Generalized Levinson theorem for singular potentials in two dimensions

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The Levinson theorem for two-dimensional scattering is generalized for potentials with inverse square singularities. By this theorem, the number of bound states N_m^{b} in a given mth partial wave is related to the phase shift $\delta_m(k)$ and the singularity strength of the potential. When the effective potential has an inverse square singularity at the origin of the form ν^2/ρ^2 and inverse square tail at infinity such as μ^2/ρ^2, Levinson's relation gives $\delta_m(0)-\delta_m(\infty)=\pi[N_m^{\mathrm{b}}+(|\nu|-|\mu|)/2]$.

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I. INTRODUCTION

The Levinson theorem sets up a relation between the number of bound (b) states N_l^{b} in a given lth partial wave and the phase shift $\delta_l(k)$. The theorem was proved for three-dimensional (3D) central potentials $V(|\boldsymbol{r}|)$, see the review [1,2]. Levinson's relation for the l-wave phase shift gives $\delta_l(0)-\delta_l(\infty)=\pi N_l^{\mathrm{b}}$. If the half-bound (hb) state occurs for the s-wave type ($l=0$), this is modified to $\delta_0(0)-\delta_0(\infty)=\pi(N_0^{\mathrm{b}}+\frac{1}{2})$. The Levinson theorem is one of the most beautiful results of scattering theory; it was a subject of studies by many authors [2–7]. Recently, the Levinson theorem was established for lower-dimensional systems, which play an important role in modern physics of condensed matter and in field theories [8–12].

In the case of 2D systems, Levinson's relation for the partial wave phase has the usual 3D form; but the half-bound state for the p-wave ($l=1$) contributes exactly like the bound state and gives an additional π to Levinson's relation [13]. Let us remind that a half-bound state is the zero-energy solution for the case when the eigenfunction is finite, but does not decay fast enough at infinity to be square integrable. In the 2D case, a possible s-wave half-bound state does not contribute at all to Levinson's relation, but only the p-wave half-bound state does. An experimental justification of the Levinson theorem in the 2D case was made in Refs. [14] for the 2D plasma. All mentioned papers, which discuss the 2D version of the Levinson theorem, consider potentials that are less singular than $|\boldsymbol{r}|^{-2}$. This is a standard assumption, which results in the above-mentioned form of the Levinson theorem.

At present singular potentials become an object of interest. Singular potentials naturally appear in singular inverse problems, i.e., in a supersymmetric approach to the inverse scattering in 3D, when bound states are removed from the regular potential [4,5]. The short distance behavior of the singular potential is defined by the inverse square asymptotes at the origin $V(r)\sim\beta_0/r^2$; therefore, the resulting effective potential for the partial wave U_l (*partial potential*) in the 3D case has the asymptotic form

$$U_l(r)=V(r)+\frac{l(l+1)}{r^2}\underset{r\to 0}{\sim}\frac{\nu(\nu+1)}{r^2},$$

with the *singularity strength* $\nu=\sqrt{(l+1/2)^2+\beta_0}-1/2\neq l$. One can see that the singular potential acts as a correction to the centrifugal barrier $l(l+1)/r^2$. The scattering problem for such potentials with an inverse square singularity was solved first by Swan, who has generalized the Levinson theorem for singular potentials in the 3D case [3]. It reads

$$\delta_l(0)-\delta_l(\infty)=\pi\left(N_l^{\mathrm{b}}+\frac{\nu-l}{2}\right). \tag{1}$$

In addition to the general importance for the scattering theory, the generalized Levinson theorem (1) is useful for the inverse scattering theory, because it gives a possibility to determine the parameter of the singular core of the potential from the scattering data.

In the present paper we establish the 2D analog of the generalized Levinson theorem (1). Singular potentials appear in different 2D systems: in the $(2+1)$-dimensional $O(3)$-models like $3D-SU(N_f)$ skyrmions in N_f-flavor meson fields [15], in the $2D-O(3)$ spin textures as charged quasiparticles in ferromagnetic quantum Hall systems [16], in different models of 2D magnets as an effective potential of soliton (vortex)-magnon interaction [17–20].

