an approximation that captures the essential tunneling mechanism in the $4\kappa \gg N\Omega$ limit [16]. We have tested the validity of this effective Hamiltonian numerically. For example, in Fig. 2 the dashed line, which is obtained from the evolution based on $H_{\rm eff}$, agrees well with the exact numerical solution.

The quantum dynamics in the subspace defined by the set of $|\phi_q\rangle$ is transparent [17]. The interaction term $a_R^\dagger b_L^\dagger a_L b_R$ suggests that every time when an atom A moves from the left well to the right well there must be an atom B that moves from the right well to the left well. This explains why the small fluctuations in the total particle number in each potential well. The reverse process is described by $a_L^\dagger b_R^\dagger a_R b_L$ in $H_{\rm eff}$. In other words, the atoms A and B move in pair in opposite directions during the tunneling process.

It is worth noting that $H_{\rm eff}$ can be cast in the form

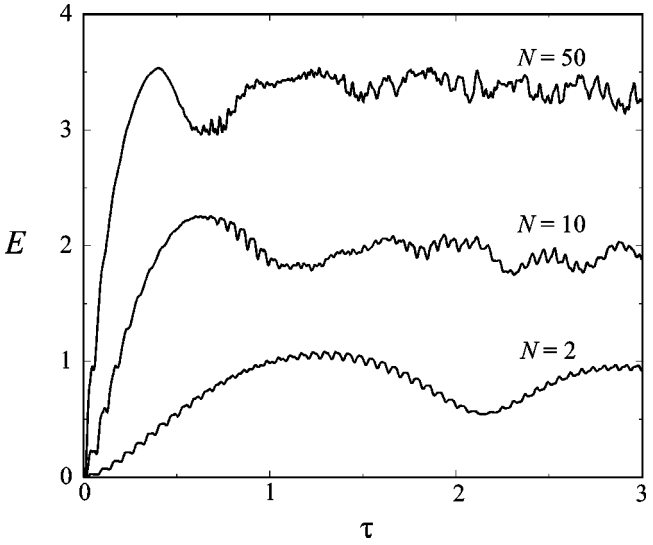

FIG. 3. Entanglement entropy E as a function of the dimensionless time τ for the different particle numbers: $N=2$, $N=10$, and $N=50$ with $\kappa = \kappa_a = \kappa_b = 10\ \Omega$.

$$H_{\rm eff} = -\frac{\Omega^2}{4\kappa}(K_+ K_- + K_- K_+), \tag{6}$$

where $K_+ = a_L^\dagger a_R + b_L^\dagger b_R$ and $K_- = a_L a_R^\dagger + b_L b_R^\dagger$ satisfy the angular-momentum commutation relations: $[K_+, K_-] = 2K_z$, $[K_z, K_\pm] = \pm K_\pm$, where $K_z = (a_L^\dagger a_L + b_L^\dagger b_L - a_R^\dagger a_R - b_R^\dagger b_R)/2$. Therefore, $K_\pm$ and K_z are analogous to collective spin operators, and analytic solutions can, in principle, be constructed using angular-momentum algebra. This shares a similar feature in the description of spinor condensates [18]. Indeed, the nonlinear interaction between collective spins is the key to generation of nonclassical correlations such as spin squeezing [19], and particularly, the notion of multiparticle entanglement has been discussed in the context of squeezed spin states [20]. In the following section, we shall see that the double-well tunneling process leads to quantum entanglement between two initially well separated multiparticle subsystems (A and B) involving variables of two potential wells.

