Perturbation theory of spin-1 Bose-Einstein condensates

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We generalize the perturbation theory of weakly interacting bosons to Bose-Einstein condensates with hyperfine spin $F=1$. Analytical expressions of second-order self-energies beyond the mean-field approximation are derived at zero temperature. We further extend the perturbation theory to the finite-temperature case, and derive the corresponding self-energies. We evaluate these self-energies near the poles of the first-order (mean-field) Green's functions within the on-shell approximation, and expand the results in powers of wave number in the long-wavelength limit. Excitation spectra for each mode are obtained analytically at low temperatures. We also derive the ground-state energy and thermodynamic quantities such as free energy of the system.

I. INTRODUCTION

Bose-Einstein condensates of alkali-metal atoms have internal degrees of freedom due to the hyperfine structures of the atoms. These are frozen in a magnetic field, but recent experiments have produced spin-1 and "(pseudospin) spin-$\frac{1}{2}$" Bose condensates. The former is produced in optically confined ^{23}Na [1–3] and ^{87}Rb [4], which is characterized by the three hyperfine spin states $|F=1,m_F=\pm 1,0\rangle$, and the latter in magnetically confined ^{87}Rb with hyperfine states $|F=2,m_F=1\rangle$ and $|F=1,m_F=-1\rangle$ [5]. Generally, only atoms in the low lying hyperfine spin states are confined in the optical potential. Atoms in the higher hyperfine spin states will leave the potential by spin-flip scattering. In the case of ^{23}Na and ^{87}Rb, which are alkali-metal atoms with nuclear spin $I=3/2$, their hyperfine spin states are $F=2$, and $F=1$ is regular, i.e., the higher spin state $F=2$ has higher energy. While the experiments have reported so far the spin-1 and spin-$\frac{1}{2}$ Bose condensates, the spin-2 Bose condensates appear feasible using the $F=2$ multiple of bosons such as ^{23}Na and ^{87}Rb.

An important feature of Bose condensates with spin degrees of freedom is that, in addition to the repulsive two-body hard-core collisions that give rise to density fluctuations, atoms in the condensates can also couple to each other through the spin-exchange interaction. The competition between these two interactions leads to complex ground-state structures. Ho [6] and Ohmi and Machida [7] showed that the ground-states of spin-1 bosons are either ferromagnetic or "polar" states, depending on the scattering lengths in different angular momentum channels. They also developed the mean-field theory for describing a vectorial Bose condensates by generalizing the Gross-Pitaevskii equation under the restriction of gauge and spin rotational symmetries. Law *et al.* [8] constructed an algebraic representation of the spin-1 Bose condensates to study the exact many-body states, and found that spin-exchange interactions cause a set of collective dynamic behaviors. Recently, the properties of spin-2 Bose condensates are also investigated. Ciobanu *et al.* [9] generalized the approach for the spin-1 Bose condensates developed by Ho to study the ground-state structure of the spin-2 Bose condensates. Koashi and Ueda [10] studied the

eigenstates and eigenspectra of spin-2 Bose condensates.

We generalize the perturbation theory of scalar Bose condensates [11–13] to vectorial spin-1 Bose condensates under the single-condensate assumption. It is now understood that the ground-state of spin-1 Bose gas can be "fragmented" [14]. Despite this fact, the phase diagram for single condensates remains highly valuable and gives the best agreement with experiments [2]. This paper is organized as follows. In Sec. II, after giving the specifications of the Hamiltonian, the normal and anomalous single-particle Green's functions (matrices in the spin variables) are defined. We give the Dyson's equations expressing the 3×3 matrix Green's functions in terms of the 3×3 self-energies. Solving these matrix Dyson's equations, we obtain the formal expression of the Green's functions in terms of the self-energies. Within the first-order calculations, in which the self-energies are approximated by the lowest-order terms of the ladder diagram, we derive the energy spectrum that coincides with the results of previous studies [6,7]. We then discuss the second-order approximation, in which the self-energies include the second-order terms of the order of $(n_0 a_{F_t}^3)^{1/2}$ $(F_t=0,2)$, where n_0 is the density of the condensate, and a_0 and a_2 are the s-wave scattering lengths for two colliding atoms with total angular momenta, $F_t=0$ (spin singlet) and $F_t=2$ (quintuple), respectively. Summing the set of diagrams, we represent the second-order self-energies and derive excitation spectra for each mode. In Sec. III, we generalize the perturbation theory in Bose system with spin degrees of freedom at zero temperature, which is developed in Sec. II to the finite-temperature case. In Secs. IV and V, we evaluate the self-energies near the poles of the first-order Green's functions, which is the on-shell approximation, and expand excitation spectra in powers of the wave number in the long-wavelength limit analytically. We also derive the analytical expressions of the ground-state energy and thermodynamic quantities such as free energy of the system. Results are summarized in Sec. VI.

II. GREEN'S FUNCTION METHOD

A. Basic equations and definitions

We consider an assembly of homogeneous dilute Bose-condensed gas with hyperfine spin state $F=1$, which con-

sists of N atoms, enclosed in a box of volume V and interacting through two-body potential $U_{\alpha\alpha',\beta\beta'}(r)$. The natural basis set to characterize such a system is the hyperfine spin states $|m_F = \pm 1,0\rangle$. In this case, the wave function has three components, and written as

$$\hat{\psi}(\mathbf{r}) = \begin{pmatrix} \hat{\psi}_+(\mathbf{r}) \\ \hat{\psi}_0(\mathbf{r}) \\ \hat{\psi}_-(\mathbf{r}) \end{pmatrix}. \qquad (1)$$

In this representation, the spin operators are

$$F_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad F_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad (2)$$

$$F_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Here, $\hat{\psi}_\alpha(\mathbf{r})$ and $\hat{\psi}_\alpha^\dagger(\mathbf{r})$ are the boson field operators, satisfying the usual Bose commutation relations which are $[\hat{\psi}_\alpha(\mathbf{r}), \hat{\psi}_\beta(\mathbf{r}')] = 0$, $[\hat{\psi}_\alpha(\mathbf{r}), \hat{\psi}_\beta^\dagger(\mathbf{r}')] = \delta(\mathbf{r}-\mathbf{r}')\delta_{\alpha\beta}$. Assuming periodic boundary conditions, the field operator can be expanded in terms of creation and annihilation operators characterized by wave vector $\mathbf{p}$, $\hat{\psi}_\alpha^{(\dagger)}(\mathbf{r}) = (1/\sqrt{V})\Sigma_\mathbf{p} a_{\mathbf{p}\alpha}^{(\dagger)} e^{i\mathbf{p}\cdot\mathbf{r}}$.

In the second quantized formalism, the grand canonical Hamiltonian of the system is given by

$$\hat{K} = \hat{H} - \mu\hat{N}$$

$$= \int d\mathbf{r}\, \hat{\psi}_\alpha^\dagger(\mathbf{r})\left[-\frac{\hbar^2\nabla^2}{2m} - \mu \right]\hat{\psi}_\alpha(\mathbf{r})$$

$$+ \int d\mathbf{r} \int d\mathbf{r}'\, \hat{\psi}_\alpha^\dagger(\mathbf{r})\hat{\psi}_\beta^\dagger(\mathbf{r}') U_{\alpha\alpha',\beta\beta'}$$

$$\times (\mathbf{r}-\mathbf{r}')\hat{\psi}_{\beta'}(\mathbf{r}')\hat{\psi}_{\alpha'}(\mathbf{r}) bf$$

$$\equiv \hat{K}_0 + \hat{V}, \qquad (3)$$

where the chemical potential μ is chosen so that $\langle\hat{N}\rangle = N$. Here and henceforth, an implicit summation is to be carried out over all repeated spin indius. With the grand canonical Hamiltonian $\hat{K}$, we introduce the time-dependent Heisenberg picture for Schrödinger operators, and the field operators have a time dependency given by $\hat{\psi}_{K\alpha}^{(\dagger)}(\mathbf{r}t) \equiv e^{i\hat{K}t/\hbar}\hat{\psi}_\alpha^{(\dagger)}(\mathbf{r})e^{-i\hat{K}t/\hbar}$. Using these operators, the single-particle Green's function is defined as

$$iG^{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = \frac{\langle\Psi_0|T[\hat{\psi}_{K\alpha}(\mathbf{r}t)\hat{\psi}_{K\beta}^\dagger(\mathbf{r}'t')]|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}, \qquad (4)$$

where $|\Psi_0\rangle$ denotes the ground-state wave function of the system and T is the time-ordered product of operators.

B. Bogoliubov prescription and ground-state structure

In an ideal Bose gas, all the particles will be in the state with zero momentum. From the operators $\hat{\psi}_\alpha(\mathbf{r})$, we separate the operators corresponding to creation and annihilation of particles in the state with $\mathbf{p} = \mathbf{0}$: $\hat{\psi}_\alpha(\mathbf{r}) = \xi_\alpha + \hat{\psi}_\alpha'(\mathbf{r})$, where $\xi_\alpha = a_{0\alpha}/\sqrt{V}$, and the prime means to omit the term $\mathbf{p} = \mathbf{0}$. The number of particles which are in $\mathbf{p} = \mathbf{0}$ state ($N_0 = V\xi_\alpha^\dagger\xi_\alpha$) becomes infinite as $V\to\infty$. If we neglect the right-hand side in the commutation relation $[\xi_\alpha, \xi_\beta^\dagger] = \delta_{\alpha\beta}/V$, the operators ξ_α and $\xi_\alpha^\dagger$ can be regarded as c number within the first approximation. This approximate procedure clearly neglects fluctuations in the occupation number of the condensate. It is convenient to write $\xi_\alpha = \sqrt{n_0}\zeta_\alpha$, where ζ_α is normalized as $\boldsymbol{\zeta}^\dagger\boldsymbol{\zeta} = 1$.

The ground-state structure of ξ_α is determined by minimizing the energy with a fixed particle number [6,7]. If we use the point-contact interaction

$$U_{\alpha\alpha',\beta\beta'}(\mathbf{r}) \simeq \frac{4\pi\hbar^2}{m}[c_n\delta_{\alpha\alpha'}\delta_{\beta\beta'} + c_s F_{\alpha\alpha'}\cdot F_{\beta\beta'}]\delta(\mathbf{r})$$

$$\equiv \frac{\hbar^2}{m}\tilde{f}^0_{\alpha\alpha',\beta\beta'}\delta(\mathbf{r}), \qquad (5)$$

which is approximately valid in a dilute gas, the energy of the condensate is expressed as $E = \int d\mathbf{r}[-\mu n_0 + (n_0^2/2)(c_n + c_s\langle\mathbf{F}\rangle^2)]$, where $\langle\mathbf{F}\rangle \equiv \zeta_\alpha^* F_{\alpha\beta}\zeta_\beta$. The coefficients c_n and c_s are related to the spin-singlet ($F_t = 0$) and quintuple ($F_t = 2$) s-wave scattering lengths a_{F_t}. Note that the assumption of a two-body point-contact interaction gives rise to a divergent ground-state energy, and renormalized coupling constants should be substituted to eliminate this divergence [15]. There are two types of ground-state structures depending on the sign of the spin-dependent part of interaction, i.e., $c_s < 0$ and $c_s > 0$. When the interaction for the spin-exchange channel is ferromagnetic $c_s < 0$, the ferromagnetic state emerges and the energy is minimized by $\langle\mathbf{F}\rangle^2 = 1$. In this case, the condensate wave function ζ_α in the ferromagnetic ground-state is given by $_f\boldsymbol{\zeta} = (1,0,0)^T$. When the interaction for the spin-exchange channel is antiferromagnetic $c_s > 0$, the polar state emerges and the energy is minimized by $\langle\mathbf{F}\rangle^2 = 0$. In this case, the condensate wave function ζ_α in the polar ground-state is given by $_p\boldsymbol{\zeta} = (0,1,0)^T$. According to the current estimate [6], the scattering lengths of ^{23}Na are $a_2 = (52\pm5)a_B$ and $a_0 = (46\pm4)a_B$; and those for ^{87}Rb are $a_2 = (107\pm4)a_B$ and $a_0 = (110\pm4)a_B$, where a_B is the Bohr radius. If the inequalities suggested by current estimate ($a_2 > a_0$ for ^{23}Na and $a_0 > a_2$ for ^{87}Rb) are true, then the condensates of ^{23}Na and ^{87}Rb are the polar state and the ferromagnetic state, respectively.

The separation of $\hat{\psi}_\alpha(\mathbf{r})$ into the condensate and noncondensate parts modifies the Hamiltonian in a fundamental way. The grand canonical Hamiltonian is now given by

$$\hat{K} = E_0 - \mu N_0 + \frac{1}{V}\sum_{\mathbf{p}\neq 0}(\hbar\omega_\mathbf{p} - \mu)a_{\mathbf{p}\alpha}^\dagger a_{\mathbf{p}\alpha} + \sum_j \hat{V}_j$$

$$\equiv E_0 - \mu N_0 + \hat{K}', \qquad (6)$$

where we denote $\hbar\omega_p = \hbar^2 p^2/2m$. The interaction Hamiltonian $\hat{V}$ separates into the several distinct parts: E_0 is the part which does not include noncondensate operator $a_{p\alpha}^{(\dagger)}$ and $\hat{V}_j$ denotes the parts which include $a_{p\alpha}^{(\dagger)}$, where the index j indicates the number of $a_{p\alpha}^{(\dagger)}$. In a normal system ($n_0 = 0$), only $\hat{V}_4$ is present. We also note that $\hat{K}$ has no term containing a single $a_{p\alpha}^{(\dagger)}$ because these would violate momentum conservation.

The Heisenberg picture after the Bogoliubov prescription is modified, and the field operator $\hat{\psi}_\alpha(\mathbf{r})$ becomes $\hat{\psi}_{K\alpha}(\mathbf{r}t)$ $= \sqrt{n_0}\zeta_\alpha + \hat{\psi}'_{K\alpha}(\mathbf{r}t)$, where $\hat{\psi}'_{K\alpha}(\mathbf{r}t) = e^{it\hat{K}'/\hbar}\hat{\psi}'_{K\alpha}(\mathbf{r})e^{-it\hat{K}'/\hbar}$. After the Bogoliubov prescription, the single-particle Green's function, Eq. (4), is modified as

$$iG^{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = n_0\zeta_\alpha\zeta_\beta^\dagger + \frac{\langle O|T[\hat{\psi}'_{K\alpha}(\mathbf{r}t)\hat{\psi}'^\dagger_{K\beta}(\mathbf{r}'t')]|O\rangle}{\langle O|O\rangle}$$

$$\equiv n_0\zeta_\alpha\zeta_\beta^\dagger + iG'^{\alpha\beta}_{11}(\mathbf{r}t,\mathbf{r}'t'), \qquad (7)$$

where the prime denotes the noncondensate part. Here, the ground-state $|O\rangle$ is not an eigenstate of $\hat{N}$ and differs from the state $|\Psi_0\rangle$ in Eq. (4).

C. Dyson's equations

Using the operator which is defined as $\hat{K}_0 = -\mu N_0 + \hat{K}'_0$, where $\hat{K}'_0 \equiv \Sigma_{p\neq 0}(\hbar\omega_p - \mu)a_{p\alpha}^\dagger a_{p\alpha}$, the field operator in the interaction picture is written as $\hat{\psi}_I(\mathbf{r}t) = e^{i\hat{K}_0 t/\hbar}\hat{\psi}(\mathbf{r})e^{-i\hat{K}_0 t/\hbar}$. The ground-state $|0\rangle$ of the operator $\hat{K}_0$ is an eigenstate of the particle number in the momentum space, which is expressed as $|0\rangle = |N,0,0,\dots\rangle$, where the number of particles in each component of spin is arbitrary and the sum of the particles in zero momentum state is equal to the total number of particles N. The zero-order Green's function, which is the Green's function for an ideal Bose-condensed gas, becomes

$$iG_0^{\alpha\beta}(\mathbf{r}t,\mathbf{r}'t') = \langle 0|T[\hat{\psi}'_{I\alpha}(\mathbf{r}t)\hat{\psi}'^\dagger_{I\beta}(\mathbf{r}'t')]|0\rangle. \qquad (8)$$

After a simple calculation, its Fourier transform is expressed as

$$G_0^{\alpha\beta}(p) = \frac{\delta_{\alpha\beta}}{\omega - \omega_p + \mu/\hbar + i\delta}, \qquad (9)$$

where we use the notation $p = (\mathbf{p}, \omega)$. Note that there is no backward propagation in time, or hole propagation in Eq. (9).

The presence of the condensate leads to the appearance of self-energy diagrams of an anomalous type. These terms stem from the interaction of particles which are not in the condensate with particles in the condensate, and they contain operator $\xi_\alpha^{(\dagger)}$ at certain vertices. By the Bogoliubov prescription, these operators behave as c numbers, and there exist the terms which contain the interaction potential, at which the number of the noncondensate particles is not conserved. It follows that we can make the following three types of clas-

sifications for all the irreducible self-energy diagrams: Diagrams with one ingoing and one outgoing noncondensate particle, and the corresponding sum of matrix elements is denoted by $\Sigma_{11}^{\alpha\beta}(p)$; diagrams with two outgoing noncondensate particles, and the corresponding sum of matrix elements is denoted by $\Sigma_{12}^{\alpha\beta}(p)$; diagrams with two ingoing non-condensate particles, and the corresponding sum of matrix elements is denoted by $\Sigma_{21}^{\alpha\beta}(p)$.

Correspondingly, we must introduce two anomalous exact Green's functions $G'^{\alpha\beta}_{12}(p)$ and $G'^{\alpha\beta}_{21}(p)$, which represent the appearance and disappearance of two noncondensate particles from the condensate. The anomalous Green's functions are defined in terms of Heisenberg operators:

$$iG'^{\alpha\beta}_{12}(\mathbf{r}t,\mathbf{r}'t') = \frac{\langle O|T[\hat{\psi}'^\dagger_{K\alpha}(\mathbf{r}t)\hat{\psi}'^\dagger_{K\beta}(\mathbf{r}'t')]|O\rangle}{\langle O|O\rangle}, \quad (10)$$

$$iG'^{\alpha\beta}_{21}(\mathbf{r}t,\mathbf{r}'t') = \frac{\langle O|T[\hat{\psi}'_{K\alpha}(\mathbf{r}t)\hat{\psi}'_{K\beta}(\mathbf{r}'t')]|O\rangle}{\langle O|O\rangle}. \quad (11)$$

Dyson's equations for a scalar Bose-condensed system are first derived by Beliaev [11,12]. We generalize these equations for this Bose-condensed system with spin degrees of freedom, with are written as follows:

$$G'^{\alpha\beta}_{11}(p) = G_0^{\alpha\beta}(p) + G_0^{\alpha\gamma}(p)\Sigma^{\gamma\delta}_{11}(p)G'^{\delta\beta}_{11}(p)$$
$$+ G_0^{\alpha\gamma}(p)\Sigma^{\gamma\delta}_{12}(p)G'^{\delta\beta}_{21}(p), \qquad (12)$$

$$G'^{\alpha\beta}_{12}(p) = G_0^{\alpha\gamma}(p)\Sigma^{\gamma\delta}_{12}(p)G'^{\delta\beta}_{11}(-p)$$
$$+ G_0^{\alpha\gamma}(p)\Sigma^{\gamma\delta}_{11}(p)G'^{\delta\beta}_{12}(p), \qquad (13)$$

$$G'^{\alpha\beta}_{21}(p) = G_0^{\alpha\gamma}(-p)\Sigma^{\gamma\delta}_{21}(p)G'^{\delta\beta}_{11}(p)$$
$$+ G_0^{\alpha\gamma}(-p)\Sigma^{\gamma\delta}_{11}(-p)G'^{\delta\beta}_{21}(p), \qquad (14)$$

$$G'^{\alpha\beta}_{11}(-p) = G_0^{\alpha\beta}(-p) + G_0^{\alpha\gamma}(-p)\Sigma^{\gamma\delta}_{21}(p)G'^{\delta\beta}_{12}(p)$$
$$+ G_0^{\alpha\gamma}(-p)\Sigma^{\gamma\delta}_{11}(-p)G'^{\delta\beta}_{11}(-p). \qquad (15)$$

Overall four-momentum conservation determines the direction of the momentum flow. Equations (12) and (15) are clearly equivalent equations for $G'^{\alpha\beta}_{11}(p)$. Dyson's equations (12)–(15) become a single-matrix equation

$$\hat{G}'(p) = \hat{G}_0(p) + \hat{G}_0(p)\hat{\Sigma}(p)\hat{G}'(p), \qquad (16)$$

where

$$\hat{G}'(p) = \begin{bmatrix} G'_{11}(p) & G'_{12}(p) \\ G'_{21}(p) & G'_{11}(-p) \end{bmatrix}, \qquad (17)$$

$$\hat{\Sigma}(p) = \begin{bmatrix} \Sigma_{11}(p) & \Sigma_{12}(p) \\ \Sigma_{21}(p) & \Sigma_{11}(-p) \end{bmatrix}, \qquad (18)$$

$$\hat{G}_0(p) = \begin{bmatrix} G_0(p) & 0 \\ 0 & G_0(-p) \end{bmatrix}. \qquad (19)$$

In Eqs. (17)–(19) matrix elements are also the 3×3 matrices, and the elements in $\hat{G}'(p)$ and $\hat{G}_0(p)$ are defined by the Fourier components of Eqs. (7), (8), (10), and (11).

