Unitary relation for the time-dependent SU(1,1) systems

Dae-Yup Song*
Department of Physics, University of Florida, Gainesville, Florida 32611, USA
(Received 2 March 2003; published 24 July 2003)

The system whose Hamiltonian is a linear combination of the generators of SU(1,1) group with time-dependent coefficients is studied. It is shown that there is a unitary relation between the system and a system whose Hamiltonian is simply proportional to the generator of the compact subgroup of SU(1,1). The unitary relation is described by the classical solutions of a time-dependent (harmonic) oscillator. Making use of the relation, the wave functions satisfying the Schrödinger equation are given, for a general unitary representation, in terms of the matrix elements of a finite group transformation (Bargmann function). The wave functions of the harmonic oscillator with an inverse-square potential is studied in detail, and it is shown that through an integral, the model provides a way of deriving the Bargmann function for the representation of positive discrete series of SU(1,1).

DOI: 10.1103/PhysRevA.68.012108 PACS number(s): 03.65.Fd, 02.20.Tw, 03.65.Ge

I. INTRODUCTION

Group theoretical methods could be useful in analyzing physical systems, and particularly the su(1,1)-type algebraic structure is known to appear in many quantum systems [1–3]. Schemes for experimental realizations of the SU(1,1) states have been proposed [4,5] and the time-dependent quadratic system (a generalized harmonic oscillator) [6] is a realization of particular representations of the SU(1,1) group. The evolution operator and transition probabilities of the harmonic oscillator with a time-dependent frequency have been known in terms of classical solutions of the oscillator [7]. The wave functions of the quadratic systems [6,7], if the centers of probability distributions of the functions remain at the origin of the space coordinate, are closely related to the SU(1,1) coherent states of Perelomov [8], which are obtained by applying displacement-type elements of the group on a fiducial vector in a representation space.

Unitary transformation methods have long been recognized as a useful tool in finding the wave functions of the coherent systems [9] and of the generalized harmonic oscillators [7,10,11]. Through a unitary transformation method, the complete set of wave functions for a general quadratic system has been given in terms of the classical solutions of the system [6,11], and the fact that wave functions are described by the classical solutions can be clearly understood from the path-integral approach for this system [6]. On the other hand, it turns out that the unitary transformation for a time-dependent quadratic system can be used for the same quadratic system with an inverse-square potential to give the wave functions [11]. Indeed, su(1,1) symmetry has been noticed in the model of the inverse-square potential [12], and the symmetry has been used to find the stationary wave functions for the case of a constant Hamiltonian [13], which would imply that a unitary transformation method may be applicable for general time-dependent SU(1,1) systems.

In this paper, we will consider the system which is described by Hamiltonian

$$H=\hbar[A_0(t)K_0+A_1(t)K_1+4a(t)K_2]+\beta(t), \tag{1}$$

where K_0,K_1, and K_2, satisfying commutation relations

$$[K_1,K_2]=-iK_0,\ \ [K_2,K_0]=iK_1,\ \ [K_0,K_1]=iK_2, \tag{2}$$

are the Hermitian generators of the SU(1,1) group, and $A_0(t),A_1(t),a(t),\beta(t)$ are real functions of time t, with $A_0(t)\neq A_1(t)$. This system has long been considered [2,14,15] particularly for applications in quantum optics, and it has been suggested that solutions of a classical equation of motion might be used in describing the wave functions [16]. Since $\beta(t)$ can be understood as a result of a simple unitary transformation which does not depend on the generators (see, e.g., Ref. [6]), from now on we will take $\beta(t)=0$. As an extension of the unitary relation in the quadratic systems [7,11], we will give the unitary transformation which relates the system of H and the system described by

$$H_0=2\hbar w_cK_0, \tag{3}$$

where w_c is a positive constant. The unitary transformation is described by the classical solutions of a time-dependent (harmonic) oscillator. With a choice of the realizations of the generators in terms of the canonical coordinates, the relation we will give becomes the known one of the quadratic systems [11]; Due to the nonuniqueness in realizing the generators, however, the relation in SU(1,1) is more general than that in the quadratic system, even for the representations of SU(1,1), which correspond to the quadratic systems.

In the following section, we will give the unitary relation between the systems of H and of H_0, and the relation will be discussed in some explicit realizations. In Sec. III, making use of the unitary relation, the wave functions satisfying the Schrödinger equation will be given in terms of the matrix elements of a finite group transformation (Bargmann function) which, in turn, will be determined by the classical solutions of a (harmonic) oscillator. In Sec. IV, the representations that correspond to harmonic-oscillator systems will be studied, and other expressions of the Bargmann function for these representations will be given, which generalizes the known results on transition probabilities [7]. It will be fur-

*Permanent address: Department of Physics, Sunchon National University, Suncheon 540-742, Korea.

ther shown that the wave functions of the system of H_0, obtained through the unitary transformation, can be written in a simple form. In Sec. V, the wave functions of the quadratic system with an inverse-square interaction will be studied, while the set of the wave functions gives a representation space of the positive discrete series $D^+(k)$ which is one of the unitary irreducible representations (UIRs) of the SU(1,1) group. It will be shown that the wave functions of the quadratic system with an inverse-square potential could be used to find the Bargmann function of $D^+(k)$ through an integral. The last section will be devoted to the discussions, and an appendix will be added to reveal the equivalent expressions of the Bargmann function.

