Internal-state dephasing of trapped ions

S. Brouard and J. Plata
Departamento de Física Fundamental II, Universidad de La Laguna, La Laguna E38204, Tenerife, Spain
(Received 25 February 2003; published 14 July 2003)

The effect of random fields on the internal dynamics of trapped ions is studied analytically. General characteristics of the dependence of the dephasing on the noise statistics are identified: the form of the initial decay of the coherences is determined by the probability distribution; effects of noise color, in particular, collapses and revivals rooted in spectral concentration of the fluctuations, are apparent in a transient regime; at large times, exponential decay sets in for widely different noise properties. The study is particularized to magnetic-field fluctuations: features distinctive of the linear and quadratic Zeeman shifts are traced. The scaling of the dephasing with the number of ions is analyzed; the implications for the realization of decoherence-free states are discussed.

DOI: 10.1103/PhysRevA.68.012311 PACS number(s): 03.67.Lx, 03.67.Pp

The applicability of trapped ions in quantum computation depends crucially on the control of the sources of decoherence. The realization of quantum logic gates can indeed be affected by different, internal and external, decohering mechanisms [1]. In particular, ambient fields can lead to dephasing of the "qubit" states. Recently, a scheme for protecting the logical information against *collective dephasing*, i.e., dephasing with the same effects on each qubit, was experimentally tested [2]. After encoding the logical data into a decoherence-free state, the resistance of the encoded qubit to decoherence due to an "engineered" noisy environment was checked. The effect of uncontrolled noise sources (mainly, ambient magnetic fields) was also monitored. The results, which verify the validity of the scheme, provide valuable information on general aspects of noise-induced dephasing. Here, focusing on those results, we study, analytically, the dependence of the dephasing characteristics on the noise statistics. Specifically, effects of noise color are investigated: we show how the initial decay of the coherences is determined by the noise distribution; the occurrence of collapses and revivals rooted in the spectral concentration is traced. Additionally, the role of non-Gaussian fluctuations and the emergence of exponential decay at large times are elucidated. Features specific to the linear and quadratic Zeeman shifts are identified. Finally, the scaling of the dephasing with the number of ions and the implications of noncollective effects are discussed. In addition to an understanding of these fundamental questions, the study gives general clues to the characterization of the noise sources from experimental data.

We describe the trapped ion as a two-level system (the two electronic levels $|g\rangle$ and $|e\rangle$ which form the qubit) with an energy splitting stochastically varied by the effect of the random fields. In the absence of the laser fields that serve to implement the logic gates, the dynamics is characterized by the Hamiltonian [1]

$$H_0=\frac{\hbar}{2}[\omega_0+\delta\omega_0(t)]\sigma_z, \tag{1}$$

where ω_0 is the mean frequency and $\delta\omega_0(t)$ is the shift induced by the fluctuations. In the rotating frame defined by the unitary transformation $U=\exp(-i\omega_0\sigma_z t/2)$, the evolution for each stochastic realization is given by $|\psi(t)\rangle=e^{-i\phi(t)\sigma_z/2}|\psi(0)\rangle$, where $\phi(t)$ is a nonstationary stochastic variable [3] defined by

$$\phi(t)=\int_0^t\delta\omega_0(t')dt'. \tag{2}$$

The reduced density matrix is obtained by averaging over fluctuations [4]. The populations do not change; the coherences are given by

$$\rho_{g,e}(t)=\rho_{g,e}(0)\langle e^{i\phi(t)}\rangle, \tag{3}$$

where it is evident that the form of the decay is determined by the statistics of $\phi(t)$.

Important characteristics of the dephasing can be anticipated from the following general arguments. First, we stress that $\phi(t)$ is the sum of elementary increments $\delta\omega_0 dt$, which, for finite correlation time τ_c, are statistically dependent. When $\delta\omega_0$ is a Gaussian variable, $\phi(t)$ is also normally distributed, and we obtain [5]

$$\rho_{g,e}(t)=\rho_{g,e}(0)e^{-\mathrm{var}[\phi(t)]/2}e^{i\langle\phi(t)\rangle t}, \tag{4}$$

which reflects exponential dependence on the phase-shift variance $\mathrm{var}[\phi(t)]\equiv\langle[\phi(t)-\langle\phi(t)\rangle]^2\rangle$. For a correlation function with the standard form $\langle\delta\omega_0(0)\delta\omega_0(\tau)\rangle=\mathrm{var}(\delta\omega_0)k(\tau)$, since $\mathrm{var}[\phi(t)]\propto\mathrm{var}(\delta\omega_0)$, exponential dependence of $\rho_{g,e}(t)$ on $\mathrm{var}(\delta\omega_0)$ is additionally found. On the other hand, for non-Gaussian $\delta\omega_0$, it is not trivial to characterize the statistics of $\phi(t)$ and, consequently, the form of the decay. Even so, analytical results valid for generic $\delta\omega_0$ can be found in the following limiting cases.