Due to the isotropy of space, all quantities depend on the magnitude of the wave vector $\boldsymbol{p}$. Using the Lehmann representation of the anomalous Green's functions $G'^{\alpha\beta}_{12}(p)$ and $G'^{\alpha\beta}_{21}(p)$, we find that they are even functions of frequency ω [16]; from these functions and Eqs. (27), (35), and (36), it is easily seen that $\Sigma^{\alpha\beta}_{12}(p)$ and $\Sigma^{\alpha\beta}_{21}(p)$ are also even functions of $p=(\boldsymbol{p},\omega)$. Since the interaction Hamiltonian preserves the total number of particles, the anomalous self-energies are symmetric in the operators ψ_α and $\psi^\dagger_\alpha$. Therefore, with any diagram for $\Sigma^{\alpha\beta}_{21}(p)$, we can associate exactly the same diagram for $\Sigma^{\alpha\beta}_{12}(p)$, obtained by replacing all ingoing lines in the diagram for $\Sigma^{\alpha\beta}_{21}(p)$ with outgoing lines, and *vice versa*. This procedure reverses the direction in which all internal lines are traversed. However, we can reverse the direction in which all internal lines are traversed by changing p to $-p$ in the matrix element corresponding to the given diagram for $\Sigma^{\alpha\beta}_{21}(p)$. Since $\Sigma^{\alpha\beta}_{21}(p)$ is an even function of p, it follows that $\Sigma^{\alpha\beta}_{21}(p)=\Sigma^{\alpha\beta}_{12}(p),G'^{\alpha\beta}_{21}(p)=G'^{\alpha\beta}_{12}(p)$.

Since rotational symmetry along z axis is not broken, the z component of the total angular momentum is conserved, and the matrix structures of $\hat{G}'(p)$ and $\hat{\Sigma}(p)$ are determined [17]. Due to spin conservation, the nonzero elements of normal Green's functions and self-energies are the diagonal elements for both the ferromagnetic and the polar states. For ferromagnetic state, since condensed particles are in $_f\zeta=(1,0,0)^T$ and the interaction Hamiltonian preserves the total number of the particles, the existence of the anomalous Green's functions and self-energies are forbidden except for $G^{++}_{12}(p)$ and $\Sigma^{++}_{12}(p)$. Then, the matrix Green's functions and self-energies are expressed as

$$_f\boldsymbol{G}'_{11}(p)=\begin{bmatrix} {}_fG'^{++}_{11}(p) & 0 & 0 \\ 0 & {}_fG'^{00}_{11}(p) & 0 \\ 0 & 0 & {}_fG'^{--}_{11}(p) \end{bmatrix}, \quad (20)$$

$$_f\boldsymbol{\Sigma}_{11}(p)=\begin{bmatrix} {}_f\Sigma^{++}_{11}(p) & 0 & 0 \\ 0 & {}_f\Sigma^{00}_{11}(p) & 0 \\ 0 & 0 & {}_f\Sigma^{--}_{11}(p) \end{bmatrix}, \quad (21)$$

$$_f\boldsymbol{G}'_{12}(p)=\begin{bmatrix} {}_fG'^{++}_{12}(p) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (22)$$

$$_f\boldsymbol{\Sigma}_{12}(p)=\begin{bmatrix} {}_f\Sigma^{++}_{12}(p) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (23)$$

By these matrix forms and Dyson's equation (16), the matrix inversion yields

$$_fG'^{++}_{11}(p)=\frac{\omega+\omega_{\boldsymbol{p}}-\mu/\hbar+{}_f\Sigma^{++}_{11}(-p)}{{}_fD(p)}, \quad (24)$$

$$_fG'^{00}_{11}(p)=\frac{1}{\omega-\omega_{\boldsymbol{p}}+\mu/\hbar-{}_f\Sigma^{00}_{11}(p)+i\delta}, \quad (25)$$

$$_fG'^{--}_{11}(p)=\frac{1}{\omega-\omega_{\boldsymbol{p}}+\mu/\hbar-{}_f\Sigma^{--}_{11}(p)+i\delta}, \quad (26)$$

$$_fG'^{++}_{12}(p)=-\frac{{}_f\Sigma^{++}_{12}(p)}{{}_fD(p)}, \quad (27)$$

where

$$_fD(p)=\left[\omega-\frac{{}_f\Sigma^{++}_{11}(p)-{}_f\Sigma^{++}_{11}(-p)}{2}\right]^2$$
$$-\left[\omega_{\boldsymbol{p}}-\frac{{}_f\Sigma^{++}_{11}(p)+{}_f\Sigma^{++}_{11}(-p)}{2}-\mu/\hbar\right]^2$$
$$+{}_f\Sigma^{++}_{12}(p)\,{}_f\Sigma^{++}_{12}(p). \quad (28)$$

For the polar state, since the condensed particles are in $_p\zeta=(0,1,0)^T$, the nonzero elements of the anomalous Green's functions and self-energies are the elements in which the sum of the spin indices are zero. Then, the matrix Green's functions and self-energies are expressed as

$$_p\boldsymbol{G}'_{11}(p)=\begin{bmatrix} {}_pG'^{++}_{11}(p) & 0 & 0 \\ 0 & {}_pG'^{00}_{11}(p) & 0 \\ 0 & 0 & {}_pG'^{--}_{11}(p) \end{bmatrix}, \quad (29)$$

$$_p\boldsymbol{\Sigma}_{11}(p)=\begin{bmatrix} {}_p\Sigma^{++}_{11}(p) & 0 & 0 \\ 0 & {}_p\Sigma^{00}_{11}(p) & 0 \\ 0 & 0 & {}_p\Sigma^{--}_{11}(p) \end{bmatrix}, \quad (30)$$

$$_p\boldsymbol{G}'_{12}(p)=\begin{bmatrix} 0 & 0 & {}_pG'^{+-}_{12}(p) \\ 0 & {}_pG'^{00}_{12}(p) & 0 \\ {}_pG'^{-+}_{12}(p) & 0 & 0 \end{bmatrix}, \quad (31)$$

$$_p\boldsymbol{\Sigma}_{12}(p)=\begin{bmatrix} 0 & 0 & {}_p\Sigma^{+-}_{12}(p) \\ 0 & {}_p\Sigma^{00}_{12}(p) & 0 \\ {}_p\Sigma^{-+}_{12}(p) & 0 & 0 \end{bmatrix}. \quad (32)$$

Since the ground-state structure of the condensate wave function $_p\zeta$ is invariant under spin reflection, the Green's functions and self-energies are also symmetric under spin reflection: $_pG'^{++}_{11}(p)={}_pG'^{--}_{11}(p)$, $_p\Sigma^{++}_{11}(p)={}_p\Sigma^{--}_{11}(p)$, $_pG'^{+-}_{12}(p)={}_pG'^{-+}_{12}(p)$, and $_p\Sigma^{+-}_{12}(p)={}_p\Sigma^{-+}_{12}(p)$. By these matrix forms and Dyson's equation (16), the matrix inversion yields

$$\mathrm{_pG'}_{11}^{++}(p)=\frac{\omega+\omega_{\boldsymbol{p}}-\mu/\hbar+_{\mathrm p}\Sigma_{11}^{++}(-p)}{_{\mathrm p}D_+(p)}, \qquad (33)$$

$$\mathrm{_pG'}_{11}^{00}(p)=\frac{\omega+\omega_{\boldsymbol{p}}-\mu/\hbar+_{\mathrm p}\Sigma_{11}^{00}(-p)}{_{\mathrm p}D_0(p)}, \qquad (34)$$

$$\mathrm{_pG'}_{12}^{+-}(p)=-\frac{_{\mathrm p}\Sigma_{12}^{+-}(p)}{_{\mathrm p}D_+(p)}, \qquad (35)$$

$$\mathrm{_pG'}_{12}^{00}(p)=-\frac{_{\mathrm p}\Sigma_{12}^{00}(p)}{_{\mathrm p}D_0(p)}, \qquad (36)$$

where

$$_{\mathrm p}D_+(p)=\left[\omega-\frac{_{\mathrm p}\Sigma_{11}^{++}(p)-_{\mathrm p}\Sigma_{11}^{++}(-p)}{2}\right]^2$$
$$-\left[\omega_{\boldsymbol{p}}-\frac{_{\mathrm p}\Sigma_{11}^{++}(p)+_{\mathrm p}\Sigma_{11}^{++}(-p)}{2}-_{\mathrm p}\mu/\hbar\right]^2$$
$$+_{\mathrm p}\Sigma_{12}^{+-}(p)_{\mathrm p}\Sigma_{12}^{+-}(p), \qquad (37)$$

$$_{\mathrm p}D_0(p)=\left[\omega-\frac{_{\mathrm p}\Sigma_{11}^{00}(p)-_{\mathrm p}\Sigma_{11}^{00}(-p)}{2}\right]^2$$
$$-\left[\omega_{\boldsymbol{p}}-\frac{_{\mathrm p}\Sigma_{11}^{00}(p)+_{\mathrm p}\Sigma_{11}^{00}(-p)}{2}-_{\mathrm p}\mu/\hbar\right]^2$$
$$+_{\mathrm p}\Sigma_{12}^{00}(p)_{\mathrm p}\Sigma_{12}^{00}(p). \qquad (38)$$

Note that the matrix Green's functions $\hat{G}'(p)$ and self-energies $\hat{\boldsymbol{\Sigma}}(p)$ are symmetric matrix for both the ferromagnetic and the polar states.

D. First-order Green's function

The typical interatomic potential involves a hard core. This brings a problem for the perturbation theory in terms of the bare potential $U_{\alpha\alpha',\beta\beta'}(\boldsymbol{r})$. Since $U_{\alpha\alpha',\beta\beta'}(\boldsymbol{r})$ can be large, the first few terms in such a perturbation expansion are no longer sufficient. Indeed, one has to sum over an infinite number of terms, i.e., the ladder-type diagram. In the ladder approximation, effective potential $\Gamma^0_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ involves the sum over all the ladder diagrams to infinite order in $U_{\alpha\alpha',\beta\beta'}(\boldsymbol{r})$, taking into account the repeated scattering of two particles in a gas. In Appendix A, we derive the effective potential $\Gamma^0_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$, and show the connection between $\Gamma^0_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ and the s-wave scattering amplitude $\tilde{f}_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}')$ within the ladder approximation, which is Eqs. (A12) or (A13).

The self-energies are determined by the special values which $\Gamma^0_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ takes when two out of four particles involved in a process belong to the condensed phase. Each particle of the condensed phase carries a factor $\sqrt{n_0}\,\zeta_\alpha$, we find

$$\Sigma_{11}^{\alpha\beta}(p)=\frac{n_0}{\hbar}\zeta_{\mu_1}^\dagger\Gamma^0_{\alpha\beta,\mu_1\nu_1}(\boldsymbol{p}/2,\boldsymbol{p}/2;P)\zeta_{\nu_1}$$
$$+\frac{n_0}{\hbar}\zeta_{\mu_1}^\dagger\Gamma^0_{\mu_1\beta,\alpha\nu_1}(-\boldsymbol{p}/2,\boldsymbol{p}/2;P)\zeta_{\nu_1}, \qquad (39)$$

$$\Sigma_{12}^{\alpha\beta}(p)=\frac{n_0}{\hbar}\Gamma^0_{\alpha\nu_1,\beta\nu_2}(\boldsymbol{p},0;0)\zeta_{\nu_1}\zeta_{\nu_2}, \qquad (40)$$

$$\Sigma_{21}^{\alpha\beta}(p)=\frac{n_0}{\hbar}\zeta_{\mu_1}^\dagger\zeta_{\mu_2}^\dagger\Gamma^0_{\mu_1\alpha,\mu_2\beta}(0,\boldsymbol{p};0). \qquad (41)$$

To obtain the chemical potential, we must let all four particles in $\Gamma^0_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ belong to the condensed phase, and divide by one power of n_0. We then have

$$\mu=n_0\zeta_{\mu_1}^\dagger\zeta_{\mu_2}^\dagger\Gamma^0_{\mu_1\nu_1,\mu_2\nu_2}(0,0;0)\zeta_{\nu_1}\zeta_{\nu_2}. \qquad (42)$$

To proceed further, we must know the explicit value of the scattering amplitude $\tilde{f}_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}')$. At small momenta, we neglect the wave-vector dependence of the lowest-order term in Eq. (A12): $\tilde{f}_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}')\simeq\tilde{f}^0_{\alpha\alpha',\beta\beta'}$, where $\tilde{f}^0_{\alpha\alpha',\beta\beta'}$ is the momentum-independent part of $\tilde{f}_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}')$ and is defined in Eq. (5). This approximation is allowed when the wavelength is long compared to the characteristic size of the interaction region, which has an order of magnitude given by the scattering amplitude $\tilde{f}^0_{\alpha\alpha',\beta\beta'}$.

Substituting the effective potential $\Gamma^0_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ in Eqs. (A12) into Eqs. (39), (40), and (42), and the value of $\Gamma^0_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ in Eqs. (A13) into Eqs. (41), we find

$$\Sigma_{11}^{\alpha\beta}(p)=\frac{n_0}{\hbar}\zeta_{\mu_1}^\dagger\left[\frac{\hbar^2}{m}(\tilde{f}^0_{\alpha\beta,\mu_1\nu_1}+\tilde{f}^0_{\mu_1\beta,\alpha\nu_1})+\left(\frac{\hbar^2}{m}\right)^2\right.$$
$$\times(\tilde{f}^0_{\alpha\mu_2,\mu_1\nu_2}\tilde{f}^0_{\mu_2\beta,\nu_2\nu_1}+\tilde{f}^0_{\mu_1\mu_2,\alpha\nu_2}\tilde{f}^0_{\mu_2\beta,\nu_2\nu_1})$$
$$\times\int\frac{d\boldsymbol{q}}{(2\pi)^3}\left(\frac{1}{\hbar\omega-\frac{\hbar^2p^2}{4m}+2\mu-\frac{\hbar^2q^2}{m}+i\delta}\right.$$
$$\left.\left.+\mathrm{P}\frac{1}{\frac{\hbar^2q^2}{m}-\frac{\hbar^2p^2}{4m}}\right)\right]\zeta_{\nu_1}, \qquad (43)$$

$$\Sigma_{12}^{\alpha\beta}(p)=\frac{n_0}{\hbar}\left[\frac{\hbar^2}{m}\tilde{f}^0_{\alpha\nu_1,\beta\nu_2}+\left(\frac{\hbar^2}{m}\right)^2\tilde{f}^0_{\alpha\mu_1,\beta\mu_2}\tilde{f}^0_{\mu_1\nu_1,\mu_2\nu_2}\right.$$
$$\times\int\frac{d\boldsymbol{q}}{(2\pi)^3}\left(\frac{1}{2\mu-\frac{\hbar^2q^2}{m}+i\delta}+\frac{1}{\frac{\hbar^2q^2}{m}-i\delta}\right)\right]$$
$$\times\zeta_{\nu_1}\zeta_{\nu_2}, \qquad (44)$$

$$\Sigma_{21}^{\alpha\beta}(p)=\frac{n_0}{\hbar}\zeta_{\mu_1}\zeta_{\mu_2}\left[\frac{\hbar^2}{m}\tilde{f}^0_{\mu_1\alpha,\mu_2\beta}+\left(\frac{\hbar^2}{m}\right)^2\tilde{f}^0_{\mu_1\nu_1,\mu_2\nu_2}\tilde{f}^0_{\nu_1\alpha,\nu_2\beta}\right.$$

$$\left.\times\int\frac{d\boldsymbol{q}}{(2\pi)^3}\left(\frac{1}{2\mu-\dfrac{\hbar^2q^2}{m}+i\delta}+\frac{1}{\dfrac{\hbar^2q^2}{m}+i\delta}\right)\right],$$

$$(45)$$

$$\mu=n_0\zeta_{\mu_1}\zeta_{\mu_2}\left[\frac{\hbar^2}{m}\tilde{f}^0_{\mu_1\nu_1,\mu_2\nu_2}+\left(\frac{\hbar^2}{m}\right)^2\tilde{f}^0_{\mu_1\mu_3,\mu_2\nu_3}\tilde{f}^0_{\mu_3\nu_1,\nu_3\nu_2}\right.$$

$$\left.\times\int\frac{d\boldsymbol{q}}{(2\pi)^3}\left(\frac{1}{2\mu-\dfrac{\hbar^2q^2}{m}+i\delta}+\frac{1}{\dfrac{\hbar^2q^2}{m}-i\delta}\right)\right]\zeta_{\nu_1}\zeta_{\nu_2},$$

$$(46)$$

where we use Eq. (A9). For dimensional reasons, all the integral terms in Eqs. (43)–(46) are of the order of $(n_0a_{F_t}^3)^{1/2}$. Within the first-order approximation, we neglect the integral terms in Eqs. (43)–(46).

For the ferromagnetic state, the condensate wave function can be chosen to $_f\zeta_\alpha$, and the first-order self-energies are

$$_f\boldsymbol{\Sigma}_{11}(p)^{(1)}=\frac{4\pi\hbar n_0}{m}\begin{pmatrix}2(c_n+c_s)&0&0\\0&c_n+c_s&0\\0&0&c_n-c_s\end{pmatrix},$$

$$(47)$$

$$_f\boldsymbol{\Sigma}_{12}(p)^{(1)}=\frac{4\pi\hbar n_0}{m}\begin{pmatrix}c_n+c_s&0&0\\0&0&0\\0&0&0\end{pmatrix},$$

$$(48)$$

$$_f\mu^{(1)}=\frac{4\pi\hbar^2n_0}{m}(c_n+c_s).$$

$$(49)$$

The resultant first-order Green's functions are derived using Eqs. (24)–(28),

$$_fG'^{++}_{11}(p)=\frac{A_p}{\omega-E_p/\hbar+i\delta}-\frac{B_p}{\omega+E_p/\hbar-i\delta},$$

$$(50)$$

$$_fG'^{++}_{12}(p)=\frac{C_p}{\omega-E_p/\hbar+i\delta}-\frac{C_p}{\omega+E_p/\hbar-i\delta},$$

$$(51)$$

$$_fG'^{00}_{11}(p)=\frac{1}{\omega-\omega_p+i\delta},$$

$$(52)$$

$$_fG'^{--}_{11}(p)=\frac{1}{\omega-\varepsilon_p/\hbar+i\delta},$$

$$(53)$$

where

$$E_p=[(\hbar\omega_p)^2+2_f\mu^{(1)}\hbar\omega_p]^{1/2},$$

$$(54)$$

$$\varepsilon_p=\hbar\omega_p+2\frac{4\pi\hbar^2n_0}{m}|c_s|,$$

$$(55)$$

$$A_p=\frac{1}{2E_p}[E_p+{_f\mu^{(1)}}+\omega_p],$$

$$(56)$$

$$B_p=\frac{1}{2E_p}[-E_p+{_f\mu^{(1)}}+\omega_p],$$

$$(57)$$

$$C_p=\frac{1}{2E_p}[-{_f\mu^{(1)}}].$$

$$(58)$$

The excitation energies of collective modes are determined by the poles of above Green's functions. The density-wave mode has a Bogoliubov spectrum E_p. The spin-wave mode has a free particle spectrum $\hbar\omega_p$; and there exists the "quadrupole" spin-wave mode, which has a free-particle-like spectrum with a finite gap $2(4\pi\hbar^2n_0/m)|c_s|$. For the system to be stable, the chemical potential $_f\mu^{(1)}$ should be positive, which means $c_n+c_s>0$.