II. A UNITARY RELATION

For the description of the unitary relation, we introduce $M(t)$ as

$$M(t)=\frac{2w_c}{A_0(t)-A_1(t)} \tag{4}$$

and $u(t),v(t)$ as the two real, linearly independent solutions of the second-order differential equation:

$$\ddot{y}+\frac{\dot{M}(t)}{M(t)}\dot{y}+\left[\frac{1}{4}(A_0^2-A_1^2)-4a^2+\frac{2}{M}\frac{d}{dt}(Ma)\right]y=0. \tag{5}$$

For $\frac{1}{4}(A_0^2-A_1^2)-4a^2+(2/M)(d/dt)(Ma)>0$, this is an equation of motion of a generalized harmonic oscillator [6]. By defining $\rho(t)$ and time constant Ω, which are positive, as

$$\rho(t)=\sqrt{u^2(t)+v^2(t)}, \tag{6}$$

$$\Omega=M(t)[u(t)\dot{v}(t)-\dot{u}(t)v(t)] \tag{7}$$

and a real function of t, $\tau(t)$, through relation

$$e^{i\tau}=\frac{u+iv}{\rho}, \tag{8}$$

one may find that unitary operator

$$\begin{aligned}U=&\exp\left[i\frac{M}{2w_c}\left(\frac{\dot{\rho}}{\rho}+2a\right)(e^{2iw_ct}K_++e^{-2iw_ct}K_-+2K_0)\right]\\&\times\exp\left[\ln\left(\sqrt{\frac{w_c}{\Omega}}\rho\right)(e^{2iw_ct}K_+-e^{-2iw_ct}K_-)\right]\\&\times\exp[2i(w_ct-\tau)K_0]\end{aligned} \tag{9}$$

satisfies relation

$$U\left(-i\hbar\frac{\partial}{\partial t}+H_0\right)U^\dagger=-i\hbar\frac{\partial}{\partial t}+H. \tag{10}$$

In Eq. (9), K_+ and K_- are defined as

$$K_+=e^{-2iw_ct}(K_1+iK_2),\quad K_-=K_+^\dagger, \tag{11}$$

so that

$$\frac{d}{dt}(e^{2iw_ct}K_+)=0,\quad \frac{d}{dt}(e^{-2iw_ct}K_-)=0. \tag{12}$$

Generators K_0, K_+, and K_- then satisfy commutation relations

$$[K_0,K_\pm]=\pm K_\pm,\quad [K_+,K_-]=-2K_0. \tag{13}$$

By making use of the commutation relations in Eq. (13), with the fact that

$$\frac{d}{dt}(M\dot{\rho})-\frac{\Omega^2}{M\rho^3}+M\left[\frac{1}{4}(A_0^2-A_1^2)-4a^2+\frac{2}{M}\frac{d}{dt}(Ma)\right]\rho=0, \tag{14}$$

one can explicitly verify the relation of Eq. (10) [17].

The Casimir operation C,

$$C=K_0^2-K_1^2-K_2^2=K_0^2-\frac{1}{2}(K_+K_-+K_-K_+), \tag{15}$$

is used in characterizing the UIRs of the SU(1,1) group, which are all infinite dimensional. If we reparametrize the eigenvalues of C as $k(k-1)$, it has been known that, for both cases of $k=1/4$ and $k=3/4$, the su(1,1) algebra can be realized by the operators of a quadratic system; If L_0, L_1, and L_2 are written as

$$L_0=\frac{1}{4\hbar}\left(\frac{p^2}{w_c}+w_cx^2\right),\quad L_1=\frac{1}{4\hbar}\left(-\frac{p^2}{w_c}+w_cx^2\right),$$

$$L_2=-\frac{1}{4\hbar}(xp+px), \tag{16}$$

with commutation relation $[x,p]=i\hbar$, one can find that $\{L_0,L_1,L_2\}$ can be a basis of the su(1,1) algebra with $C=-(3/16)I$. If this expression of the generators of the SU(1,1) group is plugged into Eqs. (9) and (10), one can find relation

$$\begin{aligned}&U_L\left(-i\hbar\frac{\partial}{\partial t}+\frac{1}{2}(p^2+w_0^2x^2)\right)U_L^\dagger\\&=-i\hbar\frac{\partial}{\partial t}+\frac{p^2}{2M(t)}+\frac{M(t)}{8}[A_0^2(t)-A_1^2(t)]x^2\\&\quad-a(t)[xp+px],\end{aligned} \tag{17}$$

with

$$\begin{aligned}U_L=&\exp\left[i\frac{M}{2\hbar}\left(\frac{\dot{\rho}}{\rho}+2a\right)x^2\right]\exp\left[-\frac{i}{2\hbar}\ln\left(\sqrt{\frac{w_c}{\Omega}}\rho\right)\right.\\&\left.\times(xp+px)\right]\exp\left[\frac{i}{2\hbar}\left(t-\frac{\tau}{w_c}\right)(p^2+w_c^2x^2)\right].\end{aligned} \tag{18}$$