(i) For $t\ll\tau_c$, the phase shift can be approximated as $\phi(t)\simeq\delta\omega_0(0)t$ [3,6]. Then, the average in Eq. (3) is completely determined by the probability distribution $W(\delta\omega_0)$, i.e.,

$$\rho_{g,e}(t)=\rho_{g,e}(0)\int e^{i\delta\omega_0 t}W(\delta\omega_0)d(\delta\omega_0). \tag{5}$$

In particular, for a Gaussian input with mean value $\langle\delta\omega_0\rangle$ and variance $\mathrm{var}(\delta\omega_0)$ we obtain [5]

$$\rho_{g,e}(t) \propto e^{-(1/2)\mathrm{var}(\delta\omega_0)t^2} e^{i\langle\delta\omega_0\rangle t}, \tag{6}$$

which corresponds to Gaussian decay with $1/e$ time τ_d given by $\tau_d = [\mathrm{var}(\delta\omega_0)/2]^{-1/2}$.

(ii) For the description of the asymptotic behavior, it is useful to write the phase shift as $\phi(t) = \int_0^{\Delta t}\delta\omega_0 dt + \int_{\Delta t}^{2\Delta t}\delta\omega_0 dt + \cdots + \int_{(n-1)\Delta t}^{t}\delta\omega_0 dt$. For $t \gg \tau_c$, an interval Δt larger than τ_c, which guarantees that the different increments are practically uncorrelated, is compatible with a large n. $\phi(t)$ is then approximated by a sum of a large number of statistically independent variables, and, according to the central limit theorem, presents an approximate normal distribution. Additionally, it is shown that $\mathrm{var}[\phi(t)] \simeq 2K_{\delta\omega_0}t$, where $K_{\delta\omega_0} = \int_0^\infty \langle \delta\omega_0(0)\,\delta\omega_0(\tau)\rangle d\tau \propto \mathrm{var}(\delta\omega_0)$ [3]. Consequently, the coherences can be approximated as

$$\rho_{g,e}(t) \propto e^{-K_{\delta\omega_0}t} e^{i\langle\delta\omega_0\rangle t}. \tag{7}$$

Hence, exponential decay with $1/e$ time scaling as $\tau_d \propto [\mathrm{var}(\delta\omega_0)]^{-1}$ becomes apparent. Note that the emergence of exponential decay can be anticipated, provided that the attenuation of the noise correlation function can be assumed; indeed, the particular spectral characteristics of $\delta\omega_0$ affect only the value of $K_{\delta\omega_0}$. Moreover, the asymptotic regime is more rapidly reached as τ_c decreases; in the white-noise limit $(\tau_c \to 0)$, the decay is exponential at any time.

In contrast with the above analysis of the initial and asymptotic behaviors, the characterization of the intermediate regime requires precise information on the noise spectrum. This point is illustrated by the following study of different types of noise relevant to the system.

(a) As a first approximation to the description of color dependent features, we consider standard colored noise [3], i.e., Gaussian noise $\xi_{sc}(t)$ defined by $\langle \xi_{sc}(t)\rangle = 0$ and

$$\langle \xi_{sc}(t)\xi_{sc}(t')\rangle = \gamma D e^{-\gamma|t-t'|}. \tag{8}$$

For $\delta\omega_0(t) \propto \xi_{sc}(t)$, we obtain [3] $\langle \phi(t)\rangle = 0$ and

$$\mathrm{var}[\phi(t)] \propto 2D\left[t + \frac{1}{\gamma}(e^{-\gamma t} - 1)\right]. \tag{9}$$

Hence, a finite value of $\tau_c = \gamma^{-1}$ implies nonlinear time dependence of $\mathrm{var}[\phi(t)]$, and, therefore, nonexponential decay of $\rho_{g,e}(t)$ [see Eq. (4)]. There is no sharp transition from Gaussian to exponential decay, but an intermediate regime determined by the noise spectrum. This transient regime can be analytically characterized for different types of Gaussian noise; its identification in the experimental data can serve to define the color properties of the noise sources. In the white-noise limit $(\gamma \to \infty)$, $\phi(t)$ is described by a Wiener process [3]; its variance is then linear time dependent and, as previously stated, the decay is always exponential.