For the polar state, one can choose the condensate wave function as $_p\zeta_\alpha$, and the first-order self-energies are

$$_p\boldsymbol{\Sigma}_{11}(p)^{(1)}=\frac{4\pi\hbar n_0}{m}\begin{pmatrix}c_n+c_s&0&0\\0&2c_n&0\\0&0&c_n+c_s\end{pmatrix},$$

$$(59)$$

$$_p\boldsymbol{\Sigma}_{12}(p)^{(1)}=\frac{4\pi\hbar n_0}{m}\begin{pmatrix}0&0&c_s\\0&c_n&0\\c_s&0&0\end{pmatrix},$$

$$(60)$$

$$_p\mu^{(1)}=\frac{4\pi\hbar^2n_0}{m}c_n.$$

$$(61)$$

The resultant first-order Green's functions are derived using Eqs. (33)–(38),

$$_pG'^{++}_{11}(p)=\frac{A^s_p}{\omega-E^s_p/\hbar+i\delta}-\frac{B^s_p}{\omega+E^s_p/\hbar-i\delta},$$

$$(62)$$

$$_pG'^{00}_{11}(p)=\frac{A^n_p}{\omega-E^n_p/\hbar+i\delta}-\frac{B^n_p}{\omega+E^n_p/\hbar-i\delta},$$

$$(63)$$

$$_pG'^{++}_{12}(p)=\frac{C^s_p}{\omega-E^s_p/\hbar+i\delta}-\frac{C^s_p}{\omega+E^s_p/\hbar-i\delta},$$

$$(64)$$

$$_pG'^{00}_{12}(p)=\frac{C^n_p}{\omega-E^n_p/\hbar+i\delta}-\frac{C^n_p}{\omega+E^n_p/\hbar-i\delta},$$

$$(65)$$

where

$$E^{s(n)}_p=\left[(\hbar\omega_p)^2+2\frac{4\pi\hbar^2n_0c_{s(n)}}{m}\hbar\omega_p\right]^{1/2},$$

$$(66)$$
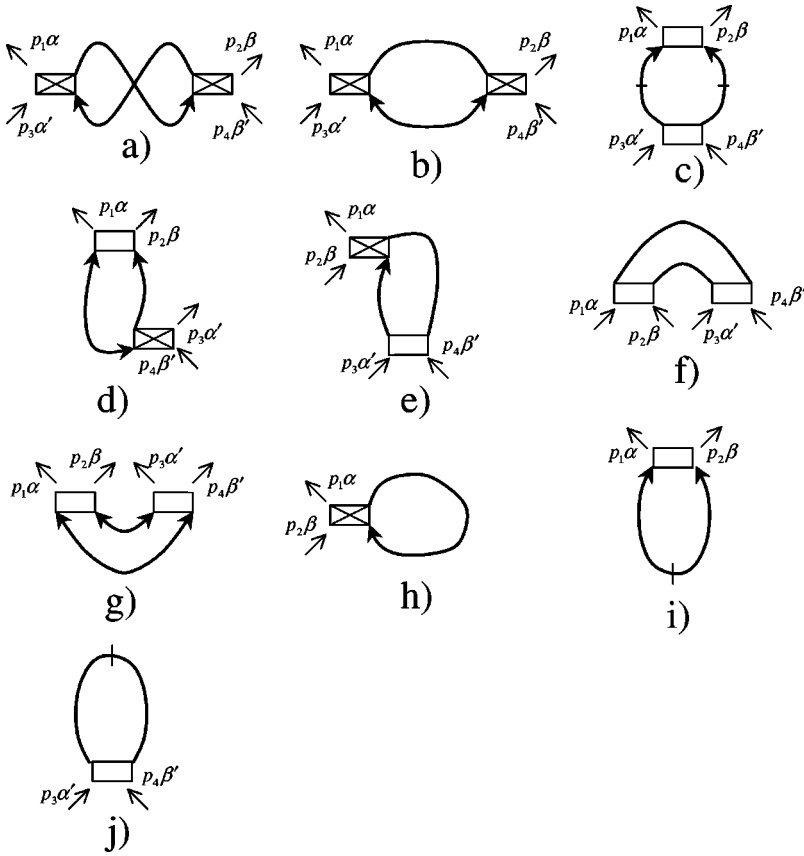
FIG. 1. Topologically different diagrams of order $(n_0 a_{F_t}^3)^{1/2}$.

$$A_p^{s(n)} = \frac{1}{2E_p^{s(n)}}\left[E_p^{s(n)} + \frac{4\pi\hbar^2 n_0 c_{s(n)}}{m} + \omega_p\right], \qquad (67)$$

$$B_p^{s(n)} = \frac{1}{2E_p^{s(n)}}\left[-E_p^{s(n)} + \frac{4\pi\hbar^2 n_0 c_{s(n)}}{m} + \omega_p\right], \qquad (68)$$

$$C_p^{s(n)} = \frac{1}{2E_p^{s(n)}}\left[-\frac{4\pi\hbar^2 n_0 c_{s(n)}}{m}\right]. \qquad (69)$$

In this case, the excitation forms are all of the Bogoliubov form, which are E_p^n and E_p^s. The former is the density-wave mode and the latter is the spin-wave mode. For the system to be stable, the chemical potential $_p\mu^{(1)}$ should be positive, which means $c_n > 0$.

Note that above excitation energies for both the ferromagnetic and the polar states coincide with the results of previous studies [6,7].

E. Second-order approximation

For the second-order approximation to the self-energies and the chemical potential, we must retain quantities of the order $(n_0 a_{F_t}^3)^{1/2}$. The integral terms in Eqs. (43)–(46) are of the order of $(n_0 a_{F_t}^3)^{1/2}$. The diagrams containing one loop with three or more continuous lines give contributions of the same order. The summation over sets of diagrams, which differ only in the number of continuous lines in a loop, is automatically performed if one replaces the zero-order

Green's function $G_0^{\alpha\beta}(p)$ with the first-order Green's function $G'^{\alpha\beta}_{ij}(p)$ $(i,j=1,2)$. We therefore consider the loops, which can be built out of $G'^{\alpha\beta}_{ij}(p)$ and $\Gamma^0_{\alpha\alpha',\beta\beta'}(\mathbf{k},\mathbf{k}';P)$. There are ten essentially different loops and we show them in Fig. 1. A rectangle denotes the constant part of the effective potential which corresponds to $(\hbar^2/m)\widetilde{f}^0_{\alpha\alpha',\beta\beta'}$, and a rectangle with a cross denotes a sum of two rectangles, one being a direct interaction and the other an exchange interaction. The two differ only by an interchange of the upper or the lower ends. The sum of the two rectangles introduces a factor

$$\Gamma^0_{\alpha\alpha',\beta\beta'}(\mathbf{p},\mathbf{p}';P) + \Gamma^0_{\beta\alpha',\alpha\beta'}(-\mathbf{p},\mathbf{p}';P)$$

$$\simeq \frac{\hbar^2}{m}(\widetilde{f}^0_{\alpha\alpha',\beta\beta'} + \widetilde{f}^0_{\beta\alpha',\alpha\beta'}) \equiv \frac{\hbar^2}{m}\widetilde{f}_{\alpha\alpha',\beta\beta'}. \qquad (70)$$

If the first-order Green's functions $G'^{\alpha\beta}_{11}(p)$, $G'^{\alpha\beta}_{12}(p)$, and $G'^{\alpha\beta}_{21}(p)$ are expanded in powers of the effective potential $\Gamma^0_{\alpha\alpha',\beta\beta'}(\mathbf{k},\mathbf{k}';P)$, then in the lowest approximation the diagrams (c,i,j) in Fig. 1 become loops with two continuous lines of $G_0^{\alpha\beta}(p)$. But all such loops are already included in $\Gamma^0_{\alpha\alpha',\beta\beta'}(\mathbf{k},\mathbf{k}';P)$ and must therefore be omitted. This omission is represented in Fig. 1 by the strokes across the continuous lines. It is convenient to separate out contributions, which are explicitly linear and quadratic in $\widetilde{f}^0_{\alpha\alpha',\beta\beta'}$,

$$\mathbf{\Sigma}_{11}(p) = \mathbf{\Sigma}_{11}(p)^{(1)} + \mathbf{\Sigma}_{11}(p)^{(2)}, \qquad (71)$$

$$\Sigma_{12}(p) = \Sigma_{12}(p)^{(1)} + \Sigma_{12}(p)^{(2)}, \tag{72}$$

$$\mu = \mu^{(1)} + \mu^{(2)}. \tag{73}$$

Here, the first-order terms $\Sigma_{11}(p)^{(1)}$, $\Sigma_{12}(p)^{(1)}$, and $\mu^{(1)}$ have been given in Eqs. (47)–(49) and (59)–(61) for the ferromagnetic and polar states, respectively. For further convenience, we call second-order contributions from the integral terms in Eqs. (43)–(46) type-I contributions, and denote their contributions to the self-energies and the chemical potential as $\Sigma(p)^{\mathrm{I}}$ and μ^{I}, respectively. Similarly, the contributions from the diagrams in Fig. 1 are called the type-II contributions and their contributions are denoted as $\Sigma(p)^{\mathrm{II}}$ and μ^{II}.

Let $F^{a,\cdots,g}(p_1,p_2,p_3,p_4)$ and $F^{h,i,j}(p_1,p_2)$ denote the contributions from the diagrams in Fig. 1, which are written as follows:

$$F^a_{\alpha\alpha',\beta\beta'}(p_1,p_2;p_3,p_4)$$
$$= \left(\frac{i}{\hbar}\right)^2 \int \frac{d^4q}{(2\pi)^4} \frac{\hbar^2}{m} \tilde{f}_{\alpha\alpha',\beta_1\beta_1'} G'^{\beta_1'\alpha_1}_{11}(q) \frac{\hbar^2}{m}$$
$$\times \tilde{f}_{\alpha_1\alpha_1',\beta\beta'} G'^{\alpha_1'\beta_1}_{11}(p_3 - p_1 + q), \tag{74}$$

$$F^b_{\alpha\alpha',\beta\beta'}(p_1,p_2;p_3,p_4)$$
$$= \left(\frac{i}{\hbar}\right)^2 \int \frac{d^4q}{(2\pi)^4} G'^{\beta_1\alpha_1}_{21}(p_3 - p_1 + q)$$
$$\times \frac{\hbar^2}{m} \tilde{f}_{\alpha\alpha',\beta_1\beta_1'} \frac{\hbar^2}{m} \tilde{f}_{\alpha_1\alpha_1',\beta\beta'} G'^{\beta_1'\alpha_1'}_{12}(q), \tag{75}$$

$$F^c_{\alpha\alpha',\beta\beta'}(p_1,p_2;p_3,p_4)$$
$$= \left(\frac{i}{\hbar}\right)^2 \int \frac{d^4q}{(2\pi)^4} \frac{\hbar^2}{m} \tilde{f}^0_{\alpha\alpha',\beta\beta_1'} [G'^{\alpha_1'\alpha_1}_{11}(q) G'^{\beta_1'\beta_1}_{11}$$
$$\times (p_3 + p_4 - q) - G'^{\alpha_1'\alpha_1}_0(q) G'^{\beta_1'\beta_1}_0$$
$$\times (p_3 + p_4 - q)] \frac{\hbar^2}{m} \tilde{f}^0_{\alpha_1\alpha',\beta_1\beta'}, \tag{76}$$

$$F^d_{\alpha\alpha'\beta,\beta'}(p_1,p_2,p_3;p_4)$$
$$= \left(\frac{i}{\hbar}\right)^2 \int \frac{d^4q}{(2\pi)^4} \frac{\hbar^2}{m} \tilde{f}^0_{\alpha\alpha_1',\beta\beta_1'} G'^{\beta_1'\alpha_1}_{11}$$
$$\times (p_1 + p_2 - q) G'^{\alpha_1'\alpha_2'}_{12}(q) \frac{\hbar^2}{m} \tilde{f}_{\alpha_1\alpha_2',\alpha'\beta'}, \tag{77}$$

$$F^e_{\alpha,\alpha'\beta\beta'}(p_1;p_2,p_3,p_4)$$
$$= \left(\frac{i}{\hbar}\right)^2 \int \frac{d^4q}{(2\pi)^4} \frac{\hbar^2}{m} \tilde{f}_{\alpha\beta,\beta_1\beta_1'} G'^{\beta_1'\alpha_1}_{11}$$
$$\times (p_3 + p_4 - q) G'^{\beta_1\beta_2}_{21}(q) \frac{\hbar^2}{m} \tilde{f}^0_{\alpha_1\alpha',\beta_2\beta'}, \tag{78}$$

$$F^f_{\alpha\alpha'\beta\beta'}(;p_1,p_2,p_3,p_4)$$
$$= \left(\frac{i}{\hbar}\right)^2 \int \frac{d^4q}{(2\pi)^4} \frac{\hbar^2}{m} \tilde{f}^0_{\alpha_1\alpha,\beta_1\beta} \frac{\hbar^2}{m} \tilde{f}^0_{\alpha_2\alpha',\beta_2\beta'}$$
$$\times G'^{\alpha_1\beta_2}_{21}(p_1 + p_2 - q) G'^{\beta_1\alpha_2}_{21}(q), \tag{79}$$

$$F^g_{\alpha\alpha'\beta\beta'}(p_1,p_2,p_3,p_4;)$$
$$= \left(\frac{i}{\hbar}\right)^2 \int \frac{d^4q}{(2\pi)^4} \frac{\hbar^2}{m} \tilde{f}^0_{\alpha\alpha_1',\beta\beta_1'} \frac{\hbar^2}{m} \tilde{f}^0_{\alpha'\alpha_2',\beta'\beta_2'} G'^{\beta_1'\alpha_2'}_{12}(q)$$
$$\times G'^{\alpha_1'\beta_2'}_{12}(p_1 + p_2 - q), \tag{80}$$

$$F^h_{\alpha,\alpha'}(p_1;p_2) = \left(\frac{i}{\hbar}\right) \int \frac{d^4q}{(2\pi)^4} \frac{\hbar^2}{m} \tilde{f}_{\alpha\alpha_1',\beta_1\beta_1'} G'^{\beta_1'\beta}_{11}(q), \tag{81}$$

$$F^i_{\alpha\beta}(p_1p_2;) = \left(\frac{i}{\hbar}\right) \int \frac{d^4q}{(2\pi)^4} \frac{\hbar^2}{m} \tilde{f}^0_{\alpha\alpha_1',\beta\beta_1'}$$
$$\times \left[G'^{\alpha_1'\beta_1'}_{12}(q) - \left(\frac{n_0}{\hbar}\right) G^{\alpha_1'\alpha_1}_0(q) G^{\beta_1'\beta_1}_0(-q) \right.$$
$$\left. \times \frac{\hbar^2}{m} \tilde{f}^0_{\alpha_1\alpha_2',\beta_1\beta_2'} \zeta_{\alpha_2'} \zeta_{\beta_2'} \right], \tag{82}$$

$$F^j_{\alpha'\beta'}(;p_3p_4) = \left(\frac{i}{\hbar}\right) \int \frac{d^4q}{(2\pi)^4} \left[G'^{\alpha_1\beta_1}_{21}(q) - \left(\frac{n_0}{\hbar}\right) \zeta^\dagger_{\alpha_2} \zeta^\dagger_{\beta_2} \right.$$
$$\left. \times \frac{\hbar^2}{m} \tilde{f}^0_{\alpha_2\alpha_1',\beta_2\beta_1'} G^{\alpha_1'\alpha_1}_0(q) G^{\beta_1'\beta_1}_0(-q) \right]$$
$$\times \frac{\hbar^2}{m} \tilde{f}^0_{\alpha_1\alpha',\beta_1\beta'}. \tag{83}$$

Here momentum conservation is assumed to hold everywhere. Everywhere on the right-hand side of Eqs. (74)–(83), the value of the first-order approximation $_f\mu^{(1)}$ or $_p\mu^{(1)}$ should should be substituted for μ.

The self-energies involve special values of the functions $F^{a,\cdots,g}(p_1,p_2,p_3,p_4)$ and $F^{h,i,j}(p_1,p_2)$, together with a factor $\sqrt{n_0}\,\zeta_\alpha$ for each particle of the condensed phase:

$$\Sigma_{11}^{\alpha\beta}(p)^{\mathrm{II}}=(-in_0)[\zeta_{\beta_1}^{\dagger}F_{\alpha\alpha_1',\beta_1\beta}^{a}(p,0;0,p)\zeta_{\alpha_1'}+\zeta_{\beta_1}^{\dagger}F_{\alpha\alpha_1',\beta_1\beta}^{b}(p,0;0,p)\zeta_{\alpha_1'}+\zeta_{\beta_1}^{\dagger}F_{\alpha\beta,\beta_1\beta_1'}^{c}(p,0;p,0)\zeta_{\beta_1'}$$

$$+\zeta_{\beta_1}^{\dagger}F_{\alpha\alpha_1',\beta_1\beta}^{c}(p,0;0,p)\zeta_{\alpha_1'}+\zeta_{\alpha_1}^{\dagger}\zeta_{\beta_1}^{\dagger}F_{\alpha\alpha_1\beta_1,\beta}^{d}(p,0,0;p)+\zeta_{\alpha_1}^{\dagger}\zeta_{\alpha_2}^{\dagger}F_{\alpha_1\alpha_2\alpha,\beta}^{d}(0,p,0;p)+F_{\alpha,\alpha_1'\beta_1\beta}^{e}(p;0,0,p)\zeta_{\alpha_1'}\zeta_{\beta_1}$$

$$+F_{\alpha,\beta\beta_1\beta_1'}^{e}(p;0,p,0)]\zeta_{\beta_1}\zeta_{\beta_1'}+F_{\alpha,\beta}^{h}(p;p), \tag{84}$$

$$\Sigma_{12}^{\alpha\beta}(p)^{\mathrm{II}}=(-in_0)[F_{\alpha\alpha_1',\beta\beta_1'}^{a}(p,-p;0,0)\zeta_{\alpha_1'}\zeta_{\beta_1'}+F_{\alpha\alpha_1',\beta\beta_1'}^{b}(p,-p;0,0)\zeta_{\alpha_1'}\zeta_{\beta_1'}+\zeta_{\beta_1}^{\dagger}F_{\alpha\beta\beta_1,\beta_1'}^{d}(p,0,-p;0)\zeta_{\beta_1'}bf$$

$$+\zeta_{\beta_1}^{\dagger}F_{\alpha\beta\beta_1,\beta_1'}^{d}(-p,0,p;0)\zeta_{\beta_1'}+\zeta_{\alpha_1}^{\dagger}F_{\alpha_1\alpha\beta,\beta_1'}^{d}(0,p,-p;0)\zeta_{\beta_1'}+\zeta_{\alpha_1}^{\dagger}F_{\alpha_1\alpha\beta,\beta_1'}^{d}(0,-p,p;0)\zeta_{\beta_1'}$$

$$+\zeta_{\beta_1}^{\dagger}\zeta_{\beta_2}^{\dagger}F_{\alpha\beta\beta_1\beta_2}^{g}(p,0,-p,0;)+\zeta_{\beta_1}^{\dagger}\zeta_{\alpha_1'}^{\dagger}F_{\alpha\alpha_1'\beta_1\beta}^{g}(p,0,0,-p;)]+F_{\alpha\beta}^{i}(p,p;), \tag{85}$$

$$\Sigma_{21}^{\alpha\beta}(p)^{\mathrm{II}}=(-in_0)[\zeta_{\alpha_1}^{\dagger}\zeta_{\beta_1}^{\dagger}F_{\alpha_1\alpha,\beta_1\beta}^{a}(0,0;p,-p)+\zeta_{\alpha_1}^{\dagger}\zeta_{\beta_1}^{\dagger}F_{\alpha_1\alpha,\beta_1\beta}^{b}(0,0;p,-p)+\zeta_{\alpha_1}^{\dagger}F_{\alpha_1,\alpha\beta\beta_1'}^{e}(0;p,-p,0)\zeta_{\beta_1'}$$

$$+\zeta_{\alpha_1}^{\dagger}F_{\alpha_1,\alpha\beta\beta_1'}^{e}(0;-p,p,0)\zeta_{\beta_1'}+\zeta_{\alpha_1}^{\dagger}F_{\alpha_1\alpha_1'\alpha\beta}^{e}(0;p,0,-p)\zeta_{\alpha_1'}+\zeta_{\alpha_1}^{\dagger}F_{\alpha_1\alpha_1'\alpha,\beta}^{e}(0;-p,0,p)\zeta_{\alpha_1'}$$

$$+F_{\alpha\beta_1\beta\beta_1'}^{f}(;p,0,-p,0)\zeta_{\beta_1}\zeta_{\beta_1'}+F_{\alpha\alpha_1'\beta_1\beta}^{f}(p,0,0,-p;)\zeta_{\beta_1}\zeta_{\alpha_1'}]+F_{\alpha\beta}^{j}(;p,-p). \tag{86}$$

The chemical potential is determined by the terms arising from the excitation of two particles out of the condensate, where they interact repeatedly and then drop back into the condensate. The sums of contributions from diagrams of these types are, respectively, $F_h(0;0)$ and $F_i(0,0;)$:

$$\mu^{\mathrm{II}}/\hbar = \zeta_{\alpha}^{\dagger}F_{\alpha\beta}^{h}(0;0)\zeta_{\beta}+\zeta_{\alpha}^{\dagger}\zeta_{\beta}^{\dagger}F_{\alpha\beta}^{i}(0,0;). \tag{87}$$

To carry out the frequency integration in Eqs. (74)–(83), it is convenient to note that $G_{12}'(p)=G_{21}'(p)$. The frequency integrations are now performed and the results should be substituted into Eqs. (84)–(87).