The paper is organized as follows. In Sec. II we formulate the scattering problem in the 2D case. We discuss the possible supersymmetric nature of singular potentials. The scattering problem is solved for the simplest example of a singular potential, i.e., for the centrifugal model, in Sec. III. The

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generalized Levinson theorem is proved in Sec. IV. A discussion and concluding remarks are presented in Sec. V.

II. SCATTERING IN TWO DIMENSIONS: NOTATIONS, SINGULAR POTENTIALS

The quantum states of a spinless particle in a central (axially symmetric) potential $V(\rho)$ in two dimensions can be described by the radial Schrödinger equation

$$H\psi_m^{\mathcal{E}}(\rho)=\mathcal{E}\psi_m^{\mathcal{E}}(\rho) \tag{2a}$$

for the Schrödinger operator $H=-\nabla_\rho^2+U_m(\rho)$ with the partial potential

$$U_m(\rho)=V(\rho)+\frac{m^2}{\rho^2}. \tag{2b}$$

The behavior of the eigenfunctions in the potential $V(\rho)$ can be analyzed at large distances from the origin, $\rho \gg R$, where R is a typical range of the potential $V(\rho)$. In view of the asymptotic behavior $U_m(\rho)\sim m^2/\rho^2$, which is valid for fast decreasing potentials $V(\rho)$, in the leading approximation in $1/\rho$ we have the usual result [21]

$$\psi_m^{\mathcal{E}} \propto J_{|m|}(k\rho)+\sigma_m(k)Y_{|m|}(k\rho), \quad k=\sqrt{\mathcal{E}}>0, \tag{3a}$$

where k is a "radial wave number," J_m and Y_m are the Bessel and the Neumann functions, respectively. The quantity $\sigma_m(k)$ stems from the scattering; it can be interpreted as the scattering amplitude [21,22]. In the limiting case, $k\rho \gg |m|$, it is convenient to consider the asymptotic form of Eq. (3a),

$$\psi_m^{\mathcal{E}} \propto \frac{1}{\sqrt{\rho}}\cos\left(k\rho-\frac{|m|\pi}{2}-\frac{\pi}{4}+\delta_m(k)\right), \tag{3b}$$

where the scattering phase, or the phase shift $\delta_m(k)=-\arctan\sigma_m(k)$. The phase shift contains all information about the scattering process.

For regular 2D potentials $V(\rho)$, the 2D analog of the Levinson theorem has the form [8,9,13]

$$\delta_m(0)-\delta_m(\infty)=\pi(N_m^{\mathrm{b}}+N_m^{\mathrm{hb}}\delta_{|m|,1}). \tag{4}$$

Here the potential $V(\rho)$ satisfies the asymptotic conditions

$$\lim_{\rho=0}\rho^2V(\rho)=0, \tag{5a}$$

$$\lim_{\rho=\infty}\rho^2V(\rho)=0, \tag{5b}$$

which provide a regular behavior at the origin, and fast decaying at infinity.

Now we switch to the singular potentials, having in mind to reestablish the Levinson theorem.

A. Potentials with inverse square singularity

Let us consider potentials with inverse square singularity. At the origin, the potential has an asymptote such as $V(\rho)\sim\beta_0/\rho^2$; the corresponding partial potential (2b)

$$U_m(\rho)\underset{\rho\to 0}{\sim}\frac{\nu^2}{\rho^2},$$

with

$$\nu=\sqrt{m^2+\beta_0}\neq m. \tag{6}$$

Singular potentials such as form (6) appear in various 2D nonlinear field theories, e.g., for the scattering problem of linear excitations by topological solitons [15–19].