(b) In recent experiments [2], a noisy reservoir was engineered by applying an off-resonant laser with a randomly varying intensity: through the Stark shift, a "controlled" random variation of $\omega_0(t)$ was achieved. Since, in the experiments, the stochastic signal was generated by filtering white noise, it is of interest to explicitly consider a Gaussian input $\xi_{fw}(t)$ defined by $\langle \xi_{fw}(t)\xi_{fw}(t')\rangle = (D/\pi)(\sin[\Omega_c(t-t')]/(t-t'))$, which corresponds to the spectral density $S(\Omega) = 2D$ for $\Omega < \Omega_c$ and $S(\Omega) = 0$ for $\Omega > \Omega_c$ [3]. Assuming that $\xi_{fw}(t)$ describes the fluctuations in intensity, i.e., for $\delta\omega_0(t) \propto \xi_{fw}(t)$, we find

$$\mathrm{var}[\phi(t)] \propto 2\frac{D}{\pi}\left[\mathrm{Si}(\Omega_c t)t + \frac{1}{\Omega_c}(\cos\Omega_c t - 1)\right], \tag{10}$$

where $\mathrm{Si}(\Omega_c t) \equiv \int_0^{\Omega_c t}(\sin z/z)dz$ is the sine integral function [7]. $\mathrm{Si}(\Omega_c t)$ presents a linear time increase followed by nonexponentially damped oscillations around $\pi/2$. Indeed, taking into account that $\mathrm{Si}(\Omega_c t) \sim \Omega_c t$ for $t \ll \Omega_c^{-1}$ and $\mathrm{Si}(\Omega_c t) \sim (\pi/2) - (1/\Omega_c t)\cos\Omega_c t$ for $t \gg \Omega_c^{-1}$, the time dependence of the coherences, in agreement with our previous general discussion, is found to be Gaussian at short times and exponential in the asymptotic regime. Qualitative differences with the behavior corresponding to standard colored noise are apparent in the intermediate regime: oscillations with Ω_c can be observable depending on the relative magnitude of Ω_c^{-1} and the decoherence time. Here, it is worth recalling that the values of Ω_c and the noise strength can be controlled in the experiments [2]. For $\Omega_c \to \infty$, the results for white noise are consistently recovered.

(c) Spectral concentration is a characteristic of important components of the ambient fields [1]. Useful insight into the implications of this property can be obtained from the study of a purely monochromatic signal. Hence, we assume a harmonic shift $\delta\omega_0(t) = A\sin(\Omega_0 t + \varphi)$ with fixed amplitude A and uniformly distributed initial phase φ, i.e., $W(\varphi) = (1/2\pi)$. $\delta\omega_0(t)$ is then a stationary non-Gaussian variable. For each φ, one obtains $\phi(t) = (A/\Omega_0)[\cos\varphi - \cos(\Omega_0 t + \varphi)]$. The expansion of $e^{i\phi(t)}$ in terms of the Bessel functions [5] and the subsequent average over φ, leads to

$$\rho_{g,e}(t) = \rho_{g,e}(0)J_0\left(\frac{2A}{\Omega_0}\sin\frac{\Omega_0 t}{2}\right), \tag{11}$$

where significant differences with the effect of Gaussian noise are evident. From the oscillatory behavior of the argument of the Bessel function, a nontrivial evolution can be anticipated: recurrent complete collapses and revivals appear if zeros and peaks of $J_0(z)$ are reached. As there is no attenuation of the correlation function of $\delta\omega_0(t)$, there is no emergence of exponential decay at large times. For $t \ll \Omega_0^{-1}$, we find $\rho_{g,e}(t) \simeq \rho_{g,e}(0)J_0(At)$; then, if the asymptotic limit of $J_0(At)$ is reached inside the considered regime, a $t^{-1/2}$ dependence is eventually present. On the other hand, for $t \gg \Omega_0^{-1}$, through a coarse graining over $2\pi/\Omega_0$, one obtains $\rho_{g,e}(t) = \rho_{g,e}(0)J_0^2(A/\Omega_0)$ for the average behavior. The effect of a high-frequency signal can, therefore, be described as a "renormalization" of the coherences, which is, in fact, irrelevant if $A/\Omega_0 \ll 1$. In the laser-induced coupling of internal and motional modes needed to implement the logic gates, this effect leads to an effective reduction of the Rabi frequency [1].