For $\frac{1}{4}(A_0^2-A_1^2)-4a^2+(2/M)(d/dt)(Ma)>0$, the relation of Eq. (17) becomes the relation between a general quadratic system and a simple harmonic oscillator [11,7]. For $\frac{1}{4}(A_0^2-A_1^2)-4a^2+(2/M)(d/dt)(Ma)\leqslant 0$, one may find that the relation in Eq. (17) is true, though, in these cases, U_L may not be useful in finding wave functions for a general quadratic system, which are localized for all time t. Since L_i and K_i share the same algebraic structure, proving Eq. (17) constitutes a proof of the general relation of Eq. (10).

It should be, however, mentioned that, even for the quadratic systems of $C=-(3/16)I$, Eq. (10) is more general than Eq. (17), as much as the realization of the algebra is not unique. For example, generators $\tilde{L}_0$, $\tilde{L}_1$, and $\tilde{L}_2$ of the SU(1,1) group can be realized as

$$\tilde{L}_0=L_0, \quad \tilde{L}_1=-L_1, \quad \tilde{L}_2=-L_2. \tag{19}$$

In this realization, Eq. (10) is written as

$$\begin{aligned}U_{\tilde{L}}\left(-i\hbar\frac{\partial}{\partial t}+\frac{1}{2}(p^2+w_c^2x^2)\right)U_{\tilde{L}}^{\dagger}\\=-i\hbar\frac{\partial}{\partial t}+\frac{M}{8w_c^2}[A_0^2(t)-A_1^2(t)]p^2+\frac{w_c^2}{2M(t)}x^2\\+a(t)[xp+px],\end{aligned} \tag{20}$$

with

$$\begin{aligned}U_{\tilde{L}}=\exp\left[i\frac{M}{2\hbar w_c^2}\left(\frac{\dot{\rho}}{\rho}+2a\right)p^2\right]\exp\left[\frac{i}{2\hbar}\ln\left(\sqrt{\frac{w_c}{\Omega}}\rho\right)\right.\\\left.\times(xp+px)\right]\exp\left[\frac{i}{2\hbar}\left(t-\frac{\tau}{w_c}\right)(p^2+w_0^2x^2)\right].\end{aligned} \tag{21}$$

III. WAVE FUNCTIONS OF THE SU(1, 1) SYSTEMS

Making use of the Baker-Campbell-Hausdorff (or, disentanglement) formula (see, e.g., Ref. [3]) with the commutation relations in Eq. (13), one can find that operator U is written as

$$U=(e^{\xi K_+}e^{\gamma K_0}e^{-\bar{\xi}K_-})e^{i\varphi K_0}, \tag{22}$$

where

$$\xi=\frac{-\frac{\Omega}{\rho}+w_c\rho+iM(\dot{\rho}+2a\rho)}{\frac{\Omega}{\rho}+w_c\rho-iM(\dot{\rho}+2a\rho)}e^{2iw_ct}, \tag{23}$$

$$\gamma=\ln(1+|\xi|^2), \tag{24}$$

$$\varphi=2(w_ct-\tau)-i\ln\frac{\frac{\Omega}{\rho}+w_c\rho+iM(\dot{\rho}+2a\rho)}{\frac{\Omega}{\rho}+w_c\rho-iM(\dot{\rho}+2a\rho)}. \tag{25}$$

In Eq. (22), $\bar{\xi}$ denotes the complex conjugate of ξ. Element g of SU(1,1) may be written in the form

$$g(\alpha,\beta)=\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix}, \quad |\alpha|^2-|\beta|^2=1. \tag{26}$$

Making use of the realization of generators

$$K_0=\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}, \quad K_+=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad K_-=\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix}, \tag{27}$$

one can find that $g(\alpha,\beta)$ is parametrized as

$$\begin{aligned}\alpha=\frac{e^{i\varphi/2}}{\sqrt{1-|\xi|^2}}=\frac{e^{i(w_ct-\tau)}}{2}\left[\frac{1}{\rho}\sqrt{\frac{\Omega}{w_c}}+\rho\sqrt{\frac{w_c}{\Omega}}\right.\\\left.+\frac{iM}{\sqrt{w_c\Omega}}(\dot{\rho}+2a\rho)\right],\end{aligned} \tag{28}$$

$$\begin{aligned}\beta=\frac{\xi e^{-i\varphi/2}}{\sqrt{1-|\xi|^2}}=\frac{e^{i(w_ct+\tau)}}{2}\left[-\frac{1}{\rho}\sqrt{\frac{\Omega}{w_c}}+\rho\sqrt{\frac{w_c}{\Omega}}\right.\\\left.+\frac{iM}{\sqrt{w_c\Omega}}(\dot{\rho}+2a\rho)\right].\end{aligned} \tag{29}$$