Moreover, singular potentials naturally appear from regular ones under the Darboux transformations [4,5,23]. Let us recall the principle of the Darboux (supersymmetric) transformations for the 2D case [18]. We suppose that spectral problem (2a) has at least one bound state $\mathcal{E}_0<0$. Assuming that we start from the regular potential under conditions (5), then the eigenfunction $\psi_0\equiv\psi_m^{\mathcal{E}_0}(\rho)$ may have the following asymptotic behavior:

$$\psi_0(\rho)\propto\begin{cases}\rho^{|m|} & \text{when} \quad \rho\to 0,\\ \rho^{-1/2}\exp(-\kappa\rho) & \text{when} \quad \rho\to\infty,\end{cases} \tag{7}$$

where $\kappa=\sqrt{-\mathcal{E}_0}>0$.

To explain the method we introduce the Hermitian conjugate lowering and raising operators [18]

$$A=-\frac{d}{d\rho}+W(\rho), \quad A^\dagger=\frac{d}{d\rho}+\frac{1}{\rho}+W(\rho), \tag{8}$$

where the superpotential

$$W(\rho)=\frac{d}{d\rho}\ln\psi_0 \tag{9}$$

is such that $A\psi_0=0$. By introducing these operators we can represent the Schrödinger operator H in the factorized form

$$H=A^\dagger A+\mathcal{E}_0, \tag{10}$$

the factorization energy $\mathcal{E}_0$ coincides with the energy of the bound state. Such a factorization makes it possible to reformulate initial problem (10) in terms of the eigenfunction $\tilde{\psi}_m=A\psi_m$ of the spectral problem

$$\tilde{H}=AA^\dagger+\mathcal{E}_0=-\nabla_\rho^2+\tilde{U}_m(\rho), \tag{11}$$

where the partial potential

$$\tilde{U}_m(\rho)=U_m(\rho)+\frac{1}{\rho^2}-2\frac{d}{d\rho}W(\rho). \tag{12}$$

Taking into account conditions (7), one can derive the asymptotic behavior of the partial potential $\tilde{U}_m$,

$$\tilde{U}_m(\rho)\sim\begin{cases}\dfrac{\nu^2}{\rho^2} & \text{with} \quad \nu=|m|-1 \quad \text{when} \quad \rho\to 0,\\[2ex] \dfrac{m^2}{\rho^2} & \text{when} \quad \rho\to\infty.\end{cases} \tag{13}$$

We see that the eigenspectrum of the new spectral problem (11) does not contain the bound state ψ_0. The resulting potential has a singularity; in fact, the partial potential $\tilde{U}_m(\rho)$ corresponds to the particle potential $V(\rho)=\beta/\rho^2$ with the parameter $\beta=1-2|m|$. After a series of n transformations such as Eq. (11), we remove n bound states from the spectrum, which results in $\tilde{U}_m\sim\nu^2/\rho^2$, with $\nu=|m|-n$.

B. Potentials with inverse square tail

Let us discuss potentials with an inverse square tail, when far from the origin the potential $V(\rho)\sim\beta_\infty/\rho^2$; the corresponding partial potential

$$U_m(\rho)\underset{\rho\to\infty}{\sim}\frac{\mu^2}{\rho^2},$$

with

$$\mu=\sqrt{m^2+\beta_\infty}\neq m. \tag{14}$$

Potentials such as form (14) are of interest in field theories: in the $(2+1)$-nonlinear σ model of the $\boldsymbol{n}$ field [15,18], in models of 2D easy-axis [19] and easy-plane ferromagnets in the cone state [20].

To study the scattering problem let us consider the asymptotic behavior of the eigenfunctions. Obviously, at large distances $\rho\gg R$, where the scattering approximation is valid, one can use the partial wave expansion by the cylinder functions of the integer indices only; then the eigenfunction $\psi_m^{\mathcal{E}}$ can be written as $J_{|m|}+\sigma_m Y_{|m|}$ with the asymptotic form (3b).