Adding the type-I terms in Eqs. (43)–(46) to type-II terms in Eqs. (84)–(87), we obtain the second-order self-energies and chemical potential. For the ferromagnetic state,

$$_{\mathrm{f}}\Sigma_{11}^{++}(p)^{(2)}=\frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2\int\frac{d\mathbf{q}}{(2\pi)^3}\left[\frac{_{\mathrm{f}}N_{11}(E_q,E_k)}{\hbar\omega-E_q-E_k+i\delta}\right.$$

$$\left.-\frac{_{\mathrm{f}}N_{11}(-E_q,-E_k)}{\hbar\omega+E_q+E_k-i\delta}+2\frac{(c_n+c_s)^2}{\frac{\hbar^2q^2}{m}-\frac{\hbar^2p^2}{4m}}\right]$$

$$+2\frac{4\pi\hbar}{m}(c_n+c_s)\int\frac{d\mathbf{q}}{(2\pi)^3}B_q, \tag{88}$$

$$_{\mathrm{f}}\Sigma_{11}^{00}(p)^{(2)}=\frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2(c_n+c_s)^2\int\frac{d\mathbf{q}}{(2\pi)^3}$$

$$\times\left[\frac{A_q+B_q+2C_q}{\hbar\omega-E_q-\hbar\omega_k+i\delta}+\frac{1}{\frac{\hbar^2q^2}{m}-\frac{\hbar^2p^2}{4m}}\right]$$

$$+\frac{4\pi\hbar}{m}(c_n+c_s)\int\frac{d\mathbf{q}}{(2\pi)^3}B_q, \tag{89}$$

$$_{\mathrm{f}}\Sigma_{11}^{--}(p)^{(2)}=\frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2\int\frac{d\mathbf{q}}{(2\pi)^3}\left[(c_n-c_s)^2\right.$$

$$\times\frac{A_q+B_q+2C_q}{\hbar\omega-E_q-\varepsilon_k+i\delta}+2c_s^2$$

$$\times\frac{1}{\hbar\omega-\hbar\omega_q-\hbar\omega_k+i\delta}+[(c_n-c_s)^2$$

$$+2c_s^2]\frac{1}{\frac{\hbar^2q^2}{m}-\frac{\hbar^2p^2}{4m}}\right]$$

$$+\frac{4\pi\hbar}{m}(c_n-c_s)\int\frac{d\mathbf{q}}{(2\pi)^3}B_q, \tag{90}$$

$$_f\Sigma_{12}^{++}(p)^{(2)} = \frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left[\frac{_fN_{12}(E_q,E_q)}{\hbar\omega-E_q-E_k+i\delta}\right.$$

$$\left. -\frac{_fN_{12}(-E_q,-E_q)}{\hbar\omega+E_q+E_k-i\delta}+\frac{(c_n+c_s)^2}{\frac{\hbar^2q^2}{m}}\right]$$

$$+\frac{4\pi\hbar}{m}(c_n+c_s)\int \frac{d\boldsymbol{q}}{(2\pi)^3}C_q, \qquad (91)$$

$$_f\mu^{(2)}=\frac{4\pi\hbar^2}{m}(c_n+c_s)\int \frac{d\boldsymbol{q}}{(2\pi)^3}$$

$$\times\left[C_q+2B_q+n_0\frac{4\pi\hbar^2}{m}(c_n+c_s)\frac{1}{\frac{\hbar^2q^2}{m}}\right], \quad (92)$$

where $\boldsymbol{k}=\boldsymbol{p}-\boldsymbol{q}$, and

$$_fN_{11}(E_q,E_k)=2(c_n+c_s)^2(2A_qB_k+2C_qC_k+A_qA_k$$
$$+4C_qA_k), \qquad (93)$$

$$_fN_{12}(E_q,E_k)=2(c_n+c_s)^2(2B_qA_k+2C_qA_k+2C_qB_k$$
$$+3C_qC_k). \qquad (94)$$

For the polar state,

$$_p\Sigma_{11}^{++}(p)^{(2)} = \frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left[\frac{_pN_{11}(E_q^s,E_k^n)}{\hbar\omega-E_q^s-E_k^n+i\delta}\right.$$

$$\left. -\frac{_pN_{11}(-E_q^s,-E_k^n)}{\hbar\omega+E_q^s+E_k^n-i\delta}+\frac{(c_n+c_s)^2}{\frac{\hbar^2q^2}{m}-\frac{\hbar^2p^2}{4m}}\right]$$

$$+\frac{4\pi\hbar}{m}\int \frac{d\boldsymbol{q}}{(2\pi)^3}[(c_n+c_s)B_q^n$$

$$+(3c_n+c_s)B_q^s], \qquad (95)$$

$$_p\Sigma_{11}^{00}(p)^{(2)} = \frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left[\frac{_pN_{11}^n(E_q^n,E_k^n)}{\hbar\omega-E_q^n-E_k^n+i\delta}\right.$$

$$\left. -\frac{_pN_{11}^n(-E_q^n,-E_k^n)}{\hbar\omega+E_q^n+E_k^n-i\delta}+\frac{_pN_{11}^s(E_q^s,E_k^s)}{\hbar\omega-E_q^s-E_k^s+i\delta}\right.$$

$$\left. -\frac{_pN_{11}^s(-E_q^s,-E_k^s)}{\hbar\omega+E_q^s+E_k^s-i\delta}+2\frac{(c_n^2+2c_s^2)}{\frac{\hbar^2q^2}{m}-\frac{\hbar^2p^2}{4m}}\right]$$

$$+2\frac{4\pi\hbar}{m}\int \frac{d\boldsymbol{q}}{(2\pi)^3}[c_nB_q^n+(c_n+c_s)B_q^s], \qquad (96)$$

$$_p\Sigma_{12}^{+-}(p)^{(2)} = \frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left[\frac{_pN_{12}(E_q^s,E_k^n)}{\hbar\omega-E_q^s-E_k^n+i\delta}\right.$$

$$\left. -\frac{_pN_{12}(-E_q^s,-E_k^n)}{\hbar\omega+E_q^s+E_k^n-i\delta}\right]+\frac{4\pi\hbar}{m}\int \frac{d\boldsymbol{q}}{(2\pi)^3}$$

$$\times\left[(c_n-c_s)C_q^s+c_sC_q^n\right.$$

$$\left. +\frac{n_0}{2}\frac{4\pi\hbar^2}{m}c_s(2c_n-c_s)\frac{1}{\hbar\omega_q}\right], \qquad (97)$$

$$_p\Sigma_{12}^{00}(p)^{(2)} = \frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left[\frac{_pN_{12}^n(E_q^n,E_k^n)}{\hbar\omega-E_q^n-E_k^n+i\delta}\right.$$

$$\left. -\frac{_pN_{12}^n(-E_q^n,-E_k^n)}{\hbar\omega+E_q^n+E_k^n-i\delta}+\frac{_pN_{12}^s(E_q^s,E_k^s)}{\hbar\omega-E_q^s-E_k^s+i\delta}\right.$$

$$\left. -\frac{_pN_{12}^s(-E_q^s,-E_k^s)}{\hbar\omega+E_q^s+E_k^s-i\delta}\right]+\frac{4\pi\hbar}{m}\int \frac{d\boldsymbol{q}}{(2\pi)^3}$$

$$\times\left[c_nC_q^n+2c_sC_q^s+\frac{n_0}{2}\frac{4\pi\hbar^2}{m}\frac{(c_n^2+2c_s^2)}{\hbar\omega_q}\right], \qquad (98)$$

$$_p\mu^{(2)}=\frac{4\pi\hbar^2}{m}\int \frac{d\boldsymbol{q}}{(2\pi)^3}\left[c_n(C_q^n+2B_q^n+2B_q^s)+2c_s(C_q^s+B_q^s)\right.$$

$$\left. +\frac{n_0}{2}\frac{4\pi\hbar^2}{m}\frac{(c_n^2+2c_s^2)}{\hbar\omega_q}\right], \qquad (99)$$

where

$$_pN_{11}(E_q^s,E_k^n)=(c_n+c_s)^2A_q^s(A_k^n+B_k^n+2C_k^n)$$
$$+4c_s(c_n+c_s)C_q^s(C_k^n+A_k^n)+4c_s^2B_q^sA_k^n, \qquad (100)$$

$$_pN_{11}^n(E_q^n,E_k^n)=2c_n^2(2A_q^nB_k^n+2C_q^nC_k^n+A_q^nA_k^n+4C_q^nA_k^n), \qquad (101)$$

$$_pN_{11}^s(E_q^s,E_k^s)=2(c_n+c_s)^2(A_q^sB_k^s+C_q^sC_k^s)+4c_s^2A_q^sA_k^s$$
$$+8c_s(c_n+c_s)C_q^sA_k^s, \qquad (102)$$

$$_pN_{12}(E_q^s,E_k^n)=(c_n+c_s)^2C_q^s(A_k^n+B_k^n+2C_k^n)+4c_s^2C_q^sC_k^n$$
$$+2c_s(c_n+c_s)(A_q^sB_k^n+B_q^sA_k^n+A_q^sC_k^n$$
$$+B_q^sC_k^n), \qquad (103)$$

$$_pN_{12}^n(E_q^n,E_k^n)=2c_n^2(2B_q^nA_k^n+2C_q^nA_k^n+2C_q^nB_k^n+3C_q^nC_k^n), \qquad (104)$$

$$_pN_{12}^s(E_q^s,E_k^s)=2(c_n+c_s)^2(B_q^sA_k^s+C_q^sC_k^s)+4c_s(c_n+c_s)$$

$$\times(C_q^sA_k^s+C_q^sB_k^s)+4c_s^2C_q^sC_k^s. \tag{105}$$

Once we have obtained a specific form for self-energies and chemical potential, we then substitute these terms into Eqs. (24)–(28) and Eqs. (33)–(38) to calculate the Green's functions. Since the second-order terms are small corrections, we can express the Green's functions in a form analogous to the first-order Green's functions in Eqs. (50)–(53) and (62)–(65). For the ferromagnetic state,

$$_fG'^{++}_{11}(p)=\frac{A_p+{}_f\alpha^{++}(p)}{\omega-E_p/\hbar-{}_f\Lambda^{++}(p)^+}$$

$$-\frac{B_p+{}_f\alpha^{++}(p)}{\omega+E_p/\hbar+{}_f\Lambda^{++}(p)^-}, \tag{106}$$

$$_fG'^{00}_{11}(p)=\frac{1}{\omega-\omega_p-{}_f\Lambda^{00}(p)+i\delta}, \tag{107}$$

$$_fG'^{--}_{11}(p)=\frac{1}{\omega-\omega_p-{}_f\Lambda^{--}(p)+i\delta}, \tag{108}$$

$$_fG'^{++}_{12}(p)=\frac{C_p+{}_f\beta^{++}(p)}{\omega-E_p/\hbar-{}_f\Lambda^{++}(p)^+}$$

$$-\frac{C_p+{}_f\beta^{++}(p)}{\omega+E_p/\hbar+{}_f\Lambda^{++}(p)^-}, \tag{109}$$

where

$$_f\Lambda^{++}(p)^\pm=\frac{\hbar\omega_p}{2E_p}[{}_f\Sigma^{++}_{11}(p)^{(2)}+{}_f\Sigma^{++}_{11}(-p)^{(2)}-2{}_f\mu^{(2)}/\hbar]$$

$$+\frac{{}_f\mu^{(1)}}{2E_p}[{}_f\Sigma^{++}_{11}(p)^{(2)}+\Sigma^{++}_{11}(-p)^{(2)}$$

$$-2{}_f\mu^{(2)}/\hbar-{}_f\Sigma^{++}_{12}(p)^{(2)}]\pm\frac{1}{2}[{}_f\Sigma^{++}_{11}(p)^{(2)}$$

$$-{}_f\Sigma^{++}_{11}(-p)^{(2)}], \tag{110}$$

$$_f\alpha^{++}(p)/\hbar=\frac{{}_f\mu^{(1)}}{4E_p^3}\{2{}_f\Sigma^{++}_{12}(p)^{(2)}\hbar\omega_p-{}_f\mu^{(1)}[{}_f\Sigma^{++}_{11}(p)^{(2)}$$

$$+{}_f\Sigma^{++}_{11}(-p)^{(2)}-2{}_f\mu^{(2)}/\hbar-{}_f\Sigma^{++}_{12}(p)^{(2)}]\}, \tag{111}$$

$$_f\beta^{++}(p)/\hbar=-\frac{1}{2E_p}{}_f\Sigma^{++}_{12}(p)^{(2)}+\frac{{}_f\mu^{(1)}}{4E_p^3}\{\hbar\omega_p[{}_f\Sigma^{++}_{11}(p)^{(2)}$$

$$+{}_f\Sigma^{++}_{11}(-p)^{(2)}-2{}_f\mu^{(2)}/\hbar]$$

$$-{}_f\mu^{(1)}[{}_f\Sigma^{++}_{11}(p)^{(2)}+{}_f\Sigma^{++}_{11}(-p)^{(2)}$$

$$-2{}_f\mu^{(2)}/\hbar-{}_f\Sigma^{++}_{12}(p)^{(2)}]\}, \tag{112}$$

$$_f\Lambda^{00}(p)=\Sigma^{00}_{11}(p)^{(2)}-{}_f\mu^{(2)}/\hbar, \tag{113}$$

$$_f\Lambda^{--}(p)=\Sigma^{--}_{11}(p)^{(2)}-{}_f\mu^{(2)}/\hbar. \tag{114}$$

For the polar state, Green's functions including the second-order corrections are

$$_pG'^{++}_{11}(p)=\frac{A_p^s+{}_p\alpha^s(p)}{\omega-E_p^s/\hbar-{}_p\Lambda^s(p)^+}-\frac{B_p^s+{}_p\alpha^s(p)}{\omega+E_p^s/\hbar+{}_p\Lambda^s(p)^-}, \tag{115}$$

$$_pG'^{00}_{11}(p)=\frac{A_p^n+{}_p\alpha^n(p)}{\omega-E_p^n/\hbar-{}_p\Lambda^n(p)^+}-\frac{B_p^n+{}_p\alpha^n(p)}{\omega+E_p^n/\hbar+{}_p\Lambda^n(p)^-}, \tag{116}$$

$$_pG'^{+-}_{12}(p)=\frac{C_p^s+{}_p\beta^s(p)}{\omega-E_p^s/\hbar-{}_p\Lambda^s(p)^+}-\frac{C_p^s+{}_p\beta^s(p)}{\omega+E_p^s/\hbar+{}_p\Lambda^s(p)^-}, \tag{117}$$

$$_pG'^{00}_{12}(p)=\frac{C_p^n+{}_p\beta^n(p)}{\omega-E_p^n/\hbar-{}_p\Lambda^n(p)^+}-\frac{C_p^n+{}_p\beta^n(p)}{\omega+E_p^n/\hbar+{}_p\Lambda^n(p)^-}, \tag{118}$$

where

$$_p\Lambda^s(p)^\pm=\frac{\hbar\omega_p}{2E_p^s}[{}_p\Sigma^{++}_{11}(p)^{(2)}+{}_p\Sigma^{++}_{11}(-p)^{(2)}-2{}_p\mu^{(2)}/\hbar]$$

$$+\frac{{}_p\mu^{(1)}}{2E_p}[{}_p\Sigma^{++}_{11}(p)^{(2)}+{}_p\Sigma^{++}_{11}(-p)^{(2)}-2{}_p\mu^{(2)}/\hbar$$

$$-{}_p\Sigma^{+-}_{12}(p)^{(2)}]\pm\frac{1}{2}[{}_p\Sigma^{++}_{11}(p)^{(2)}$$

$$-{}_p\Sigma^{++}_{11}(-p)^{(2)}], \tag{119}$$

$$_p\alpha^s(p)/\hbar=\frac{{}_p\mu^{(1)}}{4E_p^{s3}}\{2{}_p\Sigma^{+-}_{12}(p)^{(2)}\hbar\omega_p-{}_p\mu^{(1)}[{}_p\Sigma^{++}_{11}(p)^{(2)}$$

$$+{}_p\Sigma^{++}_{11}(-p)^{(2)}-2{}_p\mu^{(2)}/\hbar-{}_p\Sigma^{+-}_{12}(p)^{(2)}]\}, \tag{120}$$

$$_p\beta^s(p)/\hbar=-\frac{1}{2E_p^s}{}_p\Sigma^{+-}_{12}(p)^{(2)}+\frac{{}_p\mu^{(1)}}{4E_p^{s3}}\{\hbar\omega_p[{}_p\Sigma^{++}_{11}(p)^{(2)}$$

$$+{}_p\Sigma^{++}_{11}(-p)^{(2)}-2{}_p\mu^{(2)}/\hbar]$$

$$-{}_p\mu^{(1)}[{}_p\Sigma^{++}_{11}(p)^{(2)}+{}_p\Sigma^{++}_{11}(-p)^{(2)}-2{}_p\mu^{(2)}/\hbar$$

$$-{}_p\Sigma^{+-}_{12}(p)^{(2)}]\}, \tag{121}$$

and we can obtain the terms $_p\Lambda^n(p)^\pm$, $_p\alpha^n(p)$, and $_p\beta^n(p)$ by replacing the subscripts of self-energies, which are $^{++}$ and $^{+-}$, with ^{00} and E_p^s with E_p^n in corresponding terms.

III. FINITE-TEMPERATURE EXTENSION

So far, we have studied Bose-condensed system with spin degrees of freedom at the absolute zero of temperature. For finite temperatures, the problem becomes more complicated. But to a great extent, the discussion and calculations at finite temperatures duplicate the corresponding material in the preceding section. To avoid the duplication, we discuss the difference between $T=0$ and $T \neq 0$.

A. Temperature Green's function

If we consider the finite-temperature case, the definition of the Green's functions becomes slightly different. With the grand canonical Hamiltonian $\hat{K}$, the grand partition function and statistical operator are written as $Z_G = e^{-\beta\Omega} = \mathrm{tr} e^{-\beta\hat{K}}$ and $\hat{\rho}_G = Z_G^{-1} e^{-\beta\hat{K}} = e^{\beta(\Omega-\hat{K})}$, respectively where we use the short-hand notation $\beta = 1/k_B T$, and Ω is the thermodynamic potential. We introduce the modified τ-dependent Heisenberg picture, and the field operator assumes the form $\hat{\psi}_{K\alpha}(\boldsymbol{r}\tau) = e^{\hat{K}\tau/\hbar}\hat{\psi}_\alpha(\boldsymbol{r})e^{-\hat{K}\tau/\hbar}$. Then, the single-particle temperature Green's function is defined as

$$G_{\alpha\beta}(\boldsymbol{r}\tau,\boldsymbol{r}'\tau') = -\mathrm{tr}[\hat{\rho}_G T_\tau[\hat{\psi}_{K\alpha}(\boldsymbol{r}\tau)\hat{\psi}^\dagger_{K\beta}(\boldsymbol{r}'\tau')]]. \quad (122)$$

Here, the symbol T_τ orders the operators according to their value of τ. The trace tr implies a sum over a complete set of states in the Hilbert space, each contribution being weighted with the operator $\hat{\rho}_G$.

The single-particle temperature Green's function for noncondensate part is also defined as follows

$$G'^{\,\alpha\beta}_{11}(\boldsymbol{r}\tau,\boldsymbol{r}'\tau') = -\mathrm{tr}[\hat{\rho}_G T_\tau[\hat{\psi}'_{K\alpha}(\boldsymbol{r}\tau)\hat{\psi}'^\dagger_{K\beta}(\boldsymbol{r}'\tau')]], \quad (123)$$

where the trace means the sum over a complete set of eigenstates after the Bogoliubov prescription. The diagrams in the technique for absolute zero and the diagrams in Matsubara's technique differ only in which integration over frequencies for $T=0$ is replaced by summation over discrete "frequencies" in $i\omega_n$ for $T \neq 0$. More precisely, any expression for a correction to a temperature Green's function, corresponding to a certain diagram, can be obtained from the expression for the Green's function for $T=0$, corresponding to the same diagram, if in the latter expression we replace ω with $i\omega_n$ and integration by the summation, according to the rule $(i/2\pi)\int d\omega \cdots \rightarrow -(1/\beta\hbar)\sum_n \cdots$. This fact allows us to immediately extend many results in the preceding section to the case $T \neq 0$, provided we simply change the notation. In particular, Dyson's equation still holds in Matsubara's technique and takes the same form as for the case $T=0$, and can be obtained by replacing ω with $i\omega_n$ in Eq. (16).

B. Second-order approximation at finite temperatures

As we note in previous sections, at finite temperatures, any expression for the self-energies, corresponding to a certain diagram, can be obtained from the expression for the self-energies at $T=0$, corresponding to the same diagram, if

in the latter expression we replace ω with $i\omega_n$ and integration with summation. Effective potential at finite temperatures $\Gamma^T_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ are derived in Appendix B.