Among the representations of SU(1, 1) group, we only consider the UIRs [18]. In a UIR, a basis state can be denoted as $|m,q_0,k\rangle$ satisfying

$$C(e^{-2i(m+q_0)t}|m,q_0,k\rangle)=k(k-1)(e^{-2i(m+q_0)t}|m,q_0,k\rangle), \tag{30}$$

$$K_0(e^{-2i(m+q_0)t}|m,q_0,k\rangle)=(m+q_0)(e^{-2i(m+q_0)t}|m,q_0,k\rangle). \tag{31}$$

There are four classes of UIRs, and m must be integer [2,18]. When a group element is acted on a basis state of a UIR, if we assume the completeness of the representation, the result should be written as a linear combination of the basis states of the UIR. Since $e^{-2i(m+q_0)}|m,q_0,k\rangle$ satisfies the Scrödinger eqation

$$i\hbar\frac{\partial}{\partial t}(e^{-2i(m+q_0)t}|m,q_0,k\rangle)=H_0(e^{-2i(m+q_0)t}|m,q_0,k\rangle), \tag{32}$$

from the unitary relation of Eq. (10), one can find that the state given by

$$|\Psi_{m,q_0,k}\rangle=U(e^{-2i(m+q_0)t}|m,q_0,k\rangle) \tag{33}$$

should satisfy the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi_{m,q_0,k}\rangle=H|\Psi_{m,q_0,k}\rangle, \tag{34}$$

while $|\Psi_{m,q_0,k}\rangle$ may be written as

$$\begin{aligned}|\Psi_{m,q_0,k}\rangle&=V(g)(e^{-2i(m+q_0)t}|m,q_0,k\rangle)\\&=\sum_{m'}V_{m',m}^{(k,q_0)}(\alpha,\beta)(e^{-2i(m'+q_0)t}|m',q_0,k\rangle).\end{aligned} \tag{35}$$

Though Eq. (35) is valid in any UIR, from now on we only consider the representation of positive discrete series $D^+(k)$, where $k>0$, $q_0=k$, (k is real), and $m=0,1,2,3,\ldots$. Since $q_0=k$ in this representation, q_0 will be omitted or replaced by k. A basis state of $D^+(k)$ could then be written as

$$(e^{-2i(m+k)t}|m,k\rangle)=\sqrt{\frac{\Gamma(2k)}{m!\Gamma(m+2k)}}(K_+)^m(e^{-2ikt}|0,k\rangle). \tag{36}$$

The explicit expression of $V^{(k)}_{m',m}(\alpha,\beta)$, the Bargmann functions, are known in this case, and $|\Psi_{m,k}\rangle$ is written as [2,18]

$$|\Psi_{m,k}\rangle=\sum_{m'=0}^{\infty}V^{(k)}_{m',m}(\alpha,\beta)(e^{-2i(m'+k)t}|m',q_0,k\rangle). \tag{37}$$

As shown in the Appendix, $V^{(k)}_{m',m}$ can be given as

$$\begin{aligned}V^{(k)}_{m',m}(\alpha,\beta)=&\frac{\Gamma(m+m'+2k)}{\sqrt{m!\Gamma(m+2k)(m')!\Gamma(m'+2k)}}\\&\times\bar{\alpha}^{-m-m'-2k}\beta^{m'}(-\bar{\beta})^m\\&\times F\left(-m,-m';-m-m'-2k+1;\frac{\alpha\bar{\alpha}}{\beta\bar{\beta}}\right),\end{aligned} \tag{38}$$

where $F(a,b;c;z)$ is the hypergeometric function [19].

IV. GENERALIZED HARMONIC OSCILLATORS

As is well known [13], the representation spaces of $k=1/4$ and 3/4 of $D^+(k)$ reduce to the Hilbert space of a simple harmonic oscillator. $|m,1/4\rangle$ in a representation space of SU(1,1) corresponds to $|2m\rangle$ of a simple harmonic oscillator, which is an eigenstate of Hamiltonian $H_0=2\hbar w_c L_0$ with energy eigenvalue $(2m+\frac{1}{2})\hbar w_c$. For $k=3/4$, $|n,3/4\rangle$ corresponds to eigenstate $|2m+1\rangle$ of H_0 with energy eigenvalue $(2m+1+\frac{1}{2})\hbar w_c$. Since the unitary relation of Eq. (10) reduces to the one for the quadratic systems if we choose a basis of the su(1,1) algebra, as in Eq. (16), for $\frac{1}{4}(A_0^2-A_1^2)-4a^2+(2/M)(d/dt)(Ma)\rangle 0$, one can find the explicit expressions of $e^{-i(2m+1/2)t}\langle x|U_L|m,\frac{1}{4}\rangle$ and $e^{-i(2m+3/2)t}\langle x|U_L|m,\frac{3}{4}\rangle$, as in Refs. [6,11].

In this section, $V^{(k)}_{m',m}$ will be studied in more detail for both cases of $k=1/4$ and 3/4, which will generalize the known results [7,13]. It will also be shown that, if the unitary relation becomes a relation between the same system described by H_0, operator U and (thus, the corresponding Bargmann function) can be written in a very simple form.