On the other hand, in the leading approximation in $1/k\rho$, the solution of the Schrödinger equation (2a) with potential (14) can be written as

$$\psi_m^{\mathcal{E}}(\rho)\propto J_{|\mu|}(k\rho)+\tilde{\sigma}_\mu(k)Y_{|\mu|}(k\rho)$$
$$\propto\frac{1}{\sqrt{\rho}}\cos\left(k\rho-\frac{|\mu|\pi}{2}-\frac{\pi}{4}+\tilde{\delta}_\mu(k)\right), \tag{15}$$

where the indices of the cylinder functions $\mu\neq m$, see Eq. (14).

The phase shift δ_m can be calculated from $\tilde{\delta}_\mu$ by comparing Eqs. (3b) and (15),

$$\delta_m(k)=\tilde{\delta}_\mu(k)+\frac{|m|-|\mu|}{2}\pi, \tag{16}$$

in accordance with the results of Refs. [9,20]. Note that Levinson's relation has the same form for both phase shifts δ_m and $\tilde{\delta}_\mu$,

$$\delta_m(0)-\delta_m(\infty)=\tilde{\delta}_\mu(0)-\tilde{\delta}_\mu(\infty).$$

III. SCATTERING PROBLEM FOR THE CENTRIFUGAL MODEL

For the analytical description of the scattering problem, let us consider the simplest model, which includes the main features of the problem, having both inverse square singularity and inverse square tail. The partial potential of this very simple *centrifugal model* [19] has the form

$$U_m^{\mathrm{cf}}(\rho)=\begin{cases}\dfrac{\nu^2}{\rho^2} & \text{when} \quad \rho<R,\\[2ex] \dfrac{\mu^2}{\rho^2} & \text{otherwise},\end{cases} \tag{17}$$

with $\nu\neq m$, and $\mu\neq m$.

This model describes a quasifree particle in each of the regions $\rho<R$ and $\rho>R$. The only effect of the interaction with the potential U_m^{cf} is a shift of the mode indices:

$$\psi_m^{\mathrm{cf}}(r)\propto\begin{cases}J_{|\nu|}(k\rho) & \text{when} \quad \rho<R,\\ J_{|\mu|}(k\rho)+\tilde{\sigma}_\mu(k)Y_{|\mu|}(k\rho) & \text{otherwise}.\end{cases} \tag{18}$$

The usual matching condition for these solutions has the form

$$\left[\frac{\psi'}{\psi}\right]_R=0, \tag{19}$$

where $[\cdots]_R\equiv(\cdots)|_{R+0}-(\cdots)|_{R-0}$, and the prime denotes $d/d\rho$. The calculations lead to the scattering phase shift in the form

$$\delta_m^{\mathrm{cf}}(k)=\frac{|m|-|\mu|}{2}\pi-\arctan\tilde{\sigma}_\mu^{\mathrm{cf}}(\varkappa\equiv kR),$$

$$\tilde{\sigma}_\mu^{\mathrm{cf}}(\varkappa)=\frac{J'_{|\nu|}(\varkappa)J_{|\mu|}(\varkappa)-J'_{|\mu|}(\varkappa)J_{|\nu|}(\varkappa)}{J_{|\nu|}(\varkappa)Y'_{|\mu|}(\varkappa)-J'_{|\nu|}(\varkappa)Y_{|\mu|}(\varkappa)}. \tag{20}$$

Using the asymptotic form of the cylinder functions, one can find the long- and short-wavelength behavior of phase shift (20),

$$\delta_m^{\mathrm{cf}}(k)\sim\begin{cases}\dfrac{|m|-|\mu|}{2}\pi+\mathcal{A}_m\left(\dfrac{kR}{2}\right)^{2|\mu|}, & kR\ll 1\\[2ex] \dfrac{|m|-|\nu|}{2}\pi-\dfrac{\mu^2-\nu^2}{2kR}, & kR\gg 1,\end{cases} \tag{21}$$

where

$$\mathcal{A}_m = -\frac{\pi|\mu|}{(|\mu|)!^2}\frac{|\mu|+|\nu|}{|\mu|-|\nu|}.$$

The Levinson theorem for the centrifugal model can be easily derived from Eq. (21):

$$\delta_m^{\rm cf}(0)-\delta_m^{\rm cf}(\infty)=\pi\frac{|\nu|-|\mu|}{2}. \tag{22}$$