Like at the absolute zero of temperature, the contributions from the effective potential are determined by the special values, which $\Gamma^T_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ takes when two out of four particles involved in a process belong to the condensed phase:

$$\Sigma^{\alpha\beta}_{11}(p) = \frac{n_0}{\hbar}\zeta^\dagger_{\mu_1}\Gamma^T_{\alpha\beta,\mu_1\nu_1}(\boldsymbol{p}/2,\boldsymbol{p}/2;P)\zeta_{\nu_1}$$

$$+\frac{n_0}{\hbar}\zeta^\dagger_{\mu_1}\Gamma^T_{\mu_1\beta,\alpha\nu_1}(-\boldsymbol{p}/2,\boldsymbol{p}/2;P)\zeta_{\nu_1}, \quad (124)$$

$$\Sigma^{\alpha\beta}_{12}(p) = \frac{n_0}{\hbar}\Gamma^T_{\alpha\nu_1,\beta\nu_2}(\boldsymbol{p},0;0)\zeta_{\nu_1}\zeta_{\nu_2}, \quad (125)$$

$$\Sigma^{\alpha\beta}_{21}(p) = \frac{n_0}{\hbar}\zeta^\dagger_{\mu_1}\zeta^\dagger_{\mu_2}\Gamma^T_{\mu_1\alpha,\mu_2\beta}(0,\boldsymbol{p};0). \quad (126)$$

To obtain the chemical potential, we must let all four particles in $\Gamma^T_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ belong to the condensed phase, and divide by one power of n_0. We then have

$$\mu = n_0 \zeta^\dagger_{\mu_1}\zeta^\dagger_{\mu_2}\Gamma^T_{\mu_1\nu_1,\mu_2\nu_2}(0,0;0)\zeta_{\nu_1}\zeta_{\nu_2}. \quad (127)$$

Using the value of $\Gamma^0_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ in Eqs. (A12) and (A13) and substituting the value of $\Gamma^T_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ in Eq. (B6) into Eqs. (124)–(127), we find

$$\Sigma^{\alpha\beta}_{11}(p) = \frac{n_0}{\hbar}\zeta^\dagger_{\mu_1}\left\{ \frac{\hbar^2}{m}\tilde{f}_{\alpha\beta,\mu_1\nu_1} + \left(\frac{\hbar^2}{m}\right)^2 \tilde{f}_{\alpha\mu_2,\mu_1\nu_2}\tilde{f}^0_{\mu_2\beta,\nu_2\nu_1} \right.$$

$$\times \int \frac{d\boldsymbol{q}}{(2\pi)^3}$$

$$\times \left[\frac{1 + f_B(\hbar\omega_{P/2+q}-\mu) + f_B(\hbar\omega_{P/2-q}-\mu)}{i\hbar\omega_n - \dfrac{\hbar^2 p^2}{4m} + 2\mu - \dfrac{\hbar^2 q^2}{m}} \right.$$

$$\left.\left.+ \frac{\mathcal{P}}{\dfrac{\hbar^2 q^2}{m} - \dfrac{\hbar^2 p^2}{4m}} \right]\right\}\zeta_{\nu_1}, \quad (128)$$

$$\Sigma^{\alpha\beta}_{12}(p) = \frac{n_0}{\hbar}\left\{ \frac{\hbar^2}{m}\tilde{f}^0_{\alpha\nu_1,\beta\nu_2} + \left(\frac{\hbar^2}{m}\right)^2 \tilde{f}^0_{\alpha\mu_1,\beta\mu_2}\tilde{f}^0_{\mu_1\nu_1,\mu_2\nu_2} \right.$$

$$\times \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left[\frac{1 + 2f_B(\hbar\omega_q-\mu)}{2\mu - \dfrac{\hbar^2 q^2}{m}} \right.$$

$$\left.\left.+ \frac{1}{\dfrac{\hbar^2 q^2}{m} - i\delta} \right]\right\}\zeta_{\nu_1}\zeta_{\nu_2}, \quad (129)$$

$$\Sigma_{21}^{\alpha\beta}(p)=\frac{n_0}{\hbar}\zeta_{\mu_1}\zeta_{\mu_2}\left\{\frac{\hbar^2}{m}\tilde{f}^0_{\mu_1\alpha,\mu_2\beta}+\left(\frac{\hbar^2}{m}\right)^2\tilde{f}^0_{\mu_1\nu_1,\mu_2\nu_2}\tilde{f}^0_{\nu_1\alpha,\nu_2\beta}\right.$$

$$\left.\times\int\frac{d\boldsymbol{q}}{(2\pi)^3}\left[\frac{1+2f_{\mathrm{B}}(\hbar\omega_q-\mu)}{2\mu-\dfrac{\hbar^2q^2}{m}}+\frac{1}{\dfrac{\hbar^2q^2}{m}+i\delta}\right]\right\},$$

(130)

$$\mu=n_0\zeta_{\mu_1}\zeta_{\mu_2}\left\{\frac{\hbar^2}{m}\tilde{f}^0_{\mu_1\nu_1,\mu_2\nu_2}+\left(\frac{\hbar^2}{m}\right)^2\tilde{f}^0_{\mu_1\mu_3,\mu_2\nu_3}\tilde{f}^0_{\mu_3\nu_1,\nu_3\nu_2}\right.$$

$$\left.\times\int\frac{d\boldsymbol{q}}{(2\pi)^3}\left[\frac{1+2f_{\mathrm{B}}(\hbar\omega_q-\mu)}{2\mu-\dfrac{\hbar^2q^2}{m}}+\frac{1}{\dfrac{\hbar^2q^2}{m}-i\delta}\right]\right\}\zeta_{\nu_1}\zeta_{\nu_2},$$

(131)

where we use Eq. (A9) and the notation in Eq. (70). If we substitute the value of the ground-state condensate wave functions $_{\mathrm{f}}\zeta_\alpha$ and $_{\mathrm{p}}\zeta_\alpha$ into Eqs. (128)–(131), the self-energies and the chemical potential for the ferromagnetic and polar state are obtained by summing spin indexes.

Within the first-order approximation, in which we neglect the integrals in Eqs. (128)–(131), we obtain the self-energies and the chemical potential, which coincide with the results derived at absolute zero of temperature in Eqs. (47)–(49) and (59)–(61) for the ferromagnetic and polar states, respectively. As a result, the first-order temperature Green's functions are derived by replacing ω with $i\omega_n$ in Eqs. (50)–(53) and (62)–(65).

For the second-order approximation to the self-energies and the chemical potential, we must retain the terms of the order of $(n_0a_{F_t}^3)^{1/2}$. Besides the integral terms in Eqs. (128)–

(131), which are the type-I contributions, we must add the contributions from the diagrams containing one loop with continuous lines, which are the type-II contribution and give the contributions of the order of $(n_0a_{F_t}^3)^{1/2}$. The essentially different diagrams of the type-II contribution are same as the ones at $T=0$ and are shown in Fig. 1. We have to express the diagrams in Fig. 1 in terms of the temperature Green's functions and change the coefficient of the nth-order contributions $(i/\hbar)^n(-i)^C$ to $(-1/\hbar)^n(-1)^C$ in Eqs. (74)–(83), where C is the number of the condensate factors n_0 appearing in the diagram. To calculate the self-energies and the chemical potential, we must collect the special values of the functions $F^{a,\cdots,g}(p_1,p_2,p_3,p_4)$ and $F^{h,i,j}(p_1,p_2)$ for each quantity like Eqs. (74)–(83) at $T=0$. If we change the coefficient $-in_0$ to $-n_0$ in Eqs. (84)–(87), we can obtain the desired expressions at finite temperatures.

Adding the contributions from the diagrams in Fig. 1 to the second-order terms from Eqs. (128)–(131), we obtain the second-order self-energies and the chemical potential at finite temperatures. If we replace ω with $i\omega_n$ in Eqs. (88)–(92) and Eqs. (95)–(99), the "temperature-independent" parts of each quantity are the same as the ones derived at $T=0$ for both the ferromagnetic and the polar states. To distinguish the temperature-dependent parts from the temperature-independent parts, we denote the temperature-dependent parts, which depend on the Bose distribution function, by $\Sigma(p)_T$ and the temperature independent parts, which do not depend on the Bose distribution function *explicitly*, by $\Sigma(p)_0$: $\Sigma(p)=\Sigma(p)_0+\Sigma(p)_T$. Although at finite temperatures the condensate density $n_0(T)$ is temperature dependent, we neglect the implicit temperature dependence through $n_0(T)$ and regard the implicit temperature-dependent terms as temperature-independent terms. For the ferromagnetic state, the temperature-dependent parts of the self-energies and the chemical potential are written as

$$_{\mathrm{f}}\Sigma_{11}^{++}(p)_T^{(2)}=\frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2\int\frac{d\boldsymbol{q}}{(2\pi)^3}\left\{\left[\frac{_{\mathrm{f}}N_{11}(E_q,E_k)}{i\hbar\omega_n-E_q-E_k}-\frac{_{\mathrm{f}}N_{11}(-E_q,-E_k)}{i\hbar\omega_n+E_q+E_k}\right]f_+(E_q,E_k)+\left[-\frac{_{\mathrm{f}}N_{11}(E_q,-E_k)}{i\hbar\omega_n-E_q+E_k}\right.\right.$$

$$\left.+\frac{_{\mathrm{f}}N_{11}(-E_q,E_k)}{i\hbar\omega_n+E_q-E_k}\right]f_-(E_q,E_k)-\frac{(c_n+c_s)^2}{i\hbar\omega_n-\hbar\omega_q+\hbar\omega_k}f_-(\hbar\omega_q,\hbar\omega_k)-\frac{(c_n-c_s)^2}{i\hbar\omega_n-\varepsilon_q+\varepsilon_k}f_-(\varepsilon_q,\varepsilon_k)\right\}$$

$$+\frac{4\pi\hbar}{m}\int\frac{d\boldsymbol{q}}{(2\pi)^3}\{(c_n+c_s)[2(A_q+B_q)f_{\mathrm{B}}(E_q)+f_{\mathrm{B}}(\hbar\omega_q)]+(c_n-c_s)f_{\mathrm{B}}(\varepsilon_q)\},$$

(132)

$$_{\mathrm{f}}\Sigma_{11}^{00}(p)_T^{(2)}=\frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2\int\frac{d\boldsymbol{q}}{(2\pi)^3}\left\{(c_n+c_s)^2\left[\frac{A_q+B_q+2C_q}{i\hbar\omega_n-E_q-\hbar\omega_k}f_+(E_q,\hbar\omega_k)+\frac{A_q+B_q+2C_q}{i\hbar\omega_n+E_q-\hbar\omega_k}f_-(E_q,\hbar\omega_k)\right]\right.$$

$$\left.-4c_s^2\frac{1}{i\hbar\omega_n-\varepsilon_q+\hbar\omega_k}f_-(\varepsilon_q,\hbar\omega_k)\right\}+\frac{4\pi\hbar}{m}\int\frac{d\boldsymbol{q}}{(2\pi)^3}\{(c_n+c_s)[(A_q+B_q)f_{\mathrm{B}}(E_q)+f_{\mathrm{B}}(\varepsilon_q)]+2c_nf_{\mathrm{B}}(\hbar\omega_q)\},$$

(133)

$$
{}_f\Sigma_{11}^{--}(p)_T^{(2)} = \frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left[(c_n-c_s)^2\left\{\frac{A_q+B_q+2C_q}{i\hbar\omega_n-E_q-\varepsilon_k}f_+(E_q,\varepsilon_k)+\frac{A_q+B_q+2C_q}{i\hbar\omega_n+E_q-\varepsilon_k}f_-(E_q,\varepsilon_k)\right]\right.
$$

$$
\left.+\frac{2c_s^2 f_+(\hbar\omega_q,\hbar\omega_k)}{i\hbar\omega_n-\hbar\omega_q-\hbar\omega_k}\right\}+\frac{4\pi\hbar}{m}\int \frac{d\boldsymbol{q}}{(2\pi)^3}\{(c_n-c_s)(A_q+B_q)f_B(E_q)+(c_n+c_s)[f_B(\hbar\omega_q)+2f_B(\varepsilon_q)]\},
$$

$$\tag{134}$$

$$
{}_f\Sigma_{12}^{++}(p)_T^{(2)} = \frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left\{\left[\frac{{}_fN_{12}(E_q,E_k)}{i\hbar\omega_n-E_q-E_k}-\frac{{}_fN_{12}(-E_q,-E_k)}{i\hbar\omega_n+E_q+E_k}\right]f_+(E_q,E_k)+\left[-\frac{{}_fN_{12}(E_q,-E_k)}{i\hbar\omega_n-E_q+E_k}\right.\right.
$$

$$
\left.+\frac{{}_fN_{12}(-E_q,E_k)}{i\hbar\omega_n+E_q-E_k}\right]f_-(E_q,E_k)+\frac{(c_n+c_s)^2}{i\hbar\omega_n+\hbar\omega_q-\hbar\omega_k}f_-(\hbar\omega_q,\hbar\omega_k)+\frac{(c_n-c_s)^2}{i\hbar\omega_n+\varepsilon_q-\varepsilon_k}f_-(\varepsilon_q,\varepsilon_k)\right\}
$$

$$
+2\frac{4\pi\hbar}{m}(c_n+c_s)\int \frac{d\boldsymbol{q}}{(2\pi)^3}C_q f_B(E_q),
$$

$$\tag{135}$$

$$
{}_f\mu_T^{(2)} = \frac{4\pi\hbar^2}{m}\int \frac{d\boldsymbol{q}}{(2\pi)^3}\{(c_n+c_s)[2(A_q+B_q+C_q)f_B(E_q)+f_B(\hbar\omega_q)]+(c_n-c_s)f_B(\varepsilon_q)\},
$$

$$\tag{136}$$

where the two $f_\pm(x,y)$ thermal functions are defined as

$$
f_\pm(x,y)\equiv f_B(x)\pm f_B(y),
$$

$$\tag{137}$$

and the functions ${}_fN_{11(12)}(x,y)$ are defined in Eqs. (93) and (94). For polar state temperature-dependent parts of the self-energies and the chemical potential are written as

$$
{}_p\Sigma_{11}^{++}(p)_T^{(2)} = \frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left\{\left[\frac{{}_pN_{11}(E_q^s,E_k^n)}{i\hbar\omega_n-E_q^s-E_k^n}-\frac{{}_pN_{11}(-E_q^s,-E_k^n)}{i\hbar\omega_n+E_q^s+E_k^n}\right]f_+(E_q^s,E_k^n)\right.
$$

$$
+\left[-\frac{{}_pN_{11}(E_q^s,-E_k^n)}{i\hbar\omega_n-E_q^s+E_k^n}+\frac{{}_pN_{11}(-E_q^s,E_k^n)}{i\hbar\omega_n+E_q^s-E_k^n}\right]f_-(E_q^s,E_k^n)\bigg\}
$$

$$
+\frac{4\pi\hbar}{m}\int \frac{d\boldsymbol{q}}{(2\pi)^3}\{(c_n+c_s)(A_q^n+B_q^n)f_B(E_q^n)+(3c_n+c_s)(A_q^s+B_q^s)f_B(E_q^s)\},
$$

$$\tag{138}$$

$$
{}_p\Sigma_{11}^{00}(p)_T^{(2)} = \frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left\{\left[\frac{{}_pN_{11}^n(E_q^n,E_k^n)}{i\hbar\omega_n-E_q^n-E_k^n}-\frac{{}_pN_{11}^n(-E_q^n,-E_k^n)}{i\hbar\omega_n+E_q^n+E_k^n}\right]f_+(E_q^n,E_k^n)\right.
$$

$$
+\left[-\frac{{}_pN_{11}^n(E_q^n,-E_k^n)}{i\hbar\omega_n-E_q^n+E_k^n}+\frac{{}_pN_{11}^n(-E_q^n,E_k^n)}{i\hbar\omega_n+E_q^n-E_k^n}\right]f_-(E_q^n,E_k^n)+\left[\frac{{}_pN_{11}^s(E_q^n,E_k^s)}{i\hbar\omega_n-E_q^s-E_k^s}-\frac{{}_pN_{11}^s(-E_q^s,-E_k^s)}{i\hbar\omega_n+E_q^s+E_k^s}\right]f_+(E_q^s,E_k^s)
$$

$$
+\left[-\frac{{}_pN_{11}^s(E_q^s,-E_k^s)}{i\hbar\omega_n-E_q^s+E_k^s}+\frac{{}_pN_{11}^s(-E_q^s,E_k^s)}{i\hbar\omega_n+E_q^s-E_k^s}\right]f_-(E_q^s,E_k^s)\bigg\}+\frac{4\pi\hbar}{m}\int \frac{d\boldsymbol{q}}{(2\pi)^3}\{2c_n(A_q^n+B_q^n)f_B(E_q^n)
$$

$$
+2(c_n+c_s)(A_q^s+B_q^s)f_B(E_q^s)\},
$$

$$\tag{139}$$

$$
{}_p\Sigma_{12}^{+-}(p)_T^{(2)} = \frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left\{\left[\frac{{}_pN_{12}(E_q^s,E_k^n)}{i\hbar\omega_n-E_q^s-E_k^n}-\frac{{}_pN_{12}(-E_q^s,-E_k^n)}{i\hbar\omega_n+E_q^s+E_k^n}\right]f_+(E_q^s,E_k^n)\right.
$$

$$
+\left[-\frac{{}_pN_{12}(E_q^s,-E_k^n)}{i\hbar\omega_n-E_q^s+E_k^n}+\frac{{}_pN_{12}(-E_q^s,E_k^n)}{i\hbar\omega_n+E_q^s-E_k^n}\right]f_-(E_q^s,E_k^n)\bigg\}+\frac{4\pi\hbar}{m}\int \frac{d\boldsymbol{q}}{(2\pi)^3}\{2(c_n-c_s)C_q^s f_B(E_q^s)+2c_s C_q^n f_B(E_q^n)\},
$$

$$\tag{140}$$

$$
{}_\mathrm{p}\Sigma_{12}^{00}(p)_T^{(2)} = \frac{n_0}{\hbar}\left(\frac{4\pi\hbar^2}{m}\right)^2 \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left\{\left[\frac{{}_\mathrm{p}N_{12}^n(E_q^n,E_k^n)}{i\hbar\omega_n - E_q^n - E_k^n} - \frac{{}_\mathrm{p}N_{12}^n(-E_q^n,-E_k^n)}{i\hbar\omega_n + E_q^n + E_k^n}\right]f_+(E_q^n,E_k^n)\right.
$$

$$
+\left[-\frac{{}_\mathrm{p}N_{12}^n(E_q^n,-E_k^n)}{i\hbar\omega_n - E_q^n + E_k^n} + \frac{{}_\mathrm{p}N_{12}^n(-E_q^n,E_k^n)}{i\hbar\omega_n + E_q^n - E_k^n}\right]f_-(E_q^n,E_k^n) + \left[\frac{{}_\mathrm{p}N_{12}^s(E_q^s,E_k^s)}{i\hbar\omega_n - E_q^s - E_k^s} - \frac{{}_\mathrm{p}N_{12}^s(-E_q^s,-E_k^s)}{\hbar\omega_n + E_q^s + E_k^s}\right]f_+(E_q^s,E_k^s)
$$

$$
\left.+\left[-\frac{{}_\mathrm{p}N_{12}^s(E_q^s,-E_k^s)}{i\hbar\omega_n - E_q^s + E_k^s} + \frac{{}_\mathrm{p}N_{12}^s(-E_q^s,E_k^s)}{i\hbar\omega_n + E_q^s - E_k^s}\right]f_-(E_q^s,E_k^s)\right\} + \frac{4\pi\hbar}{m}\int \frac{d\boldsymbol{q}}{(2\pi)^3}\{2c_n C_q^n f_\mathrm{B}(E_q^n) + 4c_s C_q^s f_\mathrm{B}(E_q^s)\},
$$

$$
\tag{141}
$$

$$
{}_\mathrm{p}\mu_T^{(2)} = \frac{4\pi\hbar^2}{m}\int \frac{d\boldsymbol{q}}{(2\pi)^3}\{2c_n(A_q^n + B_q^n + C_q^s)f_\mathrm{B}(E_q^n)
$$

$$
+ [2(c_n + c_s)(A_q^s + B_q^s) + 4c_s C_q^s]f_\mathrm{B}(E_q^s)\}, \tag{142}
$$

where ${}_\mathrm{p}N_{11(12)}^{n(s)}(x,y)$ are defined in Eqs. (100)–(105).

IV. CALCULATIONS AT $T=0$

In previous sections, we worked out the single-particle Green's functions for Bose-condensed system with spin degrees of freedom. The formalism developed is valid for an arbitrary temperature if there exists the condensate. In this section, we use the results obtained in previous sections to study the energy spectrum at $T=0$. In particular, we give the explicit expressions for the self-energies which contain infrared divergent terms. As well-known [11,12], for scalar Bose-condensed gas these divergent contributions cancel out in the final expressions for physical quantities such as the long-wavelength excitation spectrum.

A. Excitation spectrum

At zero temperature, the Green's functions are given by Eqs. (106)–(109) and (115)–(118). The functions in Eqs. (110)–(114) and Eqs. (119)–(121) are linear combinations of the second-order corrections to the self-energies and chemical potential. At zero temperature, we then substitute the temperature-independent part of the self-energies $\boldsymbol{\Sigma}^{(2)}(p)_0$ and the chemical potential $\mu_0^{(2)}$. Note that, in second-order self-energies and chemical potential, the integrals which do not involve frequency ω can be carried out exactly.