A. For a general quadratic system

Making use of transformation formula [19]

$$F(a,b;c;z)=(1-z)^{-a}F\left(a,c-b;c;\frac{z}{z-1}\right) \tag{39}$$

and Eq. (A2), one can find that $V^{(k)}_{m',m}(\alpha,\beta)$ may be written as

$$\begin{aligned}V^{(k)}_{m',m}(\alpha,\beta)=&\frac{\Gamma(m+m'+2k)}{\sqrt{m!\Gamma(m+2k)(m')!\Gamma(m'+2k)}}\\&\times(\bar{\alpha})^{-m-m'-2k}(\beta)^{m'-m}F(-m,-m-2k\\&+1;-m-m'-2k+1;\alpha\bar{\alpha})\\=&\frac{(-1)^m}{(2k-1)!}\sqrt{\frac{\Gamma(m+2k)\Gamma(m'+2k)}{m!(m')!}}\\&\times(\bar{\alpha})^{-m'-2k}\alpha^m\beta^{m'-m}\\&\times F\left(-m,m+2k;2k;\frac{1}{\alpha\bar{\alpha}}\right).\end{aligned} \tag{40}$$

For $k=1/4$, a relation between the hypergeometric function and the associate Legendre function with non-negative integers p,q,

$$F\left(-p,q+\frac{1}{2};\frac{1}{2};x^2\right)=(-1)^q\frac{(2p)!!}{(2q-1)!!}(1-x^2)^{(p-q)/2}P^{q-p}_{q+p}(x), \tag{41}$$

obtained from a more general one in Ref. [19], can be used to find a simpler expression of $V^{(1/4)}_{m',m}(\alpha,\beta)$, as

$$V^{(1/4)}_{m',m}(\alpha,\beta)=\frac{(-1)^{m+m'}}{\sqrt{\bar{\alpha}}}\sqrt{\frac{(2m)!}{(2m')!}}\left(\frac{\alpha}{\bar{\alpha}}\right)^{(m+m')/2}\times\left(\frac{\bar{\beta}}{\beta}\right)^{(m-m')/2}P^{m'-m}_{m'+m}\left(\frac{1}{\sqrt{\alpha\bar{\alpha}}}\right). \tag{42}$$

For $k=3/4$, formula

$$F\left(-p,q+\frac{3}{2};\frac{3}{2};x^2\right)=(-1)^q\frac{(2p)!!}{(2q+1)!!}\frac{1}{x}(1-x^2)^{(p-q)/2}P^{q-p}_{q+p+1}(x), \tag{43}$$

can be used to find

$$V^{(3/4)}_{m',m}(\alpha,\beta)=\frac{(-1)^{m+m'}}{\sqrt{\bar{\alpha}}}\sqrt{\frac{(2m+1)!}{(2m'+1)!}}\left(\frac{\alpha}{\bar{\alpha}}\right)^{(m+m'+1)/2}\times\left(\frac{\bar{\beta}}{\beta}\right)^{(m-m')/2}P^{m'-m}_{m'+m+1}\left(\frac{1}{\sqrt{\alpha\bar{\alpha}}}\right). \tag{44}$$

For the case of $M(t)=1$ and $a(t)=0$, with the choice of the basis in Eq. (16), H of Eq. (1) becomes the Hamiltonian of a harmonic oscillator of unit mass and time-dependent frequency ($w(t)=\sqrt{(w_c/2)[A_0(t)+A_1(t)]}$). In this case, one can find that Eqs. (42) and (44) exactly reproduce Eq. (83) of Ref. [7].

B. For a simple harmonic oscillator

For the case of $A_0(t)=2w_c$ and $A_1(t)=a(t)=0$, H of Eq. (1) becomes H_0 of Eq. (3), and the unitary relation given in Eq. (10) becomes a relation between the same system. In this case, ρ satisfies

$$\ddot{\rho}-\frac{\Omega^2}{\rho^3}+w_c^2\rho=0, \tag{45}$$

which makes it possible to analyze unitary operator U in more detail. Making use of this fact in Eq. (45), one can find that

$$\frac{d}{dt}\ln\xi=0 \tag{46}$$

and

$$\frac{d}{dt}\varphi=0. \tag{47}$$

Though Eqs. (46) and (47) are valid for general k, to be explicit, we first consider the realization given in Eq. (16). In this realization, by defining

$$a_c=\frac{1}{\sqrt{2\hbar w_c}}(w_cx+ip),\quad a_c^\dagger=\frac{1}{\sqrt{2\hbar w_c}}(w_cx-ip), \tag{48}$$

with a real constant φ_c and a complex constant ξ_c, one can find that U_L can be written as

$$U_L=\exp\left[\frac{\xi_c a_c^\dagger a_c^\dagger}{2}e^{-2iw_ct}\right]\exp\left[\frac{\ln(1+|\xi_c|^2)}{2}\left(a_c^\dagger a_c+\frac{1}{2}\right)\right]\times\exp\left[-\frac{\bar{\xi}_c a_c a_c}{2}e^{2iw_ct}\right]\exp\left[\frac{i\varphi_c}{2}\left(a_c^\dagger a_c+\frac{1}{2}\right)\right]. \tag{49}$$

U_L of $k=1/4$ or $3/4$, thus, shows that, if $a_c^\dagger a_c^\dagger$ (a_ca_c) is applied on a state to give a new state, phase factor e^{-2iw_ct} (e^{2iw_ct}) should be multiplied at the same time, which proves that

$$|\Psi_{m,k}\rangle=\sum_{m'=0}^{\infty}c_{m,m'}(e^{-2i(m'+k)t}|m',k\rangle), \tag{50}$$

where $c_{m,m'}$ is a constant.