IV. THE LEVINSON THEOREM

Now we discuss the general case where the partial potential has the asymptotic behavior

$$U_m(\rho)\sim\begin{cases}\dfrac{\nu^2}{\rho^2} & \text{when} \quad \rho\to 0,\\[2ex] \dfrac{\mu^2}{\rho^2} & \text{when} \quad \rho\to\infty,\end{cases} \tag{23}$$

with $\nu\neq m$, and $\mu\neq m$.

Let us enter into a proof of the Levinson theorem. There are three main methods to derive the theorem: the Jost functions method [24], Green's functions method [8,12,13], and the Sturm-Liouville method [9,10], which were used for the 3D case, for the details see Ref. [9].

To generalize the Levinson theorem, we use the method of Green's functions, as it was done for regular potentials by [8]. We consider the noncritical case, when the Schrödinger equation has no half-bound states.

The idea of Lin's method is to count the number of states in the system by two different ways.

The continuous part of the spectrum is discretized to count the number of scattering states. The total (infinite) number of states in the system does not depend on the shape of the potential, i.e., as it was stressed by [8], the total number of states is not altered by an attractive field, except that some scattering states are pulled down into the bound-state region. It results in

$$\mathrm{Im}\int_{-\infty}^{\infty}d\mathcal{E}\int_0^{\infty}\rho d\rho\{G[U_m]-G[U_m^{\rm free}]\}=0, \tag{24a}$$

where $G[U_m]\equiv G_m(\rho,\rho,\mathcal{E};U_m)$ and $G[U_m^{\rm free}]\equiv G_m(\rho,\rho,\mathcal{E};U_m^{\rm free})$ are Green's functions with and without potential, respectively, and retarded Green's function is defined by

$$G_m(\rho,\rho',\mathcal{E};U_m)=\sum_{\kappa}\frac{\psi_m^{\mathcal{E}}(\rho)\psi_m^{\mathcal{E}}(\rho')}{\mathcal{E}-\mathcal{E}_{m\kappa}+\imath\epsilon}.$$

In this method, the number of bound states,

$$\pi N_m^{\rm b}=-\mathrm{Im}\int_{-\infty}^{0}d\mathcal{E}\int_0^{\infty}\rho d\rho\{G[U_m]-G[U_m^{\rm free}]\}. \tag{24b}$$

On the other hand, the continuous part of expression (24a) can be directly calculated without discretization:

$$\mathrm{Im}\int_{0}^{\infty}d\mathcal{E}\int_0^{\infty}\rho d\rho\{G[U_m]-G[U_m^{\rm free}]\}=\delta_m(0)-\delta_m(\infty). \tag{24c}$$

Combining Eqs. (24), one can obtain the Levinson theorem in form (4). However, the method of Green's functions in the form proposed by Lin [8] does not work for singular potentials of form (23). The reason is that the difference of Green's functions $G[U_m]-G[U_m^{\rm free}]$ in Eq. (24) has a singularity at the origin, hence it is not integrable.

That is why we need to generalize the method for the case of singular potentials. The idea is to compare the required partial potential U_m not with the free particle partial potential $U_m^{\rm free}$, but with another potential $U_m^{\star}$, which could compensate the singularities of U_m. As we have mentioned before, the number of states does not depend on the shape of the potential. It means that repeating the same proof, Eqs. (24), can be easily generalized for the systems $G[U_m]$ and $G[U_m^{\star}]$ with two different potentials U_m and $U_m^{\star}$:

$$\mathrm{Im}\int_{-\infty}^{\infty}d\mathcal{E}\int_0^{\infty}\rho d\rho\{G[U_m]-G[U_m^{\star}]\}=0, \tag{25a}$$