3. Quantum correlations

The degree of entanglement between the two species is measured by the entanglement entropy [21]

$$E = -\mathrm{tr}(\rho_A \ln \rho_A) = -\mathrm{tr}(\rho_B \ln \rho_B), \tag{7}$$

where ρ_A and ρ_B are reduced density matrices of the respective subsystems, i.e., $\rho_A = \mathrm{tr}_B \rho_{AB}$ and $\rho_B = \mathrm{tr}_A \rho_{AB}$ with $\rho_{AB} = |\Psi(t)\rangle\langle\Psi(t)|$ being the density matrix of the whole system. A disentangled state (for example, the initial state above) has zero entanglement entropy. The more entangled the systems, the larger the value of E is. As an illustration, we show in Fig. 3 how the entanglement is established in time for various particle numbers. As time increases, the value of E increases until a saturated value is reached. Since there are $N+1$ degenerate states $|\phi_q\rangle$ mainly involved in the

evolution, we have $E \approx -\Sigma_{n=0}^{N} |c_{n,N-n}|^2 \ln |c_{n,N-n}|^2$. Therefore, the maximum value of E is $\approx \ln(N+1)$ if all $|\phi_q\rangle$ have equal contributions to the state of the system. In the case of Fig. 3, the value of E can reach as high as 90% of the value $\ln(N+1)$.

It is important to identify the physical variables that reveal the quantum correlation between the entangled condensates. In addition to the strongly correlated particle numbers, there is a less obvious correlation in the relative phase variables. We note that the relative phase of bosonic fields between two modes (potential wells) can be defined through a complete set of relative phase states. For the condensate species j, we define the phase states by [22]

$$|\theta_r\rangle_j \equiv \frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} e^{in\theta_r} |n, N-n\rangle_j, \quad j=A,B, \tag{8}$$

where $\theta_r = 2\pi r/(N+1)$ with $r=0,1,2,\ldots,N$. The state $|\theta_r\rangle_j$ represents the state with a well-defined phase difference θ_r [of resolution $2\pi/(N+1)$] between the two wells. Since the system consists of two atomic species, $|\Psi(t)\rangle$ is a superposition of $|\theta_r\rangle_A |\theta_s\rangle_B$. The quantity $|\langle \Psi(t)|\theta_r\rangle_A |\theta_s\rangle_B|^2$ corresponds to the joint probability of finding the component A with the phase difference θ_r and component B with the phase difference θ_s between the two wells. For the entangled tunneling process discussed above, $|\Psi(t)\rangle$ involves a coherent superposition of $|n, N-n\rangle_A |N-n, n\rangle_B$ of various n. Therefore, $|\langle \Psi(t)|\theta_r\rangle_A |\theta_s\rangle_B|^2 = p(\theta_s - \theta_r)$ is a function of *difference* of the relative phases only.

The initial Fock state $|0,N\rangle_A |N,0\rangle_B$ has completely random phases, i.e., $p(\theta_r - \theta_s, 0)$ is uniform. However, we find that $p(\theta_r, \theta_s, t)$ evolves into a single narrow peak distribution as time increases (Fig. 4). At a characteristic time $\tau_s \equiv 2\kappa/\Omega^2\sqrt{N}$, the half width of the peak is $\approx 2N^{-1/2}$, and the peak position at $\theta_s - \theta_r \approx \pi$, according to our numerical calculations for the particle number N up to 400. Therefore, the entangled tunneling process tends to buildup a sharp relative phase difference $\theta_r - \theta_s$ dynamically. For $t > \tau_s$, the relative phase distribution develops complicated interference pattern due to further mixing of condensates between the two wells.

So far we have considered quantum correlations between the two condensate components. We remark that the left and right potential wells can be treated as two spatially separated subsystems. Therefore, it is natural to ask about the quantum entanglement between the left condensate (which contains atoms of both atomic species) and the right condensate. We introduce basis vectors suitable for the left-right subsystems: $|n, N-m\rangle_L |N-n, m\rangle_R \equiv |n, N-n\rangle_A |N-m, m\rangle_B$, which corresponds to the state with n A atoms and $N-m$ B atoms in the left well, and $N-n$ A atoms and m B atoms in the right well. In the limit $4\kappa \gg N\Omega$, where the effective Hamiltonian (5) applies, we have

$$|\Psi(t)\rangle = e^{-i\kappa N^2 t} \sum_{n=0}^{N} c_{n,N-n}(t) |n, N-n\rangle_A |N-n, n\rangle_B$$

$$= e^{-i\kappa N^2 t} \sum_{n=0}^{N} c_{n,N-n}(t) |n, N-n\rangle_L |N-n, n\rangle_R.$$