Magnetic-field fluctuations. Ambient magnetic fields are expected to be the main source of internal decoherence in the usual experimental arrangements. Hence, it is worth describing this dephasing mechanism in detail. The frequency shift induced by magnetic-field fluctuations can be approximated as [1]

$$\delta\omega_0(t)=\left[\frac{\partial\omega}{\partial B}\right]_{B_0}(B-B_0)+\frac{1}{2}\left[\frac{\partial^2\omega}{\partial B^2}\right]_{B_0}(B-B_0)^2, \quad (12)$$

where B_0 is the average field and $\xi\equiv B-B_0$ represents the random increment. In standard conditions, the major contribution to magnetic noise comes from ac power line noise and, therefore, is concentrated primarily at the corresponding fundamental frequency and at its harmonics. Accordingly, we consider line noise defined by $\xi_l(t)=A(t)\sin[\Omega_0 t+\varphi(t)]$ with $A(t)$ and $\varphi(t)$ varying much more slowly than $\Omega_0 t$. In time scales much smaller than the characteristic times of $A(t)$ and $\varphi(t)$, $\xi_l(t)$ can be described as a purely harmonic signal with time independent A and φ. Actually, this is the regime relevant to the experiments: the measured decoherence times are much smaller than the ac line-noise period [2]. For simplicity, a fixed amplitude and a uniformly distributed initial phase will be assumed. Moreover, we consider that the noisy input has also a broadband component. Specifically, we model the magnetic fluctuations as $\xi=\xi_{sc}(t)+\xi_l(t)$ with $\xi_{sc}(t)$ defined by Eq. (8). We concentrate now on two cases of practical interest, which illustrate how qualitatively different behaviors can emerge depending on the relative importance of the linear and quadratic Zeeman shifts.

First, to focus on the linear Zeeman effect, we consider a regime in which the quadratic term in Eq. (12) can be neglected. Because of the non-Gaussian monochromatic contribution, $\delta\omega_0(t)$ is not normally distributed. Even so, an analytical characterization of the coherences can be given. As φ is statistically independent of $\xi_{sc}(t)$, we can proceed by obtaining first the evolution at fixed φ and then averaging the phase. The frequency shift for each φ, $\delta\omega_0^{(\varphi)}(t)$, is a Gaussian variable defined by $\langle\delta\omega_0^{(\varphi)}(t)\rangle=[\partial\omega/\partial B]_{B_0}A\sin(\Omega_0 t+\varphi)$ and $\mathrm{var}[\delta\omega_0^{(\varphi)}(t)]=[\partial\omega/\partial B]_{B_0}^2\gamma D$. The coherences for a fixed φ, $\rho_{g,e}^{(\varphi)}(t)$, are straightforwardly obtained; the subsequent phase averaging leads to

$$\rho_{g,e}(t)\propto J_0[z(t)]e^{-\mathrm{var}[\phi^{(\varphi)}(t)]/2}, \quad (13)$$

where $z(t)=[\partial\omega/\partial B]_{B_0}(2A/\Omega_0)\sin(\Omega_0 t/2)$ and $\mathrm{var}[\phi^{(\varphi)}(t)]=2[\partial\omega/\partial B]_{B_0}^2 D[t+(1/\gamma)(e^{-\gamma t}-1)]$. Actually, as $t\ll\Omega_0^{-1}$ in the experiments, we can write $z(t)\simeq[\partial\omega/\partial B]_{B_0}At$. Here, the decay corresponding to broadband fluctuations, initially Gaussian and asymptotically exponential, appears modulated by the effect of line noise. For small τ_c, as the asymptotic regime rapidly sets in, the initial transient is hardly observable. A probability distribution for the amplitude can be easily incorporated into our approach: the additional average over A implies an attenuation of the features linked to the modulation factor $J_0[z(t)]$. These results can account for the roughly exponential decay found in the experiments [2], yet higher resolution is needed for a more quantitative analysis which can reveal the actual relevance of the different elements of our model.