For a general k with $A_0(t)=2w_c$, $A_1(t)=a(t)=0$, it may be easy to find that wave functions satisfying Eq. (34) should also be written as in Eq. (50). A wave function in a simple harmonic-oscillator system can be obtained by superposing the wave functions of $k=1/4$ and $k=3/4$, so a wave function in this system is written as

$$|\psi\rangle=\sum_{n=0}^{\infty}c_ne^{-i[n+(1/2)]w_ct}|n\rangle, \tag{51}$$

where c_n is a constant.

V. HARMONIC OSCILLATOR WITH AN INVERSE-SQUARE INTERACTION

It has been known that generators for $D^+(k)$ of SU(1,1) can be realized as [12]

$$\hbar L_0^k=\frac{1}{4w_c}\left(p^2+\frac{2g}{x^2}\right)+\frac{w_cx^2}{4},$$

$$\hbar L_1^k=-\frac{1}{4w_c}\left(p^2+\frac{2g}{x^2}\right)+\frac{w_cx^2}{4},$$

$$\hbar L_2^k=-\frac{1}{4}(xp+px), \tag{52}$$

where $g=2(k-\frac{1}{4})(k-\frac{3}{4})\hbar^2$. For the system of Hamiltonian $H_k=2\hbar w_cL_0^k$ on the half-line $x>0$, the wave functions are given as [20]

$$\phi_n^{s-}(k;x,t)=\left(\frac{4w_c}{\hbar}\right)^{1/4}\left(\frac{n!}{\Gamma(n+2k)}\right)^{1/2}e^{-2i(n+k)w_ct}\times\left(\frac{w_cx^2}{\hbar}\right)^{k-1/4}\exp\left(-\frac{w_cx^2}{2\hbar}\right)L_n^{2k-1}\left(\frac{w_cx^2}{\hbar}\right), \tag{53}$$

where L_n^α is the associated Laguerre polynomial defined through

$$x\frac{d^2L_n^\alpha}{dx^2}+(\alpha+1-x)\frac{dL_n^\alpha}{dx}+nL_n^\alpha(x)=0. \tag{54}$$

However,

$$\phi_n^{s-}\left(\frac{1}{4};x,t\right)=(-1)^n\left(\frac{2\sqrt{w_c}}{2^{2n}(2n)!\sqrt{\pi\hbar}}\right)^{1/2}e^{-2i[n+(1/4)]w_ct}\times\exp\left(-\frac{w_cx^2}{2\hbar}\right)H_n\left(\sqrt{\frac{w_c}{\hbar}}x\right)$$

implies that

$$\phi_n^s(k;x,t) \equiv e^{-2i(n+k)w_c t}\langle x|n,k\rangle = (-1)^n \phi_n^{s-}(k;x,t). \tag{55}$$

If we choose L_0^k, L_1^k, and L_2^k as the generators of SU(1,1), U becomes U_L, and $U_L\phi_n^{s-}(k;x,t)$ can be calculated as [11]

$$\begin{aligned}\phi_n^-(k;x,t) &= U_L \phi_n^{s-}(k;x,t)\\ &= \left(\frac{4\Omega}{\hbar\rho^2}\right)^{1/4}\left(\frac{n!}{\Gamma(n+2k)}\right)^{1/2} e^{-2i(n+k)\tau}\left(\frac{\Omega x^2}{\hbar\rho^2}\right)^{k-1/4}\\ &\quad\times \exp\left[-\frac{x^2}{2\hbar}\left(\frac{\Omega}{\rho^2} - iM\frac{\dot\rho}{\rho} - 2iMa\right)\right]\\ &\quad\times L_n^{2k-1}\left(\frac{\Omega x^2}{\hbar\rho^2}\right).\end{aligned} \tag{56}$$

Equations (37) and (55) then suggest that

$$\phi_n^-(k;x,t) = \sum_{m=0}^{\infty}(-1)^{n+m}V_{m,n}^k(\alpha,\beta)\phi_m^{s-}(k;x,t). \tag{57}$$

Making use of the integration formula [21]

$$\begin{aligned}\int_0^\infty e^{-bx}x^\alpha L_n^\alpha(\lambda x)L_m^\alpha(\mu x)dx &= \frac{\Gamma(m+n+\alpha+1)}{m!n!}\frac{(b-\lambda)^n(b-\mu)^m}{b^{m+n+\alpha+1}}\\ &\quad\times F\left(-m,-n;-m-n-\alpha;\frac{b(b-\lambda-\mu)}{(b-\lambda)(b-\mu)}\right),\end{aligned} \tag{58}$$

which is valid for $\mathrm{Re}\,\alpha > -1$ and $\mathrm{Re}\,b > 0$, one can indeed find that

$$\int_0^\infty \overline{\phi_m^{s-}}(k;x,t)\phi_n^-(k;x,t)dx = (-1)^{n+m}V_{m,n}^k(\alpha,\beta). \tag{59}$$