Since we consider the dilute limit of Bose-condensed system with spin degrees of freedom, it is convenient to study the behavior of the Green's functions near the poles that we have obtained within the first-order approximation. In this region, we may write $|\hbar\omega| = E_p^{(s,n)}$ in $\alpha(p)$ and $\beta(p)$, and we need to retain only terms of first order of about $(E_p^{(s,n)} \mp \hbar\omega)$ in $\Lambda(p)^\pm$. For small wavelengths, which means small compared to the inverse of the coherence length $|\boldsymbol{p}| \ll \tilde{p}$ [$\tilde{p} \equiv \sqrt{4\pi n_0(c_s + c_n)}$], we find the explicit expressions. For the ferromagnetic state, we obtain

$$
{}_\mathrm{f}\alpha(p)_0 \simeq \alpha^{1/2}\left[\frac{16}{3}\frac{\tilde{p}}{p} + i\frac{\pi}{8}\frac{p}{\tilde{p}}\right] \simeq -{}_\mathrm{f}\beta(p)_0, \tag{143}
$$

$$
\hbar\,{}_\mathrm{f}\Lambda^{++}(\boldsymbol{p},\pm E_p)_0^\pm \simeq \tilde{\mu}\alpha^{1/2}\left[\frac{28}{3}\frac{p}{\tilde{p}} - i\pi\frac{3}{80}\frac{p^5}{\tilde{p}^5}\right]
$$

$$
+ (E_p \mp \hbar\omega)\alpha^{1/2}\left[4 + i\pi\frac{1}{4}\frac{p^2}{\tilde{p}^2}\right]
$$

$$
\equiv {}_\mathrm{f}\Omega_p^{++} + {}_\mathrm{f}\lambda_p^{++}(E_p \mp \hbar\omega), \tag{144}
$$

$$
\hbar\,{}_\mathrm{f}\Lambda^{00}(\boldsymbol{p},\hbar\omega_p)_0 \simeq \tilde{\mu}\alpha^{1/2}\left[-\frac{32}{45}\frac{p^2}{\tilde{p}^2} + \frac{8}{3}(\hbar\omega_p - \hbar\omega)\right]
$$

$$
\equiv {}_\mathrm{f}\Omega_p^{00} + {}_\mathrm{f}\lambda_p^{00}(\hbar\omega_p - \hbar\omega), \tag{145}
$$

$$
\hbar\,{}_\mathrm{f}\Lambda^{--}(\boldsymbol{p},\varepsilon_p)_0 \simeq \tilde{\mu}\alpha^{1/2}\left[-\frac{32}{45}\frac{p^2}{\tilde{p}^2}(\alpha_n - \alpha_s)^2\right.
$$

$$
-\frac{16}{3}\alpha_s(9\alpha_n + \alpha_s) - 4\sqrt{2}i\pi|\alpha_s|^{5/2}\Big]
$$

$$
+ (\varepsilon_p - \hbar\omega)\alpha^{1/2}\left[\frac{8}{3}(\alpha_n - \alpha_s)^2\right.
$$

$$
+ \sqrt{2}\pi i|\alpha_s|^{3/2}\Big]
$$

$$
\equiv {}_\mathrm{f}\Omega_p^{--} + {}_\mathrm{f}\lambda_p^{--}(\varepsilon_p - \hbar\omega), \tag{146}
$$

where we define the notations $\alpha = ([n_0(c_n + c_s)^3/\pi)^{1/2}]$, $\alpha_{s(n)} = c_{s(n)}/(c_n + c_s)$, and $\tilde{\mu} = 4\pi\hbar^2 n_0(c_n + c_s)/m$. For the polar state, we obtain

$$
{}_\mathrm{p}\alpha^{++}(p)_0 \simeq \alpha^{1/2}\left[\frac{\tilde{p}}{p}C_1 + i\pi\frac{p}{\tilde{p}}C_2\right] \simeq -{}_\mathrm{p}\beta^{++}(p)_0,
$$

$$
\tag{147}
$$

$$
{}_\mathrm{p}\alpha^{00}(p)_0 \simeq \alpha^{1/2}\left[\frac{\tilde{p}}{p}C_3 + i\pi\frac{p}{\tilde{p}}C_4\right] \simeq -{}_\mathrm{p}\beta^{00}(p)_0, \tag{148}
$$

$$\hbar_p\Lambda^{++}(\boldsymbol{p},\pm E_{\boldsymbol{p}}^n)_0^{\pm}\simeq\alpha^{1/2}\tilde{\mu}\left[\alpha_s^{1/2}\frac{p}{\tilde{p}}C_5-i\pi\frac{p^5}{\tilde{p}^5}C_6\right]$$

$$+\alpha^{1/2}(E_{\boldsymbol{p}}^s\mp\hbar\omega)\left[C_7+i\pi\frac{p^2}{\tilde{p}^2}C_8\right]$$

$$\equiv{}_p\Omega_p^{++}+{}_p\lambda_p^{++}(E_{\boldsymbol{p}}^s\mp\hbar\omega),\qquad(149)$$

$$\hbar_p\Lambda^{00}(\boldsymbol{p},\pm E_{\boldsymbol{p}}^n)_0^{\pm}\simeq\alpha^{1/2}\tilde{\mu}\left[\alpha_n^{1/2}\frac{p}{\tilde{p}}C_9-i\pi\frac{p^5}{\tilde{p}^5}C_{10}\right]$$

$$+(E_{\boldsymbol{p}}^n\mp\hbar\omega)\alpha^{1/2}\left[C_{11}+i\pi\frac{p^2}{\tilde{p}^2}C_{12}\right]$$

$$\equiv{}_p\Omega_p^{00}+{}_p\lambda_p^{00}(E_{\boldsymbol{p}}^n\mp\hbar\omega),\qquad(150)$$

where the constants C_i $(i=1,\dots,12)$ are defined as

$$C_1=\frac{\alpha_s^{1/2}}{180\alpha_n^{1/2}(\alpha_n^{1/2}+\alpha_s^{1/2})}[1129\alpha_n^{5/2}+1129\alpha_n^2\alpha_s^{1/2}$$
$$+1214\alpha_n^{3/2}\alpha_s-146\alpha_n\alpha_s^{3/2}-191\alpha_n^{1/2}\alpha_s^2-15\alpha_s^{5/2}],$$
$$(151)$$

$$C_2=\frac{\alpha_s(\alpha_n+2\alpha_n^{1/2}\alpha_s^{1/2})}{3(\alpha_n^{1/2}+\alpha_s^{1/2})^2},\qquad(152)$$

$$C_3=\frac{1}{\alpha_n^{1/2}}\left[\frac{16}{3}\alpha_n^{5/2}+\frac{8}{3}\alpha_n\alpha_s^{3/2}+8\alpha_s^{5/2}\right],\qquad(153)$$

$$C_4=\frac{\alpha_n^{1/2}(\alpha_n+3\alpha_n^{1/2}\alpha_s^{1/2}+5\alpha_s)}{24\alpha_s^{1/2}},\qquad(154)$$

$$C_5=\frac{1}{90\alpha_n^{1/2}(\alpha_n^{1/2}+\alpha_s^{1/2})}[1031\alpha_n^{5/2}+1031\alpha_n^2\alpha_s^{1/2}$$
$$+1186\alpha_n^{3/2}\alpha_s-334\alpha_n\alpha_s^{3/2}-289\alpha_n^{1/2}\alpha_s^2+15\alpha_s^{5/2}],$$
$$(155)$$

$$C_6=\frac{2}{105\alpha_n^{1/2}(\alpha_n^{1/2}+\alpha_s^{1/2})^5}[4\alpha_n^3+20\alpha_n^{5/2}\alpha_s^{1/2}+40\alpha_n^2\alpha_s$$
$$+40\alpha_n^{3/2}\alpha_s^{3/2}+26\alpha_n\alpha_s^2-15\alpha_n^{1/2}\alpha_s^{5/2}+11\alpha_s^3],\quad(156)$$

$$C_7=\frac{8(\alpha_n+\alpha_s)(\alpha_n+\alpha_n^{1/2}\alpha_s^{1/2}+\alpha_s)}{3(\alpha_n^{1/2}+\alpha_s^{1/2})},\qquad(157)$$

$$C_8=\frac{2\alpha_s^{1/2}(\alpha_n+2\alpha_n^{1/2}\alpha_s^{1/2})}{3(\alpha_n^{1/2}+\alpha_s^{1/2})^2},\qquad(158)$$

$$C_9=\frac{1}{\alpha_n}\left[\frac{28}{3}\alpha_n^{5/2}+\frac{8}{3}\alpha_n\alpha_s^{3/2}+16\alpha_s^{5/2}\right],\qquad(159)$$

$$C_{10}=\frac{7\alpha_n^2+22\alpha_n\alpha_s+36\alpha_n^{1/2}\alpha_s^{3/2}+43\alpha_s^2}{960\alpha_n^{1/2}\alpha_s^{3/2}},\qquad(160)$$

$$C_{11}=4(\alpha_n^{3/2}+2\alpha_s^{3/2}),\qquad(161)$$

$$C_{12}=\frac{\alpha_n+3\alpha_n^{1/2}\alpha_s^{1/2}+5\alpha_s}{12\alpha_s^{1/2}}.\qquad(162)$$

For small wavelengths $|\boldsymbol{p}|\ll\tilde{p}$, the normal Green's functions near to their poles may be written in the following forms:

$$_fG_{11}^{++}(p)=(1-{}_f\lambda_p^{++})\left[\frac{A_{\boldsymbol{p}}+{}_f\alpha^{++}(p)}{\omega-E_{\boldsymbol{p}}/\hbar-{}_f\Omega_p^{++}}\right.$$
$$\left.-\frac{B_{\boldsymbol{p}}+{}_f\alpha^{++}(p)}{\omega+E_{\boldsymbol{p}}/\hbar+{}_f\Omega_p^{++}}\right],\qquad(163)$$

$$_fG_{11}^{00}(p)=\frac{(1-{}_f\lambda_p^{00})}{\omega-\omega_{\boldsymbol{p}}-{}_f\Lambda^{00}(\boldsymbol{p},\omega_{\boldsymbol{p}})+i\delta},\qquad(164)$$

$$_fG_{11}^{--}(p)=\frac{(1-{}_f\lambda_p^{--})}{\omega-\varepsilon_{\boldsymbol{p}}-{}_f\Lambda^{--}(\boldsymbol{p},\varepsilon_{\boldsymbol{p}}/\hbar)+i\delta},\qquad(165)$$

for the ferromagnetic state, and

$$_pG_{11}^{++}(p)=(1-{}_p\lambda_p^{++})\left[\frac{A_{\boldsymbol{p}}^s+{}_p\alpha^{++}(p)}{\omega-E_{\boldsymbol{p}}^s/\hbar-{}_p\Omega_p^{++}}\right.$$
$$\left.-\frac{B_{\boldsymbol{p}}^s+{}_p\alpha^{++}(p)}{\omega+E_{\boldsymbol{p}}^s/\hbar+{}_p\Omega_p^{++}}\right],\qquad(166)$$

$$_pG_{11}^{00}(p)=(1-{}_p\lambda_p^{00})\left[\frac{A_{\boldsymbol{p}}^n+{}_p\alpha^{--}(p)}{\omega-E_{\boldsymbol{p}}^n/\hbar-{}_p\Omega_p^{00}}-\frac{B_{\boldsymbol{p}}^n+{}_p\alpha^{00}(p)}{\omega+E_{\boldsymbol{p}}^n/\hbar+{}_p\Omega_p^{00}}\right]$$
$$(167)$$

for the polar state. Comparing with the first-order Green's function given by Eqs. (50)–(53) and (62)–(65), one sees that the quantities λ_p, $\alpha(p)$, and Ω_p are the second-order corrections. These corrections are small for a low-density gas since these all are proportional to $\alpha^{1/2}\ll1$.

The quasiparticle energy is determined by the pole of the Green's functions. In the wavelength range $p\ll\tilde{p}$, the quasiparticle spectrum for the ferromagnetic case is

$$E_{\boldsymbol{p}}+{}_f\Omega_p^{++}=\tilde{\mu}\frac{p}{\tilde{p}}\left[1+\frac{28}{3}\alpha^{1/2}\right]-i\pi\frac{3}{80}\alpha^{1/2}\tilde{\mu}\frac{\tilde{p}^5}{\tilde{p}^5},\quad(168)$$

$$\hbar\omega_{\boldsymbol{p}}+{}_f\Omega_p^{00}=\tilde{\mu}\frac{1}{2}\frac{p^2}{\tilde{p}^2}\left[1-\frac{64}{45}\alpha^{1/2}\right],\qquad(169)$$

$$\varepsilon_p + {}_f\Omega_p^{--} = \tilde{\mu}\left\{ \frac{1}{2}\frac{p^2}{\tilde{p}^2}\left[1 - \frac{64}{45}\alpha^{1/2}(\alpha_n - \alpha_s)^2\right]\right.$$

$$\left. + 2|\alpha_s|\left[1 + \frac{8}{3}(9\alpha_n + \alpha_s)\alpha^{1/2}\right]\right\}$$

$$- 4\sqrt{2}\pi i \alpha^{1/2}\tilde{\mu}|\alpha_s|^{5/2}, \qquad (170)$$

and for the polar case

$$E_p^s + {}_p\Omega_p^{++} = \tilde{\mu}\alpha_s^{1/2}\frac{p}{\tilde{p}}[1 + \alpha^{1/2}C_5] - i\pi\tilde{\mu}\alpha^{1/2}\frac{p^5}{\tilde{p}^5}C_6, \qquad (171)$$

$$E_p^n + {}_p\Omega_p^{00} = \tilde{\mu}\alpha_n^{1/2}\frac{p}{\tilde{p}}[1 + \alpha^{1/2}C_9] - i\pi\frac{p^5}{\tilde{p}^5}\tilde{\mu}\alpha^{1/2}C_{10}. \qquad (172)$$

Equations (168), (171), and (172) show that for a small p, one of the quasiparticles for the ferromagnetic state and two for the polar state are phonons. The second-order approximation also gives a correction to the sound velocity, and a damping rate proportional to p^5, which is connected with a process of decay of one phonon into two. This damping rate corresponds to a finite lifetime for phonons.

The mean number of particles $\bar{N}_p$ with a given wavelength p in the ground-state of the system is related to the residue of the Green's function at its upper pole. For the ferromagnetic state, ${}_f\bar{N}_p$ can be calculated using the Green's functions, Eqs. (163)–(165), we obtain

$$_f\bar{N}_p = i\lim_{\delta\to 0^+}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{i\omega\delta}{}_fG_{11}^{\alpha\alpha}(p,\omega) \simeq \frac{1}{2}\frac{\tilde{p}}{p}\left[1 + \frac{20}{3}\alpha^{1/2}\right]. \qquad (173)$$

The leading terms of the imaginary parts of $_f\lambda_p^{++}$ and $_f\alpha^{++}(p)$ cancel in Eq. (173), and $_f N_p$ is real. For the polar state, $_p\bar{N}_p$ can be calculated using Eqs. (166) and (167), we obtain

$$_p\bar{N}_p = i\lim_{\delta\to 0^+}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{i\omega\delta}{}_pG_{11}^{\alpha\alpha}(p,\omega)bf \simeq \frac{\tilde{p}}{p}\left\{\frac{\alpha_n^{1/2}}{2} + \alpha_s^{1/2}\right.$$

$$\left. + \alpha^{1/2}\left[2C_1 + C_3 - \alpha_s^{1/2}C_7 - \frac{\alpha_n^{1/2}}{2}C_{11}\right]\right\}. \qquad (174)$$

B. Ground-state energy

To find the total number of particles with $p \neq 0$, we need to know $\bar{N}_p$ for all wavelengths. We therefore use the lowest-order form for $\bar{N}_p$, this means we neglect the terms proportional to $\alpha^{1/2}$ in Eqs. (173) and (174). For the ferromagnetic state, we find

$$n - n_0 = i\lim_{\delta\to 0^+}\int\frac{d\boldsymbol{p}}{(2\pi)^3}\frac{d\omega}{2\pi}e^{i\omega\delta}{}_fG_{11}^{\alpha\alpha}(\boldsymbol{p},\omega) = \frac{8}{3}\alpha^{1/2}n_0, \qquad (175)$$

and for the polar state

$$n - n_0 = i\lim_{\delta\to 0^+}\int\frac{d\boldsymbol{p}}{(2\pi)^3}\frac{d\omega}{2\pi}e^{i\omega\delta}{}_pG_{11}^{\alpha\alpha}(\boldsymbol{p},\omega)$$

$$= \frac{8}{3}\alpha^{1/2}(\alpha_n^{3/2} + 2\alpha_s^{3/2})n_0. \qquad (176)$$

Equations (175) and (176) give the relation between the density of particles in the condensate, n_0, and the total number of particles in the system.

We then calculate the ground-state energy by integrating the chemical potential. For the ferromagnetic state using Eqs. (49) and (92), the chemical potential up to the second-order approximation is

$$_f\mu = \tilde{\mu}\left[1 + \frac{40}{3}\alpha^{1/2}\right]. \qquad (177)$$

Expressing n_0 in terms of n by means of Eq. (175), we have

$$_f\mu = \frac{4\pi\hbar^2 n(c_n + c_s)}{m}\left[1 + \frac{32}{3}\left(\frac{n(c_n + c_s)^3}{\pi}\right)^{1/2}\right]. \qquad (178)$$

By definition $\mu = (\partial/\partial n)(E_0/V)$, we obtain the ground-state energy for the ferromagnetic state

$$\frac{_fE_0}{V} = \frac{2\pi\hbar^2 n^2(c_n + c_s)}{m}\left[1 + \frac{128}{15}\left(\frac{n(c_n + c_s)^3}{\pi}\right)^{1/2}\right]. \qquad (179)$$

We can apply the same calculation for the polar state. Using Eqs. (61) and (99), the chemical potential up to the second-order approximation is

$$_p\mu = \tilde{\mu}\left[\alpha_n + \frac{8}{3}\alpha^{1/2}[5\alpha_n^{5/2} + 2\alpha_n\alpha_s^{3/2} + 8\alpha_s^{5/2}]\right]. \qquad (180)$$

Expressing n_0 in terms of n by means of Eq. (176), we obtain

$$_p\mu = \frac{4\pi\hbar^2 n(c_n + c_s)}{m}\alpha_n\left[1 + \frac{32}{3}(\alpha_n^{3/2} + 2\alpha_s^{5/2}\alpha_n^{-1})\right.$$

$$\left. \times\left(\frac{n(c_n + c_s)^3}{\pi}\right)^{1/2}\right]. \qquad (181)$$

Then, we obtain the ground-state energy for the polar state

$$\frac{_pE_0}{V} = \frac{2\pi\hbar^2 n^2(c_n + c_s)}{m}\alpha_n\left[1 + \frac{128}{15}(\alpha_n^{3/2} + 2\alpha_s^{5/2}\alpha_n^{-1})\right.$$

$$\left. \times\left(\frac{n(c_n + c_s)^3}{\pi}\right)^{1/2}\right]. \qquad (182)$$

V. CALCULATIONS AT $T \neq 0$

In this section, we consider the case of finite temperatures. The general expressions for the self-energies and the chemical potential can be separated by two parts. One part is denoted by $\Sigma(p)_0$ and μ_0 which do not depend *explicitly* on the Bose distribution function, and we obtain these expressions in Sec. II. Another temperature-dependent part is denoted by $\Sigma(p)_T$ and μ_T which does depend on the Bose distribution function, and we obtain these expressions in Sec. III. In Sec. IV, we worked out the expressions for $\Sigma(p)_0$ and μ_0 near the poles, which are obtained within the first-order approximation. In this section, we try to work out the expressions for $\Sigma(p)_T$ and μ_T near those poles.

At finite temperatures, the temperature Green's functions are obtained by replacing ω with $i\omega_n$ in Eqs. (106)–(109) and (115)–(118) for the ferromagnetic and polar states, respectively. Since the functions $\alpha(p)$, $\beta(p)$, and $\Lambda(p)$ are linear combinations of the self-energies and the chemical potential, the temperature-dependent parts of these functions can be separated out. To work out the temperature-dependent corrections, we must substitute the temperature-dependent parts $\Sigma_T^{(2)}$ and $\mu_T^{(2)}$ into the functions $\alpha(p)$, $\beta(p)$, and $\Lambda(p)$.

To obtain the excitation spectrum at finite temperatures, we have to made the analytic continuation $i\omega_n \rightarrow \omega + i\delta$ in the expressions of Σ_T, implying that we are now dealing with the self-energies for the retarded Green's function. Compared to the zero-temperature case, the finite-temperature contributions are multiplied by the Bose distribution function $f_B(e_q) = [\exp(e_q / k_B T) - 1]^{-1}$. For $k_B T \ll \tilde{\mu}$, the dominant contributions come from excitations with energies $e_q \sim k_B T$. For $k_B T \gg \tilde{\mu}$, excitations with energies $e_q \sim \tilde{\mu}$ are important, and we approximate the Bose distribution function $f_B(e_q) \sim k_B T / e_q$.

A. Density of condensate

Using the Green's functions, one can calculate the density of particles in the condensate. For a given total density n, we have

$$_f(n - n_0) = -\frac{1}{\hbar \beta} \sum_n \int \frac{dq}{(2\pi)^3} e^{i\omega_n \delta} {}_f G_{11}^{\alpha\alpha}(q, i\omega_n)$$

$$= \frac{8}{3} n_0 \alpha^{1/2} + \int \frac{dq}{(2\pi)^3} [(A_q + B_q) f_B(E_q)$$

$$+ f_B(\hbar \omega_q) + f_B(\varepsilon_q)], \qquad (183)$$

$$_p(n - n_0) = -\frac{1}{\hbar \beta} \sum_n \int \frac{dq}{(2\pi)^3} e^{i\omega_n \delta} {}_p G_{11}^{\alpha\alpha}(q, i\omega_n)$$

$$= \frac{8}{3} n_0 \alpha^{1/2} (\alpha_n^{3/2} + 2\alpha_s^{3/2}) + \int \frac{dq}{(2\pi)^3}$$

$$\times [(A_q^n + B_q^n) f_B(E_q^n) + 2(A_q^s + B_q^s) f_B(E_q^s)], \qquad (184)$$

for the ferromagnetic and polar states, respectively. The temperature-independent terms of right-hand sides of Eqs. (183) and (184) coincide with Eqs. (175) and (176), respectively. Equations (183) and (184) are nonlinear equations for n_0, since $E_p^{(n,s)}$ and $_{f,p}\mu^{(1)}$ are also the functions of n_0. In general, such nonlinear equations must be solved numerically. However, in two important special cases, analytical results can be obtained, as we now discuss.