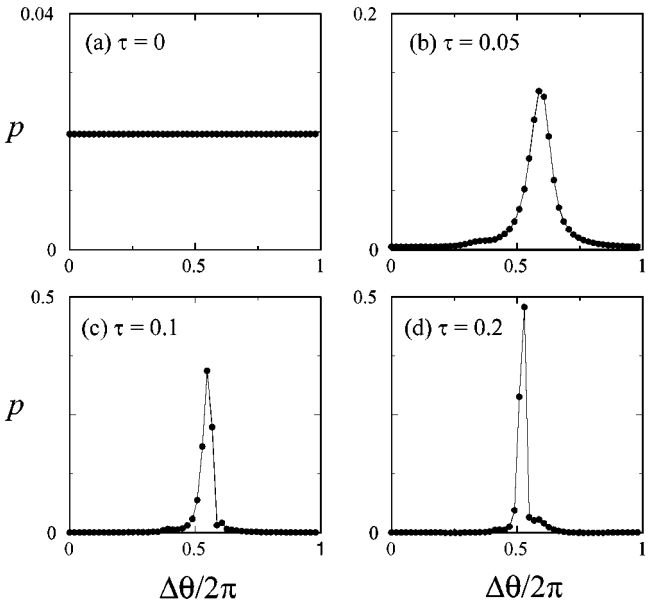

FIG. 4. An illustration of the time development of the probability distribution of relative phase difference $\Delta\theta \equiv \theta_r - \theta_s$ (see text) at four different dimensionless times. Here, the particle number $N = 50$ with $\kappa = \kappa_a = \kappa_b = 50\ \Omega$.

We see that $|\Psi(t)\rangle$ in the new basis is represented by the same set of amplitudes $c_{n,N-n}(t)$. Therefore, the entanglement entropy E between the left and right subsystems is the same as that between the two species in the $4\kappa \gg N\Omega$ limit. Similarly, the phase correlation discussed above can be applied to the left-right systems.

IV. CONCLUDING REMARKS

Finally, we would like to address the conditions for the entangled tunneling to occur. Our discussion above has been restricted to the simplest symmetric situation: $N_a = N_b = N$ and $\kappa_a = \kappa_b = \kappa$, in order to illustrate the essential mechanism under the condition $4\kappa \gg N\Omega$. The same analysis can be performed to study general situations. We have examined the system with unequal particle numbers $(N_b - N_a) \equiv D > 0$ and unequal coupling strengths $\kappa_a = \kappa + \delta$, $\kappa_b = \kappa - \delta$ with $|\delta| \ll \kappa$ (see Appendix B). We find that if the tunneling strength Ω is sufficiently weak or the self-interaction strength κ is sufficiently strong such that

$$4|\kappa(D-1) - [N_a(D-2)|\delta|]/2| \gg \Omega N_b, \tag{9}$$

then the system mainly evolves among the states $|n, N_a - n\rangle_A |N_a - n, N_b - N_a + n\rangle_B$, where $n = 0,1,2,\ldots,N_a$. In other words, the tunneling under condition (9) is characterized by entanglement generation term $a_R^\dagger b_L^\dagger a_L b_R$ as before. However, we point out that the tunneling are generally less efficient for the cases with nonzero δ and D. This is because the states $|n, N_a - n\rangle_A |N_a - n, N_b - N_a + n\rangle_B$ are not as degenerate as that in the symmetric case with δ and D are both zero.

To summarize, we have studied the quantum dynamics of double-well tunneling involving Bose condensates of two in-

teracting components, based on the two-mode approximation model in the strong-coupling regime. We find that the interplay of intraspecies and interspecies interactions permits a set of energy degenerate states, a small tunneling coupling can push the system to ''explore'' through these degenerate states and thus result in a substantial tunneling not limited by the self-trapping effect. The most interesting feature is the strongly correlated tunneling motion. We have shown that high degree of quantum entanglement between two macroscopically coherent systems can be achieved. The quantum correlations manifest themselves in the particle number sum as well as the relative phase difference variables associated with the condensates in the two potential wells.