If we consider a system described by Hamiltonian $H_k(\epsilon) = [1/2(1+i\epsilon)][p^2 + (2g/x^2)] + (1+i\epsilon)(w_c^2x^2/2)$ with real positive ϵ, one can show that kernel (propagator) $K(x_b,t_b;x_a,t_a)$ of the system (see, e.g., Ref. [2]) reduces to the kernel of free particle of unit mass, in the limit of $t_b \to t_a + 0$ and $\epsilon \to 0$, which would imply completeness of set $\{\phi_n^{s-}(k;x,t)|n=0,1,2,\ldots\}$. Indeed, if we assume completeness, the fact in Eq. (59) amounts to a proof for the relation in Eq. (57).

One can also take L_0^k, $\tilde L_1^k$, $\tilde L_2$ as the generators of SU(1,1), while

$$\tilde L_1^k = -L_1^k,\quad \tilde L_2^k = -L_2^k. \tag{60}$$

Since $\tilde L_+^k = e^{-2iw_ct}(\tilde L_1^k + i\tilde L_2^k) = -L_+^k$, Eqs. (36) and (55) imply that

$$\phi_n^{s-}(k;x,t) = \sqrt{\frac{\Gamma(2k)}{m!\Gamma(m+2k)}}(\tilde L_+^k)^m\phi_0^{s-}(k;x,t). \tag{61}$$

If we use these generators in the unitary relation of Eq. (10), the relation and Eq. (37) imply that wave function

$$\phi_n(k;x,t) = \sum_{m=0}^{\infty} V_{m,n}^k(\alpha,\beta)\phi_m^{s-}(k;x,t) \tag{62}$$

satisfies the Schrödinger equation

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\phi_n(k;x,t) &= \left[\frac{M}{8w_c^2}[A_0^2(t) - A_1^2(t)]\right.\\ &\quad\left.\times\left(-\hbar^2\frac{\partial^2}{\partial x^2} + \frac{2g}{x^2}\right)\right]\phi_n(k;x,t)\\ &\quad+\left[\frac{w_c^2}{2M(t)}x^2 - ia(t)\hbar\right.\\ &\quad\left.\times\left(2x\frac{\partial}{\partial x} + 1\right)\right]\phi_n(k;x,t).\end{aligned} \tag{63}$$

VI. DISCUSSIONS

We have shown that for the systems of su(1,1) symmetry, there is a unitary relation between the system whose Hamiltonian is given as a linear combination of the generators of SU(1,1) group with time-dependent coefficients, and a system of the Hamiltonian which is simply proportional to the generator of the compact subgroup. The unitary relation is obtained through an extension of that between the general quadratic system and a simple harmonic oscillator. However, it should be mentioned that the relation is still formal, in the sense that the explicit form of the relation is given providing classical solutions $u(t)$ and $v(t)$ are known. For the case that M is constant and $a=0$, if $A_0^2 < A_1^2$, Eq. (5) becomes the equation of motion of an inverted harmonic oscillator, so that ρ diverges as time goes to infinity. If ρ diverges, for a quadratic system, the probability distribution of a wave function obtained through the unitary relation spreads out all over the space, while it may be possible that a meaningful system could be defined algebraically with diverging ρ.

Another point worthy of being mentioned is that the formal relation is true even for the case of negative $M(t)$. In fact, for a constant Hamiltonian H, the Schrödinger equation is invariant under the exchange of $H \leftrightarrow -H$ and $t \leftrightarrow -t$. In the case of $A_0(t) = -2w_c$ and $A_1(t) = a(t) = 0$ where $M(t) = -1$ and thus $H = -H_0$, one can take the classical solutions as $u(t) = \sin w_ct$, $v(t) = \cos w_ct$, so that $U = -\exp[2it(2w_cK_0)]$. By applying this U on a stationary state $e^{-2i(m+q_0)w_ct}|m,q_0,k\rangle$, one will have the state $-e^{2i(m+q_0)w_ct}|m,q_0,k\rangle$. This fact, therefore, suggests that the invariance may be included in the unitary relation.

It would be interesting to find similar unitary relations in the systems with other symmetries. The unitary relation for the SU(1,1) system has been found, based on the relation in harmonic oscillators which may be the simplest system with the symmetry. This implies that, if we find a unitary relation in a simple system of a symmetry, the relation could be generalized for other systems with the same symmetry. Though SU(1,1) is a noncompact group, the generalization itself would be possible for a compact group. In addition, it would be interesting to find the implications of the unitary relation in a system where the su(1,1) symmetry is a part of the symmetry of the system.

ACKNOWLEDGMENTS

The author thanks John Klauder and the Physics Department of the University of Florida for discussions and hospitality. This work was supported in part by the Korea Research Foundation Grant No. (KRF-2002-013-D00025) and by NSF Grant No. 1614503-12.