We first consider the low-temperature limit $k_B T \ll \tilde{\mu}$. Since the Bose distribution function decreases exponentially for $E_p^{(n,s)} \ll k_B T$, the dominant contributions in the integral in Eqs. (183) and (184) come from the region $q \ll \tilde{p}$. Then, the integral can be approximated and we obtain

$$_f(n - n_0) \simeq \frac{8}{3} n_0 \alpha^{1/2} + 2n_0 \alpha^{1/2} \left[\frac{\pi^2}{3} \frac{(k_B T)^2}{\tilde{\mu}^2} \right.$$

$$\left. + \sqrt{2\pi} \zeta(3/2) \frac{(k_B T)^{3/2}}{\tilde{\mu}^{3/2}} + \frac{\pi^4}{20} \frac{(k_B T)^4}{\tilde{\mu}^4} \right],$$

$$(185)$$

$$_p(n - n_0) \simeq \frac{8}{3} n_0 \alpha^{1/2} (\alpha_n^{3/2} + 2\alpha_s^{3/2}) + 2n_0 \alpha^{1/2}$$

$$\times \left[\frac{\pi^2}{3} \left(\frac{1}{\alpha_n^2} + \frac{2}{\alpha_s^2} \right) \frac{(k_B T)^2}{\tilde{\mu}^2} \right.$$

$$\left. + \frac{\pi^4}{20} \left(\frac{1}{\alpha_n^{5/2}} + \frac{2}{\alpha_s^{5/2}} \right) \frac{(k_B T)^4}{\tilde{\mu}^4} \right]. \qquad (186)$$

We next discuss the region $k_B T \gg \tilde{\mu}$. To calculate the density of particles in the condensate at high temperatures, we use the noncondensate density of an ideal gas:

$$\int \frac{dq}{(2\pi)^3} \frac{1}{e^{\beta \hbar \omega_q} - 1} = \zeta(3/2) \left(\frac{k_B T}{2\pi \hbar^2} \right)^{3/2}. \qquad (187)$$

If we apply the high-temperature approximation for the distribution function $f_B(e_q) \sim k_B T / e_q$ to Eqs. (183) and (184) directly, the integral diverges. To avoid an unphysical divergence, we first subtract the noncondensate density of an ideal Bose gas from $_{f,p}(n - n_0)$, and then carry out the integral analytically to obtain

$$_f(n - n_0) \simeq \frac{8}{3} n_0 \alpha^{1/2} + 4n_0 \alpha^{1/2} \left[-\pi \frac{k_B T}{\tilde{\mu}} \right.$$

$$\left. + 3\sqrt{\frac{\pi}{2}} \zeta(3/2) \frac{(k_B T)^{3/2}}{\tilde{\mu}^{3/2}} \right], \qquad (188)$$

$$_p(n-n_0) \simeq \frac{8}{3} n_0 \alpha^{1/2} (\alpha_n^{3/2} + 2\alpha_s^{3/2}) + 4 n_0 \alpha^{1/2}$$

$$\times \left[-\pi(\alpha_n^{1/2} + 2\alpha_s^{1/2}) \frac{k_B T}{\tilde{\mu}} \right.$$

$$\left. + 3\sqrt{\frac{\pi}{2}} \zeta(3/2) \frac{(k_B T)^{3/2}}{\tilde{\mu}^{3/2}} \right]. \tag{189}$$

B. Energy shifts

We discuss energy shifts in this section. We consider only the case $p \ll \tilde{p}$ and use the on-shell approximation. For the ferromagnetic state, there exist three different kinds of excitations. For the corrections to the density modes, the contributions from the interactions between the density-wave modes are the same as the ones in the scalar Bose-condensed system [13]. So the parts that we have to analyze here are the contributions from the interactions between the spin wave and "quadrupolar" spin-wave modes, and their contributions are written as follows:

$$\hbar_f \Lambda_{\mathrm{spin}}^{++}(p)_T = n_0 \frac{\hbar \omega_p}{E_p} \left(\frac{4\pi\hbar^2}{m} \right)^2 \int \frac{d\mathbf{q}}{(2\pi)^3}$$

$$\times \left[-(c_n + c_s)^2 \frac{f_-(\hbar\omega_q, \hbar\omega_k)}{i\hbar\omega_n - \hbar\omega_q + \hbar\omega_k} \right.$$

$$\left. -(c_n - c_s)^2 \frac{f_-(\varepsilon_q, \varepsilon_k)}{i\hbar\omega_n - \varepsilon_q + \varepsilon_k} \right]. \tag{190}$$

For the low-temperature limit $k_B T \ll \tilde{\mu}$, the dominant contributions come from the region $q \ll \tilde{p}$, and the integrals can be approximated to obtain

$$\mathrm{Re}\,\hbar_f \Lambda^{++}(\mathbf{p}, \pm E_p/\hbar)_T^{\pm}$$

$$\simeq \tilde{\mu} \alpha^{1/2} \frac{p}{\tilde{p}} \left[\frac{\pi^2}{3} \frac{(k_B T)^2}{\tilde{\mu}^2} + \sqrt{2\pi}\zeta(3/2) \frac{(k_B T)^{3/2}}{\tilde{\mu}^{3/2}} - \frac{6\pi^4}{5} \right.$$

$$\left. \times \frac{(k_B T)^4}{\tilde{\mu}^4} \ln\frac{k_B T}{\tilde{\mu}} \right], \tag{191}$$

$$\mathrm{Re}\,\hbar_f \Lambda^{00}(\mathbf{p}, \omega_p)_T \simeq \tilde{\mu} \alpha^{1/2} \frac{4}{3}\sqrt{2\pi}\zeta(3/2) \frac{p^2}{\tilde{p}^2} \frac{(k_B T)^{3/2}}{\tilde{\mu}^{3/2}}, \tag{192}$$

$$\mathrm{Re}\,\hbar_f \Lambda^{--}(\mathbf{p}, \varepsilon_p/\hbar)_T \simeq \tilde{\mu}\alpha^{1/2} \left[-8\alpha_s \sqrt{\frac{\pi}{2}}\zeta(3/2) \frac{(k_B T)^{3/2}}{\tilde{\mu}^{3/2}} \right.$$

$$-\frac{4\pi^2}{3}\alpha_s \frac{(k_B T)^2}{\tilde{\mu}^2}$$

$$+ 12\sqrt{\frac{\pi}{2}}\zeta(5/2) \frac{(k_B T)^{5/2}}{\tilde{\mu}^{5/2}}$$

$$\left. + \frac{2}{\alpha_s}\sqrt{\frac{\pi}{2}}\zeta(5/2) \frac{(k_B T)^{5/2}}{\tilde{\mu}^{5/2}} \frac{p^2}{\tilde{p}^2} \right], \tag{193}$$

$$\mathrm{Re}\,\hbar_p \Lambda^{++}(\mathbf{p}, \pm E_p^s/\hbar)_T^{\pm}$$

$$\simeq \tilde{\mu}\alpha^{1/2}\alpha_s^{1/2} \frac{p}{\tilde{p}} \left[\frac{\alpha_s^{1/2} + 2\alpha_n^{1/2}}{3\alpha_n^{1/2}\alpha_s^{1/2}} \pi^2 \frac{(k_B T)^2}{\tilde{\mu}^2} + \pi^4 \frac{(k_B T)^4}{\tilde{\mu}^4} C_{13} \right], \tag{194}$$

$$\mathrm{Re}\,\hbar_p \Lambda^{00}(\mathbf{p}, \pm E_p^n/\hbar)_T^{\pm} \simeq \tilde{\mu}\alpha^{1/2}\alpha_n^{1/2} \frac{p}{\tilde{p}} \left[\frac{\alpha_s^{1/2} + 2\alpha_n^{1/2}}{3\alpha_s^{1/2}\alpha_n^{1/2}} \pi^2 \frac{(k_B T)^2}{\tilde{\mu}^2} \right.$$

$$\left. - \frac{6}{5\alpha_n^{5/2}} \pi^4 \frac{(k_B T)^4}{\tilde{\mu}^4} \ln\frac{k_B T}{\tilde{\mu}} \right], \tag{195}$$

where the function C_{13} is defined as

$$C_{13} = \frac{1}{180\alpha_n^{5/2}\alpha_s^{5/2}(\alpha_n^{1/2} + \alpha_s^{1/2})} [-26\alpha_n^3 - 26\alpha_n^{5/2}\alpha_s^{1/2}$$

$$- 56\alpha_n^2\alpha_s - 56\alpha_n^{3/2}\alpha_s^{3/2} - 56\alpha_n\alpha_s^2 + \alpha_n^{1/2}\alpha_s^{5/2} + \alpha_s^3]. \tag{196}$$

In this regime of temperatures, the condensate density is given by the expressions (185) and (186) in terms of the particle density n, and the energy shift turns out to be

$$E_p + \mathrm{Re}\,\hbar_f \Lambda^{++}(\mathbf{p}, \pm E_p/\hbar) \simeq \frac{\hbar^2}{m}\sqrt{4\pi n(c_s + c_n)}p \left\{ 1 + \alpha^{1/2} \right.$$

$$\left. \times \left[8 - \frac{6\pi^4}{5} \frac{(k_B T)^4}{\tilde{\mu}^4} \ln\frac{k_B T}{\tilde{\mu}} \right] \right\}, \tag{197}$$

$$\hbar\omega_p + \mathrm{Re}\,\hbar_f \Lambda^{00}(\mathbf{p}, \omega_p) \simeq \hbar\omega_p + \tilde{\mu}\alpha^{1/2}\frac{p^2}{\tilde{p}^2}$$

$$\times \left[-\frac{32}{45} + \frac{4}{3}\sqrt{2\pi}\zeta(3/2) \frac{(k_B T)^{3/2}}{\tilde{\mu}^{3/2}} \right], \tag{198}$$

$$\varepsilon_p + \mathrm{Re}\,\hbar_f \Lambda^{--}(\mathbf{p}, \varepsilon_p/\hbar)$$

$$\simeq \hbar\omega_p \left\{ 1 + \alpha^{1/2} \left[-\frac{64}{45}(\alpha_n - \alpha_s)^2 + \frac{4}{\alpha_s}\sqrt{\frac{\pi}{2}} \right. \right.$$

$$\left. \left. \times \zeta(5/2) \frac{(k_B T)^{5/2}}{\tilde{\mu}^{5/2}} \right] \right\} - 2\frac{4\pi\hbar^2 n c_s}{m} \left\{ 1 + \alpha^{1/2} \right.$$

$$\left. \times \left[\frac{32\alpha_n}{3} - \frac{6}{\alpha_s}\sqrt{\frac{\pi}{2}}\zeta(5/2) \frac{(k_B T)^{5/2}}{\tilde{\mu}^{5/2}} \right] \right\}, \tag{199}$$

$$E_{\boldsymbol{p}}^s + \hbar_{\mathrm{p}}\Lambda^{++}(\boldsymbol{p}, \pm E_{\boldsymbol{p}}^s/\hbar)^{\pm}$$

$$\simeq \frac{\hbar^2}{m}\sqrt{4\pi n(c_s+c_n)}\,\alpha_s^{1/2}p$$

$$\times\left\{1 + \alpha^{1/2}\left[C_{14} + C_{15}\pi^4\frac{(k_\mathrm{B}T)^4}{\tilde{\mu}^4}\right]\right\}, \tag{200}$$

$$E_{\boldsymbol{p}}^n + \mathrm{Re}\,\hbar_{\mathrm{p}}\Lambda^{00}(\boldsymbol{p}, \pm E_{\boldsymbol{p}}^n/\hbar)^{\pm}$$

$$\simeq \frac{\hbar^2}{m}\sqrt{4\pi n(c_s+c_n)}\,\alpha_n^{1/2}p\left\{1 + \alpha^{1/2}\left[\frac{8}{\alpha_n}(\alpha_n^{5/2} + 2\alpha_s^{5/2})\right.\right.$$

$$\left.\left. - \frac{6\pi^4}{5\alpha_n^2}\frac{(k_\mathrm{B}T)^4}{\tilde{\mu}^4}\ln\frac{k_\mathrm{B}T}{\tilde{\mu}}\right]\right\}, \tag{201}$$

where the constants C_{14}, C_{15} are defined as follows:

$$C_{14} = C_5 - \frac{4}{3}(\alpha_n^{3/2} + 2\alpha_s^{3/2}), \tag{202}$$

$$C_{15} = C_{13} - \frac{1}{20}\left(\frac{1}{\alpha_n^{5/2}} + \frac{2}{\alpha_s^{5/2}}\right). \tag{203}$$

For the high-temperature limit $k_\mathrm{B}T \gg \tilde{\mu}$, the dominant contributions come from excitations with energies $e_q \sim \tilde{\mu}$. Using the high-temperature form of the Bose distribution function $f_\mathrm{B}(e_q) \sim k_\mathrm{B}T/e_q$, energy shifts are written as follows:

$$\mathrm{Re}\,\hbar_{\mathrm{f}}\Lambda^{++}(\boldsymbol{p}, \pm E_{\boldsymbol{p}}/\hbar)_T^{\pm} \simeq \tilde{\mu}\alpha^{1/2}\frac{p}{\tilde{p}}\left[-13\frac{k_\mathrm{B}T}{\tilde{\mu}}\right], \tag{204}$$

$$\mathrm{Re}\,\hbar_{\mathrm{f}}\Lambda^{00}(\boldsymbol{p}, \omega_{\boldsymbol{p}})_T \simeq \tilde{\mu}\alpha^{1/2}\frac{8}{3}\frac{p^2}{\tilde{p}^2}\left[\pi\frac{k_\mathrm{B}T}{\tilde{\mu}}\right.$$

$$\left. + \sqrt{2\pi}\zeta(3/2)\frac{(k_\mathrm{B}T)^{3/2}}{\tilde{\mu}^{3/2}}\right], \tag{205}$$

$$\mathrm{Re}\,\hbar_{\mathrm{f}}\Lambda^{--}(\boldsymbol{p}, \varepsilon_{\boldsymbol{p}}/\hbar)_T \simeq 8\pi\alpha_s[5\alpha_n + \alpha_s - 2(3\alpha_n$$

$$+ \alpha_s)\sqrt{|\alpha_s|}]\frac{k_\mathrm{B}T}{\tilde{\mu}} + \frac{p^2}{\tilde{p}^2}$$

$$\times\left[\frac{8\pi}{3}(\alpha_n - \alpha_s)^2[1 - 2\sqrt{|\alpha_s|}(1\right.$$

$$- 2\alpha_s)]\frac{k_\mathrm{B}T}{\tilde{\mu}} + \frac{4}{3}(\alpha_n - \alpha_s)^2(1 - 4\alpha_s)$$

$$\left.\times\frac{\pi}{2}\zeta(3/2)\frac{(k_\mathrm{B}T)^{3/2}}{\tilde{\mu}^{3/2}}\right], \tag{206}$$

$$\mathrm{Re}\,\hbar_{\mathrm{p}}\Lambda^{++}(\boldsymbol{p}, \pm E_{\boldsymbol{p}}^s/\hbar)_T^{\pm}$$

$$\simeq \tilde{\mu}\alpha^{1/2}\frac{p}{\tilde{p}}\left[\frac{2\pi\alpha_s^{1/2}}{3(\alpha_n - \alpha_s)}[4\alpha_s^{1/2}(4\alpha_n - \alpha_s)\right.$$

$$- \alpha_n(11\alpha_n + \alpha_s)]\frac{k_\mathrm{B}T}{\tilde{\mu}}$$

$$\left. + \frac{4}{3}\alpha_s^{1/2}\sqrt{\frac{\pi}{2}}\zeta(3/2)\frac{(k_\mathrm{B}T)^{3/2}}{\tilde{\mu}^{3/2}}\right], \tag{207}$$

$$\mathrm{Re}\,\hbar_{\mathrm{p}}\Lambda^{00}(\boldsymbol{p}, \pm E_{\boldsymbol{p}}^s/\hbar)_T^{\pm} \simeq \tilde{\mu}\alpha^{1/2}\frac{p}{\tilde{p}}\frac{k_\mathrm{B}T}{\tilde{\mu}}C_{16}, \tag{208}$$

where the constants C_{16} are given in the integral form which depends on α_n and α_s, and we omit the precise expression here.

C. Free energy

Free energy of the system can be calculated from the chemical potential, that is, by definition $\mu = (\partial/\partial n)(F/V)_{V,T}$. Thus, we first derive the analytical expression of the chemical potential. The temperature-dependent part of chemical potential is obtained using Eqs. (136) and (142) for the ferromagnetic and polar states, respectively. At low temperatures $k_\mathrm{B}T \ll \tilde{\mu}$,

$$_\mathrm{f}\mu_T^{(2)} \simeq 4\alpha^{1/2}\tilde{\mu}\left[\frac{\pi^2}{6}\frac{(k_\mathrm{B}T)^2}{\tilde{\mu}^2} + \sqrt{\frac{\pi}{2}}\zeta(3/2)\frac{(k_\mathrm{B}T)^{3/2}}{\tilde{\mu}^{3/2}}\right.$$

$$\left. + \frac{7\pi^4}{120}\frac{(k_\mathrm{B}T)^4}{\tilde{\mu}^4}\right], \tag{209}$$

$$_\mathrm{p}\mu_T^{(2)} \simeq 8\tilde{\mu}\alpha^{1/2}\alpha_n^{1/2}\left[\frac{\pi^2}{12}\frac{2\alpha_n^{1/2} + \alpha_s^{1/2}}{\alpha_s^{1/2}}\frac{(k_\mathrm{B}T)^2}{\tilde{\mu}^2}\right.$$

$$\left. + \frac{6\alpha_n^{5/2} + 8\alpha_n^{3/2}\alpha_s + 7\alpha_s^{5/2}}{240\alpha_n^{5/2}\alpha_s^{5/2}}\pi^4\frac{(k_\mathrm{B}T)^4}{\tilde{\mu}^4}\right]. \tag{210}$$

Expressing n_0 in terms of n by means of Eqs. (185) and (186), we obtain

$$_\mathrm{f}\mu = {}_\mathrm{f}\mu(T=0) + \alpha^{1/2}\frac{4\pi\hbar^2 n(c_n+c_s)}{m}\left[\frac{2\pi^4}{15}\frac{(k_\mathrm{B}T)^4}{\tilde{\mu}^4}\right], \tag{211}$$

$$_\mathrm{p}\mu = {}_\mathrm{p}\mu(T=0) + \alpha^{1/2}\frac{4\pi\hbar^2 nc_n}{m}\left[\frac{2\pi^4}{15}\frac{2\alpha_n^{3/2} + \alpha_s^{3/2}}{\alpha_n^{3/2}\alpha_s^{3/2}}\frac{(k_\mathrm{B}T)^4}{\tilde{\mu}^4}\right], \tag{212}$$

where $_{\mathrm{f},\mathrm{p}}\mu(T=0)$ is the temperature-independent part of the chemical potential and is given in Eqs. (178) and (181). At high temperatures $k_\mathrm{B}T \gg \tilde{\mu}$,

$$_f\mu = {}_f\mu(T=0) - \alpha^{1/2}\frac{4\pi\hbar^2 n(c_n+c_s)}{m}\left[8\pi\frac{k_BT}{\tilde{\mu}}\right.$$

$$\left. - 2(\alpha_n - \alpha_s)\sqrt{2\pi}\zeta(3/2)\frac{(k_BT)^{3/2}}{\tilde{\mu}^{3/2}}\right], \qquad (213)$$

$$_p\mu = {}_p\mu(T=0) - \alpha^{1/2}\frac{4\pi\hbar^2 n(c_n+c_s)}{m}\left[8\pi\frac{k_BT}{\tilde{\mu}}(\alpha_n^{3/2}\right.$$

$$\left. + 2\alpha_s^{3/2}) - \sqrt{2\pi}\zeta(3/2)\frac{(k_BT)^{3/2}}{\tilde{\mu}^{3/2}}(4\alpha_n - \alpha_s)\right], \quad (214)$$

where we use the density of particles, n, instead of the density of the condensate, n_0.

We can calculate the free energy from the chemical potential. By definition $\mu = (\partial/\partial n)(F/V)_{V,T}$ and using the expressions of the chemical potential in terms of n in Eqs. (211)–(214), the free energy is written as follows:

$$\frac{_fF}{V} = \frac{_fE_0}{V} - \left(\frac{n(c_n+c_s)^3}{\pi}\right)^{1/2}\frac{4\pi\hbar^2 n^2(c_n+c_s)}{m}\frac{45\pi^4}{45}$$

$$\times\left(\frac{k_BT}{4\pi\hbar^2 n(c_n+c_s)/m}\right)^4, \qquad (215)$$

$$\frac{_pF}{V} = \frac{_pE_0}{V} - \left(\frac{n(c_n+c_s)^3}{\pi}\right)^{1/2}\frac{4\pi\hbar^2 n^2 c_n}{m}\frac{45\pi^4}{45}\frac{\alpha_n^{3/2}+2\alpha_s^{3/2}}{\alpha_n^{3/2}\alpha_s^{3/2}}$$

$$\times\left(\frac{k_BT}{4\pi\hbar^2 n(c_n+c_s)/m}\right)^4, \qquad (216)$$

at low temperatures $k_BT \ll \tilde{\mu}$, and

$$\frac{_fF}{V} = \frac{_fE_0}{V} - \left(\frac{n(c_n+c_s)^3}{\pi}\right)^{1/2}\frac{4\pi\hbar^2 n^2 c_n}{m}$$

$$\times\left[\frac{16\pi}{3}\frac{k_BT}{4\pi\hbar^2 n(c_n+c_s)/m} - 2\sqrt{2\pi}\zeta(3/2)\right.$$

$$\left. \times\left(\frac{k_BT}{4\pi\hbar^2 n(c_n+c_s)/m}\right)^{3/2}(\alpha_n - \alpha_s)\right], \qquad (217)$$

$$\frac{_pF}{V} = \frac{_pE_0}{V} - \left(\frac{n(c_n+c_s)^3}{\pi}\right)^{1/2}\frac{4\pi\hbar^2 n^2 c_n}{m}$$

$$\times\left[\frac{16\pi}{3}\frac{k_BT}{4\pi\hbar^2 n(c_n+c_s)/m}(\alpha_n^{3/2}+2\alpha_s^{3/2})\right.$$

$$\left. - \sqrt{2\pi}\zeta(3/2)\left(\frac{k_BT}{4\pi\hbar^2 n(c_n+c_s)/m}\right)^{3/2}(4\alpha_n - \alpha_s)\right], \qquad (218)$$

at high temperatures $k_BT \gg \tilde{\mu}$. From Eqs. (215)–(218), we can easily obtain all the other thermodynamic functions. From the expressions of free energy, the temperature dependences of thermodynamic quantities turn out to be the same as the ones in phonon gas, and the spin-exchange interaction is seen to modify the temperature-dependent parts.