ACKNOWLEDGMENTS

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APPENDIX A: ENERGY EIGENVECTORS OF THE TWO-ATOM SYSTEM

For a system with only one A atom and one B atom, there are four eigenvectors of H

$$|\lambda_1\rangle=\frac{1}{\sqrt{2}}(|1,0\rangle_A|1,0\rangle_B-|0,1\rangle_A|0,1\rangle_B), \tag{A1}$$

$$|\lambda_2\rangle=\frac{1}{\sqrt{2}}(|1,0\rangle_A|0,1\rangle_B-|0,1\rangle_A|1,0\rangle_B), \tag{A2}$$

$$|\lambda_3\rangle=\frac{1}{\sqrt{C_1}}\Bigg[|1,0\rangle_A|1,0\rangle_B+|0,1\rangle_A|0,1\rangle_B+\frac{2\Omega}{\kappa+\sqrt{\kappa^2+4\Omega^2}}(|1,0\rangle_A|0,1\rangle_B+|0,1\rangle_A|1,0\rangle_B)\Bigg], \tag{A3}$$

$$|\lambda_4\rangle=\frac{1}{\sqrt{C_2}}\Bigg[|1,0\rangle_A|1,0\rangle_B+|0,1\rangle_A|0,1\rangle_B+\frac{2\Omega}{\kappa-\sqrt{\kappa^2+4\Omega^2}}(|1,0\rangle_A|0,1\rangle_B+|0,1\rangle_A|1,0\rangle_B)\Bigg], \tag{A4}$$

where $C_1=2+8\Omega^2/(\kappa+\sqrt{\kappa^2+4\Omega^2})^2$ and $C_2=2+8\Omega^2/(\kappa-\sqrt{\kappa^2+4\Omega^2})^2$ are normalization constants. The eigenvalues are given by $\lambda_1=(\kappa_a+\kappa_b+2\kappa)/2$, $\lambda_2=(\kappa_a+\kappa_b)/2$, $\lambda_3=(\kappa_a+\kappa_b+\kappa+\sqrt{\kappa^2+4\Omega^2})/2$, and $\lambda_4=(\kappa_a+\kappa_b+\kappa-\sqrt{\kappa^2+4\Omega^2})/2$. Note that in the limit $\kappa\gg\Omega$, the four eigenvectors are exactly the four mutually orthogonal Bell's states.

APPENDIX B: DERIVATION OF THE EFFECTIVE HAMILTONIAN

Let us consider a system with N_a atoms of component A and N_b atoms of component B. We also allow unequal interaction strengths: $\kappa_a=\kappa+\delta$, $\kappa_b=\kappa-\delta$, and $\gamma/2=\kappa$. Without loss of generality, we let $N_b>N_a$ and $D\equiv N_b-N_a\neq 1$. The state vector of the system is given by

$$|\Psi(t)\rangle=e^{-i(\kappa N_a^2+\kappa N_b^2-\delta D N_b)t/2}\times\sum_{n=0}^{N_a}\sum_{m=0}^{N_b}c_{n,m}(t)|n,N_a-n\rangle|m,N_b-m\rangle.$$

The amplitudes $c_{n,m}(t)$ with $n+m=N_a$ are governed by

$$\begin{aligned}i\dot{c}_{n,m}=&\frac{\Omega}{2}[\sqrt{(n+1)(N_a-n)}c_{n+1,m}+\sqrt{n(N_a-n+1)}c_{n-1,m}]\\&+\frac{\Omega}{2}[\sqrt{(m+1)(N_b-m)}c_{n,m+1}\\&+\sqrt{m(N_b-m+1)}c_{n,m-1}]\\&+\frac{\kappa}{2}[(n+m)^2+(N_a+N_b-n-m)^2-N_a^2-N_b^2]c_{n,m}\\&+\frac{\delta D}{2}(n-m)c_{n,m},\end{aligned} \tag{B1}$$

with the initial condition $c_{N_a,0}(0)=1$.