To discuss effects specific to the quadratic Zeeman shift, we assume now an average field B_0 that makes the $[\partial\omega/\partial B]_{B_0}$ term vanish. This regime is especially interesting, given its probable use in practical ion-trap quantum computers [2]. Here, $\delta\omega_0^{(\varphi)}(t)$, given by a nonlinear transformation of a normal variable, has non-Gaussian character. (Problems involving *Gaussian quadratic noise* have been tackled in different contexts [6,8,9]). Consequently, nontrivial dependence of $\rho_{g,e}^{(\varphi)}(t)$ on $\mathrm{var}[\phi^{(\varphi)}(t)]$ can be expected. In particular, non-Gaussian initial decay can be predicted. Additionally, the nonzero mean value of ξ_{sc}^2 induces an oscillation of the coherences with $\Lambda\equiv(1/2)\times[\partial^2\omega/\partial B^2]_{B_0}\gamma D$, added to that associated to the mean frequency ω_0. The complex evolution of $\rho_{g,e}^{(\varphi)}(t)$ simplifies considerably for small τ_c: exponential decay with a rate proportional to the variance of $\delta\omega_0^{(\varphi)}(t)$ becomes rapidly apparent. Again, a modulation of this behavior results from the averaging over phase and amplitude.

Dephasing of multiple-ion states. The generalization of the study to a system of N ions is straightforward. Working with the product states $|K\rangle\equiv\otimes_{j=1}^N|\beta_j^{(K)}\rangle$ (j labels the ion and the two possible values of $\beta_j^{(K)}$, 0 and 1, stand, respectively, for g and e), we find

$$\rho_{K,K'}(t)=\rho_{K,K'}(0)\left\langle\exp\left[-i\sum_{j=1}^{N}(\beta_j^{(K)}-\beta_j^{(K')})\phi_j(t)\right]\right\rangle, \quad (14)$$

where the individual phase shifts $\phi_j(t)$ contain the effect of nonuniform fields.

In the case of collective dephasing, i.e., if the ions can be assumed to experience the same field [$\phi_j(t)=\phi(t)$ for all j], one obtains $\rho_{K,K'}(t)=\rho_{K,K'}(0)\langle e^{-iG_{K,K'}^{(N)}\phi(t)}\rangle$, where $G_{K,K'}^{(N)}\equiv\Sigma_{j=1}^N(\beta_j^{(K)}-\beta_j^{(K')})\leqslant N$. Now our previous results can be applied to analyze the scaling of decoherence with N. In particular, when $\phi(t)$ presents a zero-mean normal distribution, we find [5]

$$\rho_{K,K'}(t)=\rho_{K,K'}(0)\exp[-(G_{K,K'}^{(N)})^2\mathrm{var}[\phi(t)]/2]. \quad (15)$$

As the characteristic times $\tau_d^{(N)}$ vary with form of the decay, the different regimes must be explicitly taken into account. Namely, for the initial Gaussian form, one obtains $\tau_d^{(N)}=(G_{K,K'}^{(N)})^{-1}\tau_d\geqslant\tau_d/N$. On the other hand, for the asymptotic exponential decay, we have $\tau_d^{(N)}=(G_{K,K'}^{(N)})^{-2}\tau_d\geqslant\tau_d/N^2$. These results reflect a quite favourable situation for practical applications. Furthermore, condition $G_{K,K'}^{(N)}=0$ guarantees the coherence of a superposition of $|K\rangle$ and $|K'\rangle$ against all forms of collective dephasing. In particular, states $|\psi_+\rangle=(|ge\rangle+i|eg\rangle)/\sqrt{2}$ and $|\psi_-\rangle=(|ge\rangle-i|eg\rangle)/\sqrt{2}$ span a decoherence-free subspace for a system of two ions [2].