$$\mathrm{Im}\int_{-\infty}^{0}d\mathcal{E}\int_0^{\infty}\rho d\rho\{G[U_m]-G[U_m^{\star}]\}=-\pi(N_m^{\rm b}-N_m^{\rm b\star}), \tag{25b}$$

$$\mathrm{Im}\int_{0}^{\infty}d\mathcal{E}\int_0^{\infty}\rho d\rho\{G[U_m]-G[U_m^{\star}]\}=\delta_m(0) -\delta_m(\infty)-\delta_m^{\star}(0)+\delta_m^{\star}(\infty), \tag{25c}$$

where $N_m^{\rm b\star}$ and $\delta_m^{\star}(k)$ are the number of bound states and the scattering phase shift for the system with the partial potential $U_m^{\star}=V^{\star}+m^2/\rho^2$.

Note that choosing $V^{\star}=0$, one can obtain Levinson's relation for the regular potentials in the form of Lin [8], see Eqs. (24), which leads to the Levinson theorem (4).

However, in the case of a singular potential, we need to choose $V^{\star}$ in the form, which has the same singularities as the potential V. To solve the problem we set $U_m^{\star}=U_m^{\rm cf}$; hence both partial potentials U_m and the centrifugal potential $U_m^{\rm cf}$ have the same features. Therefore, Eqs. (24) with account of Levinson's relation (22) lead to the following form:

$$\delta_m(0)-\delta_m(\infty)=\pi\left(N_m^{\rm b}+\frac{|\nu|-|\mu|}{2}\right). \tag{26}$$

Let us discuss the result. To explain the meaning of the extra term $(\pi/2)(|\nu|-|\mu|)$ in the generalized Levinson relation (26), let us remind that in the partial wave method the scattering data are classified by the azimuthal quantum number m, which is the strength of the centrifugal potential.

In the presence of the potential with an inverse square tail at infinity such as $U_m\sim\mu^2/\rho^2$, the effective singularity strength is shifted by the value $|\mu|-|m|$, and the long-wavelength scattering data are changed by $(\pi/2)(|m|-|\mu|)$, see Eq. (16). The same situation takes place for the potentials with an inverse square singularity at the origin such as $U_m \sim \nu^2/\rho^2$. The effective singularity strength is shifted now by the value $|\nu|-|m|$, which results in a change in the short-wavelength scattering phase shift by $(\pi/2)(|m|-|\nu|)$. As a result, the correction to Levinson's relation is

$$\pi\frac{|m|-|\mu|}{2}-\pi\frac{|m|-|\nu|}{2}=\pi\frac{|\nu|-|\mu|}{2}.$$

Such a correction looks like a modification in the classification of the scattered states, both at the origin ($\psi_m \to \psi_\nu$) and at the infinity ($\psi_m \to \psi_\mu$). However, we need to stress that the singularity strengths ν and μ can assume any real values, while the quantum number m is always an integer.

V. CONCLUSION

In conclusion, we have established the analog of the Levinson theorem in the case of two-dimensional scattering for central potentials, which are independent of both the energy and the azimuthal momentum m, but have inverse square singularities and tails.

The presence of m-dependent potentials can essentially change the scattering picture: the symmetry $\delta_m(k)=\delta_{-m}(k)$ is broken, so it is not enough to take into account partial waves with $m \geqslant 0$ only. As a result, Levinson's relation (26) has a different form for opposite m. Moreover, the threshold behavior for the half-bound states changes, so the contribution of the half-bound states in form (4) may be not adequate.

The generalized Levinson theorem (26) can be applied to different physical problems. For example, it becomes a central point in the singular inverse method [4], giving a possibility to derive the potential from the scattering phase shift. At the same time it provides a method to count bound states. The method can be used in various 2D field theories with applications to physics of 2D plasma [14], nuclear physics [15], quantum Hall effect [16], and 2D magnetism [17–20].

The method of the 2D radial Darboux transformations, considered in the paper, can be applied to the supersymmetric quantum mechanics, e.g. for the problem of phase-equivalent potentials, even for energy-dependent potentials [4,5,23].

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