APPENDIX

In order to show that the Bargmann function given in Eq. (38) is equal to the standard expression [2,18], the hypergeometric function in the equation can be written as

$$\begin{aligned}
&F\left(-m,-m';-m-m'-2k+1;\frac{\alpha\bar{\alpha}}{\beta\bar{\beta}}\right)\\
&\quad=F\left(-m,-m';-m-m'-2k+1;1+\frac{1}{\beta\bar{\beta}}\right)\\
&\quad=\frac{\Gamma(m'+2k)\Gamma(m+2k)}{\Gamma(2k)\Gamma(m+m'+2k)}F\left(-m,-m';2k;-\frac{1}{\beta\bar{\beta}}\right),
\end{aligned}\tag{A1}$$

where the last equality is obtained through a formula given in Ref. [19]. With the Appell's symbol (a,s) defined for non-negative integers s by

$$(a,s)\equiv\left\{\begin{array}{cc}1 & (s=0)\\ a(a+1)\cdots(a+s-1) & (s>0)\end{array}\right\},$$

formula

$$F(-l,b;c;-y)=\frac{(b,l)}{(c,l)}y^{l}F(-l,1-l-c;1-l-b;-1/y),\tag{A2}$$

is known for a non-negative integer l in Ref. [18], which is valid as long as $(b,l)\neq 0$. For $m\geqslant m'$, Eq. (A2) can be used to find

$$\begin{aligned}
&F\left(-m,-m';-m-m'-2k+1;\frac{\alpha\bar{\alpha}}{\beta\bar{\beta}}\right)\\
&\quad=\frac{m!\Gamma(m+2k)}{(m-m')!\Gamma(m+m'+2k)}(-\beta\bar{\beta})^{-m'}\\
&\qquad\times F(-m',-m'-2k+1;1-m'+m;-\beta\bar{\beta}).
\end{aligned}\tag{A3}$$

For $m'\geqslant m$, Eq. (A2) can also be used to give

$$\begin{aligned}
&F\left(-m,-m';-m-m'-2k+1;\frac{\alpha\bar{\alpha}}{\beta\bar{\beta}}\right)\\
&\quad=F\left(-m',-m;-m-m'-2k+1;\frac{\alpha\bar{\alpha}}{\beta\bar{\beta}}\right)\\
&\quad=\frac{(m')!\Gamma(m'+2k)}{(m'-m)!\Gamma(m+m'+2k)}(-\beta\bar{\beta})^{-m}\\
&\qquad\times F(-m,-m-2k+1;1-m+m';-\beta\bar{\beta}).
\end{aligned}\tag{A4}$$

After some algebra with the above formulas, one can find that, the Bargmann function of $D^{+}(k)$ given in Eq. (38) is equivalent to the standard expression [2,18]: For $m'\geqslant m$,

$$\begin{aligned}
V^{(k)}_{m',m}(\alpha,\beta)=&A_{m',m}(\bar{\alpha})^{-m'-m-2k}(\beta)^{m'-m}\\
&\times F(-m,1-m-2k;1+m'-m;-\beta\bar{\beta})
\end{aligned}\tag{A5}$$

and for $m'\leqslant m$,

$$\begin{aligned}
V^{(k)}_{m',m}(\alpha,\beta)=&A_{m,m'}(\bar{\alpha})^{-m'-m-2k}(-\bar{\beta})^{m-m'}\\
&\times F(-m',1-m'-2k;1+m-m';-\beta\bar{\beta}),
\end{aligned}\tag{A6}$$

where

$$A_{m'm}=\frac{1}{(m'-m)!}\left(\frac{(m')!\Gamma(m'+2k)}{m!\Gamma(m+2k)}\right)^{1/2}.\tag{A7}$$

Since the hypergeometric series of any hypergeometric function used in this paper terminates, the series always converges. Making use of the expression of U given in Eq. (22) and the basis state in Eq. (36), the expression of Bargmann function [Eqs. (A5) and (A6)] can also be directly derived for $D^{+}(k)$ as

$$\begin{aligned}
V^{(k)}_{m',m}(\alpha,\beta)&=e^{-2i(m-m')t}\langle m',k|U|m,k\rangle\\
&=e^{i[(m+k)\varphi-2(m-m')t]}\\
&\quad\times\langle m',k|e^{\xi K_{+}}e^{\gamma K_{0}}e^{-\bar{\xi}K_{-}}|m,k\rangle\\
&=e^{i[(m+k)\varphi-2(m-m')t]}\\
&\quad\times\langle m',k|e^{-\bar{\xi}K_{-}}e^{-\gamma K_{0}}e^{\xi K_{+}}|m,k\rangle\\
&=\frac{e^{i(m+k)\varphi}\Gamma(2k)}{\sqrt{(m')!m!\Gamma(m+2k)\Gamma(m'+2k)}}\\
&\quad\times\sum_{p,q=0}^{\infty}\frac{(-\bar{\xi})^{p}\xi^{q}}{p!q!}\langle 0,k|(K_{-})^{m'+p}e^{-\gamma K_{0}}\\
&\quad\times(K_{+})^{q+m}|0,k\rangle,
\end{aligned}\tag{A8}$$

while the remaining procedures are straightforward.

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