In the experiments with engineered reservoirs [2], a scheme for protecting the logical information against collective noise, based on encoding the bare state $|\psi_{bare}\rangle=|g\rangle$

$\otimes(e^{i\alpha}|g\rangle+|e\rangle)/\sqrt{2}$ into the decoherence-free state $|\psi_{DFS}\rangle=(|\psi_-\rangle+e^{i\alpha}|\psi_+\rangle)/\sqrt{2}$, was tested ($\alpha$ denotes a rotation previous to the encoding). The results indicate a roughly uniform effect of all the relevant noise sources, in particular, of ambient magnetic fields. Let us analyze how departures from collective decoherence affect the applicability of the scheme. A simple discussion of this issue can be made from the evaluation of the fidelity, defined as $F_{|\psi\rangle}(t)\equiv\langle|\langle\psi(0)|\psi(t)\rangle|^2\rangle_f$ [4] (the average over fluctuations is denoted here by $\langle\cdots\rangle_f$ to distinguish it from the quantum average). When nonuniformity in the fields is considered, we have

$$F_{|\psi_{DFS}\rangle}(t)=\frac{1}{2}[1+\cos^2\alpha+(1-\cos^2\alpha)\langle\cos\Delta\phi(t)\rangle_f], \tag{16}$$

where $\Delta\phi(t)=\phi_1(t)-\phi_2(t)$. Hence, the decay of the fidelity is determined by $\langle\cos\Delta\phi(t)\rangle_f$ and therefore, by the statistics of $\Delta\phi(t)$. For a comparison, we recall that $F_{|\psi_{bare}\rangle}(t)=(1/2)[1+\langle\cos\phi_2(t)\rangle_f]$. Given the variety of experimental conditions that can be relevant to the problem, it is worth analyzing different possible situations.

(a) We first consider differential dephasing rooted in a spatial variation in the intensity of the fields. This may be applicable to the experiments where residual decoherence in state $|\psi_{DFS}\rangle$ was linked to departures from equal illumination by the applied random fields [2]. In this case, $\Delta\phi(t)$ has the same statistics as $\phi_j(t)$ $(j=1,2)$. Therefore, for zero-mean Gaussian variables, we obtain [5]

$$\langle\cos\Delta\phi(t)\rangle_f=e^{-\mathrm{var}[\Delta\phi(t)]/2}. \tag{17}$$

As $\mathrm{var}[\Delta\phi(t)]$ can be much smaller than $\mathrm{var}[\phi_j(t)]$, an effective reduction of the dephasing can be achieved working with $|\psi_{DFS}\rangle$. Moreover, features distinctive of noise color can be more apparent in the larger times of decay that can be reached.

(b) A different behavior occurs when there is spatial loss of correlation, namely, when sources of microscopic uncorrelated noise are present. In particular, if variables $\phi_1(t)$ and $\phi_2(t)$ are uncorrelated, i.e., if $\langle\phi_1(t)\phi_2(t')\rangle=0$, we have $\mathrm{var}[\Delta\phi(t)]=\mathrm{var}[\phi_1(t)]+\mathrm{var}[\phi_2(t)]$, which makes the intended protection of coherence ineffective. Additionally, spatial variations in the spectrum of $\phi(t)$ can induce significant differences between the color dependent transient of $F_{|\psi_{DFS}\rangle}(t)$ and that of $F_{|\psi_{bare}\rangle}(t)$.

The actual relevance of the different scenarios must be clarified by further experimental work.

As a final remark, we stress the generality of our results and their potential applicability to the characterization of the noise sources.

One of the authors (S.B.) wants to thank the Ministerio de Ciencia y Tecnologia of Spain (Grant No. BFM 2001-3349) for financial support.

[1] For a review, see D.J. Wineland, C. Monroe, W.M. Itano, D. Leibfried, B.E. King, and D.M. Meekhof, J. Res. Natl. Inst. Stand. Technol. **103**, 259 (1998), and references therein.

[2] D. Kielpinski, V. Meyer, M.A. Rowe, C.A. Sackett, W.M. Itano, C. Monroe, and D.J. Wineland, Science **291**, 1013 (2001) (published online 5 January 2001; 10.1126/science.1057357); D. Kielpinski, Ph.D. thesis, University of Colorado, 2001.

[3] R.L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963).

[4] D.F.V. James, Phys. Rev. Lett. **81**, 317 (1998).

[5] I.S. Gradshteyn and I.M. Ryshik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1994).

[6] S.O. Rice, Bell Syst. Tech. J. **23**, 282 (1944); **24**, 46 (1945); N. Wax, *Selected Papers on Noise and Stochastic Processes* (Dover, New York, 1954).

[7] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (Dover Publications, New York, 1970).

[8] L.S. Brown, Ann. Phys. (N.Y.) **159**, 62 (1985).

[9] S. Brouard and J. Plata, Phys. Rev. A **65**, 053412 (2002).