VI. CONCLUDING REMARKS

We have studied excitation spectrum and thermodynamic properties of spin-1 Bose condensates under the single-condensate assumption. Since the excitation spectra at a mean-field level had been discussed in previous studies, we here included the higher-order scattering effects, which give energy shifts, damping rates, and thermodynamic quantities. We generalized the field theory, which is used in scalar Bose-condensed system to spin-1 Bose-condensed system. After the Bogoliubov prescription, we obtained the matrix Dyson's equations expressing the Green's functions in terms of self-energies. Solving these matrix Dyson's equations, we derived the formal expressions of the Green's functions in terms of self-energies. Within the first-order calculations, we obtained the energy spectrum that coincides with the results of previous studies. We then discussed the second-order approximation, in which all the diagrams of the order of $(n_0 a_{F_t}^3)^{1/2}$ are summed and are included in the self-energies, where a_{F_t} ($F_t=0,2$) is the s-wave scattering length in the channel with total spin F_t. We further extended Green's function method to the finite-temperature case, and derived the corresponding self-energies. The main results are the analytical expressions of the excitation spectrum and thermodynamic quantities. We evaluated the self-energies within the on-shell approximation, and expanded the results in powers of wave number in the long-wavelength limit. Excitation spectra for each mode were obtained analytically at low temperatures. We also derived the analytical expressions of the ground-state energy and thermodynamic quantities. From the expressions of free energy, the temperature dependences of thermodynamic quantities turn out to be the same as the ones in phonon gas, and the spin-exchange interaction is seen to modify the temperature dependent parts.

ACKNOWLEDGMENT

APPENDIX A: EFFECTIVE POTENTIAL

In this appendix, we derive the effective potential within the ladder approximation. Since there exists a set of ladder diagrams, all of which are of the same order in the gas density, the ladder diagrams are all of equal importance. We need to sum over all the ladder diagrams, yielding the t matrix $\Gamma^0_{\alpha\alpha',\beta\beta'}(p_1,p_2;p_3,p_4)$. The equation for $\Gamma^0_{\alpha\alpha',\beta\beta'}(p_1,p_2;p_3,p_4)$ can be written explicitly as

$$\Gamma^0_{\alpha\alpha',\beta\beta'}(p_1,p_2;p_3,p_4)$$

$$= U_{\alpha\alpha',\beta\beta'}(p_1-p_3) + \frac{i}{\hbar}\int \frac{d^4q}{(2\pi)^4}$$

$$\times U_{\alpha\alpha_1',\beta\beta_1'}(p_1-q)G_0^{\alpha_1'\alpha_1}(q)$$

$$\times G_0^{\beta_1'\beta_1}(p_1+p_2-q)$$

$$\times \Gamma^0_{\alpha_1\alpha',\beta_1\beta'}(q,p_1+p_2-q;p_3,p_4). \quad (A1)$$

It is convenient to write $\Gamma^0_{\alpha\alpha',\beta\beta'}(p_1,p_2;p_3,p_4)$ in the center of mass frame of the scattering pair of atoms. We define the total wave vector and the total frequency of the scattering pair as $P=p_1+p_2=p_3+p_4$, where the last equality follows from the conservation of total four-momentum in a homogeneous system, and we also define the relative wave vectors $k=(p_1-p_2)/2, k'=(p_3-p_4)/2$. Since the instantaneous interaction potential $U_{\alpha\alpha',\beta\beta'}(r)$ is not time dependent, its Fourier component $U_{\alpha\alpha',\beta\beta'}(q)$ does not depend on the fourth component ω of q, and neither does $\Gamma^0_{\alpha_1\alpha',\beta_1\beta'}(q,p_1+p_2-q;p_3,p_4)$. One can thus perform the frequency integral. Writing Eq. (A1) in terms of variables in the center of mass frame, we have

$$\Gamma^0_{\alpha\alpha',\beta\beta'}(k,k';P)=U_{\alpha\alpha',\beta\beta'}(k-k')$$

$$+\int \frac{dq}{(2\pi)^3}U_{\alpha\alpha_1,\beta\beta_1}(k-q)$$

$$\times \frac{1}{\dfrac{\hbar^2\kappa^2}{m}-\dfrac{\hbar^2q^2}{m}+i\delta}\Gamma^0_{\alpha_1\alpha',\beta_1\beta'}(q,k';P),$$

$$(A2)$$

where $\hbar^2\kappa^2/m=\hbar\varpi-\hbar^2P^2/4m+2\mu$.

We show the connection between the free-space scattering amplitude and t the matrix $\Gamma^0_{\alpha\alpha',\beta\beta'}(k,k';P)$. The following argument is a generalization of the scalar Bose-condensed system [11,12]. Introducing the quantity $\chi^0_{\alpha\alpha',\beta\beta'}(k,k';P)$ defined by

$$\Gamma^0_{\alpha\alpha',\beta\beta'}(k,k';P)\equiv \int \frac{dq}{(2\pi)^3}U_{\alpha\alpha_1,\beta\beta_1}(q)\chi^0_{\alpha_1\alpha',\beta_1\beta'}$$

$$\times (k-q,k';P), \quad (A3)$$

one can verify that $\chi^0_{\alpha\alpha',\beta\beta'}(k,k';P)$ satisfies the following integral equations:

$$\chi^0_{\alpha\alpha',\beta\beta'}(k,k';P)=(2\pi)^3\delta(k-k')\delta_{\alpha\alpha'}\delta_{\beta\beta'}$$

$$+\frac{1}{\dfrac{\hbar^2\kappa^2}{m}-\dfrac{\hbar^2k^2}{m}+i\delta}\int \frac{dq}{(2\pi)^3}$$

$$\times U_{\alpha\alpha_1,\beta\beta_1}(q)\chi^0_{\alpha_1\alpha',\beta_1\beta'}(k-q,k';P).$$

$$(A4)$$

Equation (A4), multiplied by $\hbar^2\kappa^2/m-\hbar^2k^2/m+i\delta$, is the same as the inhomogeneous Schrödinger equation for the relative motion of two particles with spin. It is therefore easy to express its solution in terms of $\psi^p_{\alpha\alpha',\beta\beta'}(k)$, which is the normalized wave function of the relative motion of particles and satisfies the equation

$$\left[\frac{\hbar^2p^2}{m}-\frac{\hbar^2k^2}{m}+i\delta\right]\psi^p_{\alpha\alpha',\beta\beta'}(k)$$

$$-\int \frac{dq}{(2\pi)^3}U_{\alpha\alpha_1,\beta\beta_1}(q)\psi^p_{\alpha_1\alpha',\beta_1\beta'}(k-q)=0. \quad (A5)$$

The scattering amplitude $\tilde{f}_{\alpha\alpha',\beta\beta'}(k,p)$ is related to the wave function $\psi^p_{\alpha\alpha',\beta\beta'}(k)$ by

$$\psi^p_{\alpha\alpha',\beta\beta'}(k)=(2\pi)^3\delta(k-p)\delta_{\alpha\alpha'}\delta_{\beta\beta'}+\frac{\tilde{f}_{\alpha\alpha',\beta\beta'}(k,p)}{p^2-k^2+i\delta},$$

$$(A6)$$

where

$$\tilde{f}_{\alpha\alpha',\beta\beta'}(k,p)\equiv -4\pi f_{\alpha\alpha',\beta\beta'}(k,p)$$

$$=\frac{m}{\hbar^2}\int \frac{dq}{(2\pi)^3}U_{\alpha\alpha_1,\beta\beta_1}(q)\psi^p_{\alpha_1\alpha',\beta_1\beta'}(k-q).$$

$$(A7)$$

If we use the condition that the interaction potential is real and its Fourier component satisfies $U_{\alpha\alpha',\beta\beta'}(k-k')=U^*_{\alpha\alpha',\beta\beta'}(k'-k)$, which yields

$$\tilde{f}_{\alpha\alpha',\beta\beta'}(k,k')-\tilde{f}^*_{\alpha\alpha',\beta\beta'}(k',k)$$

$$=\int \frac{dp}{(2\pi)^3}\left[\frac{\tilde{f}^*_{\alpha\alpha_1,\beta\beta_1}(k',p)\tilde{f}_{\alpha_1\alpha',\beta_1\beta'}(k,p)}{p^2-k^2+i\delta}\right.$$

$$\left.-\frac{\tilde{f}_{\alpha\alpha_1,\beta\beta_1}(k,p)\tilde{f}^*_{\alpha_1\alpha',\beta_1\beta'}(k',p)}{p^2-k'^2-i\delta}\right]. \quad (A8)$$

From Eq. (A8), the lowest order of the imaginary part of $\tilde{f}_{\alpha\alpha',\beta\beta'}(p,p')$ for $|p|=|p'|$ is expressed as

$$\text{Im}\widetilde{f}_{\alpha\alpha',\beta\beta'}(\boldsymbol{p},\boldsymbol{p}') \simeq -\pi\frac{\hbar^2}{m}\widetilde{f}^0_{\alpha\alpha_1,\beta\beta_1}\widetilde{f}^0_{\alpha_1\alpha',\beta_1\beta'}$$

$$\times \int \frac{d\boldsymbol{q}}{(2\pi)^3}\delta(\hbar^2 q^2/m - \hbar^2 k^2/m). \quad (A9)$$

Equation (A9) is a generalization of the ordinary optical theorem for the scattering amplitude. Multiplying Eq. (A4) by $[\hbar^2\kappa^2/m - \hbar^2 k^2/m + i\delta]\psi^{p*}_{\alpha\alpha',\beta\beta'}(\boldsymbol{k})$ and integrating over $\boldsymbol{k}$,

$$\left[\frac{\hbar^2\kappa^2}{m} - \frac{\hbar^2 p^2}{m} + i\delta\right]\int \frac{d\boldsymbol{k}}{(2\pi)^3}\psi^{p*}_{\alpha\alpha_1,\beta\beta_1}(\boldsymbol{k})$$

$$\times \chi^0_{\alpha_1\alpha',\beta_1\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$$

$$= \left[\frac{\hbar^2\kappa^2}{m} - \frac{\hbar^2 k'^2}{m} + i\delta\right]\psi^{p*}_{\alpha\alpha',\beta\beta'}(\boldsymbol{k}'), \quad (A10)$$

where we use the complex conjugate of Eq. (A5). We next multiply both sides of Eq. (A10) by $\psi^p_{\alpha\alpha',\beta\beta'}(\boldsymbol{k})$, and integrate over $\boldsymbol{p}$. Using the completeness relation of $\psi^p_{\alpha\alpha',\beta\beta'}(\boldsymbol{k})$, one arrives at

$$\chi^0_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P) = \left[\frac{\hbar^2\kappa^2}{m} - \frac{\hbar^2 k'^2}{m} + i\delta\right]\int \frac{d\boldsymbol{p}}{(2\pi)^3}$$

$$\times \frac{\psi^p_{\alpha\alpha_1,\beta\beta_1}(\boldsymbol{k})\psi^{p*}_{\alpha_1\alpha',\beta_1\beta'}(\boldsymbol{k}')}{\dfrac{\hbar^2\kappa^2}{m} - \dfrac{\hbar^2 p^2}{m} + i\delta}.$$

$$(A11)$$

Using the complex conjugate of Eq. (A6) and the definitions of $\chi^0_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ and $\widetilde{f}_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}')$ in Eqs. (A3) and (A7), one finds

$$\Gamma^0_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P) = \frac{\hbar^2}{m}\widetilde{f}_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}') + \frac{\hbar^2}{m}\int \frac{d\boldsymbol{p}}{(2\pi)^3}$$

$$\times \widetilde{f}_{\alpha\alpha_1,\beta\beta_1}(\boldsymbol{k},\boldsymbol{p})\left[\frac{1}{\kappa^2 - p^2 + i\delta}\right.$$

$$\left. + \frac{1}{p^2 - k'^2 - i\delta}\right]\widetilde{f}^*_{\alpha_1\alpha',\beta_1\beta'}(\boldsymbol{k}',\boldsymbol{p}).$$

$$(A12)$$

This last expression gives $\Gamma^0_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ completely in terms of the scattering amplitude $\widetilde{f}_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}')$. Using Eq. (A8), we have the equivalent expression of Eq. (A12), which is

$$\Gamma^0_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P) = \frac{\hbar^2}{m}\widetilde{f}^*_{\alpha\alpha',\beta\beta'}(\boldsymbol{k}',\boldsymbol{k}) + \frac{\hbar^2}{m}\int \frac{d\boldsymbol{p}}{(2\pi)^3}$$

$$\times \frac{\widetilde{f}_{\alpha\alpha_1,\beta\beta_1}(\boldsymbol{k},\boldsymbol{p})\widetilde{f}^*_{\alpha_1\alpha',\beta_1\beta'}(\boldsymbol{k}',\boldsymbol{p})}{\kappa^2 - p^2 + i\delta} + \frac{\hbar^2}{m}\int \frac{d\boldsymbol{p}}{(2\pi)^3}$$

$$\times \frac{\widetilde{f}^*_{\alpha\alpha_1,\beta\beta_1}(\boldsymbol{k}',\boldsymbol{p})\widetilde{f}_{\alpha_1\alpha',\beta_1\beta'}(\boldsymbol{k},\boldsymbol{p})}{p^2 - k^2 + i\delta}.$$

$$(A13)$$

APPENDIX B: EFFECTIVE POTENTIAL AT $T \neq 0$

The diagrammatic definition of effective potential at finite temperatures, $\Gamma^T_{\alpha\alpha',\beta\beta'}(p_1,p_2;p_3,p_4)$, can be written explicitly as

$$\Gamma^T_{\alpha\alpha',\beta\beta'}(p_1,p_2;p_3,p_4)$$

$$= U_{\alpha\alpha',\beta\beta'}(\boldsymbol{p}_1 - \boldsymbol{p}_3) - \frac{1}{\hbar^2\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}U_{\alpha\alpha_1,\beta\beta_1}$$

$$\times (\boldsymbol{p}_1 - \boldsymbol{p}_3)G^{\alpha_1'\alpha_1}_0(k)G^{\beta_1'\beta_1}_0(p_1 + p_2 - k)\Gamma^T_{\alpha_1\alpha',\beta_1\beta'}$$

$$\times (k,p_1 + p_2 - k;p_3,p_4). \quad (B1)$$

Here, the four-dimensional vector p represents the momentum $\boldsymbol{p}$ and Matsubara frequency $i\omega_n$. Defining the total wave vector and total Matsubara frequency of the scattering pair as $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_3 + \boldsymbol{p}_4$ and $i\varpi_n = i\omega_1 + i\omega_2 = i\omega_3 + i\omega_4$, and the relative wave vectors as $\boldsymbol{k} = \frac{1}{2}(\boldsymbol{p}_1 - \boldsymbol{p}_2)$, $\boldsymbol{k}' = \frac{1}{2}(\boldsymbol{p}_3 - \boldsymbol{p}_4)$, the sum of ladder diagrams $\Gamma^T_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ can be rewritten as

$$\Gamma^T_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$$

$$= U_{\alpha\alpha',\beta\beta'}(\boldsymbol{k} - \boldsymbol{k}') + \int \frac{d\boldsymbol{q}}{(2\pi)^3}$$

$$\times U_{\alpha\alpha_1,\beta\beta_1}(\boldsymbol{k} - \boldsymbol{q})\Gamma^T_{\alpha_1\alpha',\beta_1\beta'}(\boldsymbol{q},\boldsymbol{k}';P)$$

$$\times \frac{1 + f_B(\hbar\omega_{P/2+q} - \mu) + f_B(\hbar\omega_{P/2-q} - \mu)}{\dfrac{\hbar^2\kappa^2}{m} - \dfrac{\hbar^2 q^2}{m}}, \quad (B2)$$

where $\hbar^2\kappa_n^2/m = i\hbar\varpi_n - \hbar^2 P^2/4m + 2\mu$.

We next show the connection between the scattering amplitude $\widetilde{f}_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}')$ in Eq. (A7) and the effective potential $\Gamma^T_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$. The following analysis at $T \neq 0$ is a generalization of the discussion at $T = 0$ given in Appendix A. Introducing the quantity $\chi^T_{\alpha\alpha',\beta\beta'}(\boldsymbol{k},\boldsymbol{k}';P)$ defined by

$$\Gamma^{T}_{\alpha\alpha',\beta\beta'}(k,k';P) \equiv \int \frac{dq}{(2\pi)^3} U_{\alpha\alpha_1,\beta\beta_1}(q)\chi^{T}_{\alpha_1\alpha',\beta_1\beta'}$$
$$\times (k-q,k';P), \tag{B3}$$

one can verify that $\chi^{T}_{\alpha\alpha',\beta\beta'}(k,k';P)$ satisfies the following integral equation:

$$\chi^{T}_{\alpha\alpha',\beta\beta'}(k,k';P) - \frac{1}{\frac{\hbar^2\kappa_n^2}{m} - \frac{\hbar^2 k^2}{m}} \int \frac{dq}{(2\pi)^3}$$

$$\times U_{\alpha\alpha_1,\beta\beta_1}(q)\chi^{T}_{\alpha_1\alpha',\beta_1\beta'}(k-q,k';P)$$

$$= (2\pi)^3\delta(k-k')\delta_{\alpha\alpha'}\delta_{\beta\beta'}$$

$$+ \frac{f_{\mathrm{B}}(\hbar\omega_{P/2+q}-\mu) + f_{\mathrm{B}}(\hbar\omega_{P/2-q}-\mu)}{\frac{\hbar^2\kappa_n^2}{m} - \frac{\hbar^2 k^2}{m}} \int \frac{dq}{(2\pi)^3}$$

$$\times U_{\alpha\alpha_1,\beta\beta_1}(q)\chi^{T}_{\alpha_1\alpha',\beta_1\beta'}(k-q,k';P). \tag{B4}$$

Note that Eqs. (B3) and (B4) have analogous structures as Eqs. (A3) and (A4), respectively. A comparison with Eq. (A4) shows that the operator on the left side of Eq. (B4) is just the inverse of $\chi^{0}_{\alpha\alpha',\beta\beta'}(k,k';P)$, which means that $\chi^{T}_{\alpha\alpha',\beta\beta'}(k,k';P)$ can be expressed in terms of $\chi^{0}_{\alpha\alpha',\beta\beta'}(k,k';P)$ as follows:

$$\chi^{T}_{\alpha\alpha',\beta\beta'}(k,k';P)$$

$$= \chi^{0}_{\alpha\alpha',\beta\beta'}(k,k';P) + \int \frac{dq}{(2\pi)^3}\chi^{0}_{\alpha\alpha_1,\beta\beta_1}(k,q;P)$$

$$\times \frac{f_{\mathrm{B}}(\hbar\omega_{P/2+q}-\mu) + f_{\mathrm{B}}(\hbar\omega_{P/2-q}-\mu)}{\frac{\hbar^2\kappa_n^2}{m} - \frac{\hbar^2 q^2}{m}}$$

$$\times \Gamma^{T}_{\alpha_1\alpha',\beta_1\beta'}(q,k';P). \tag{B5}$$

This equation can be verified by carrying out the operation indicated on the left side of Eq. (B4) and by using Eq. (A4). We take the convolution with $U_{\alpha\alpha',\beta\beta'}(q)$ [see Eq. (B3)], which yields our final equation for the scattering amplitude at finite temperatures as follows:

$$\Gamma^{T}_{\alpha\alpha',\beta\beta'}(k,k';P)$$

$$= \Gamma^{0}_{\alpha\alpha',\beta\beta'}(k,k';P) + \int \frac{dq}{(2\pi)^3}\Gamma^{0}_{\alpha\alpha_1,\beta\beta_1}(k,q;P)$$

$$\times \frac{f_{\mathrm{B}}(\hbar\omega_{P/2+q}-\mu) + f_{\mathrm{B}}(\hbar\omega_{P/2-q}-\mu)}{\frac{\hbar^2\kappa_n^2}{m} - \frac{\hbar^2 q^2}{m}}$$

$$\times \Gamma^{T}_{\alpha_1\alpha',\beta_1\beta'}(q,k';P). \tag{B6}$$

The above equation expresses $\Gamma^{T}_{\alpha\alpha',\beta\beta'}(k,k';P)$ in terms of $\Gamma^{0}_{\alpha\alpha',\beta\beta'}(k,k';P)$, while $\Gamma^{0}_{\alpha\alpha',\beta\beta'}(k,k';P)$ can be expressed in terms of the free-space scattering amplitude $\tilde{f}_{\alpha\alpha',\beta\beta'}(k,k')$ using Eq. (A12) or (A13). In other words, through the intermediate function $\Gamma^{0}_{\alpha\alpha',\beta\beta'}(k,k';P)$, we can relate $\Gamma^{T}_{\alpha\alpha',\beta\beta'}(k,k';P)$ to $\tilde{f}_{\alpha\alpha',\beta\beta'}(k,k')$ directly.

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