Since κ is a large parameter here, the states with $n+m=N_a\pm 1$ have a much different energy than that of the states with $n+m=N_a$. Any transition (due to Ω) from the manifold $n+m=N_a$ to the manifolds $n+m=N_a\pm 1$ must quickly return to $n+m=N_a$. In other words, the states with $n+m=N_a\pm 1$ are intermediate states that are hardly occupied at any time. The amplitudes associated with the $n+m=N_a\pm 1$ manifold are $c_{n+1,m}$, $c_{n-1,m}$, $c_{n,m+1}$, and $c_{n,m-1}$, and they can be found approximately by adiabatic approximation under the conditions $4|N_1|\gg\Omega N_b$ and $4|N_2|\gg\Omega N_b$:

$$c_{n+1,m}\approx-\frac{\Omega}{2N_1}[\sqrt{(n+1)(N_a-n)}c_{n,m}+\sqrt{m(N_b-m+1)}c_{n+1,m-1}], \tag{B2}$$

$$c_{n-1,m}\approx-\frac{\Omega}{2N_2}[\sqrt{n(N_a-n+1)}c_{n,m}+\sqrt{(m+1)(N_b-m)}c_{n-1,m+1}], \tag{B3}$$

$$c_{n,m+1}\approx-\frac{\Omega}{2N_1}[\sqrt{(m+1)(N_b-m)}c_{n,m}+\sqrt{n(N_a-n+1)}c_{n-1,m+1}], \tag{B4}$$

$$c_{n,m-1} \approx -\frac{\Omega}{2N_2}[\sqrt{m(N_b-m+1)}c_{n,m} + \sqrt{(n+1)(N_a-n)}c_{n+1,m-1}], \quad \text{(B5)}$$

where $N_1=\kappa(1-D)+\delta D/2-\delta(n-m)(D-2)/2$ and $N_2=\kappa(D+1)-\delta D/2-\delta(n-m)(D+2)/2$ are defined.

Using (B2)–(B5), the amplitude equation of $c_{n,N-n}$ is reduced to

$$\begin{aligned} i\dot{c}_{n,N_a-n} \approx & -\frac{\Omega^2(N_1+N_2)}{4N_1N_2}\sqrt{(n+1)(N_a-n)m(N_b-m+1)} \\ & \times c_{n+1,N_a-n-1} - \frac{\Omega^2(N_1+N_2)}{4N_1N_2} \\ & \times \sqrt{n(N_a-n+1)(m+1)(N_b-m)}c_{n-1,N_a-n+1} \\ & - \frac{\Omega^2}{4N_1}[(n+1)(N_a-n) \\ & + (m+1)(N_b-m)]c_{n,N_a-n} - \frac{\Omega^2}{4N_2}[n(N_a-n+1) \\ & + m(N_b-m+1)]c_{n,N_a-n} + \frac{\delta D}{2}(n-m)c_{n,N_a-n}. \end{aligned} \quad \text{(B6)}$$

In the case with equal particle number $D=0$ and equal coupling $\delta=0$, Eq. (B6) corresponds to the Schrödinger equation governed by the effective Hamiltonian (5). For the cases with unequal particle numbers and coupling strengths, i.e., $D\neq 0$ and $\delta\neq 0$, Eq. (B6) describes how the amplitudes of degenerate states couple together under the conditions $4|N_1| \gg \Omega N_b$ and $4|N_2| \gg \Omega N_b$. These inequalities lead to condition (9) in Sec. IV. We also remark that condition (9) does not apply to the special case $N_b-N_a=\pm 1$ in which some of the states in $n+m=N_a$ and $n+m=N_a\pm 1$ manifolds are accidentally degenerate.

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