Spin squeezing and pairwise entanglement for symmetric multiqubit states

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We show that spin squeezing implies pairwise entanglement for arbitrary symmetric multiqubit states. If the squeezing parameter is less than or equal to 1, we demonstrate a quantitative relation between the squeezing parameter and the concurrence for the even and odd states. We prove that the even states generated from the initial state with all qubits being spin down, via the one-axis twisting Hamiltonian, are spin squeezed if and only if they are pairwise entangled. For the states generated via the one-axis twisting Hamiltonian with an external transverse field for any number of qubits greater than 1 or via the two-axis countertwisting Hamiltonian for any even number of qubits, the numerical results suggest that such states are spin squeezed if and only if they are pairwise entangled.

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I. INTRODUCTION

The spin squeezed states [1–20] are quantum correlated states with reduced fluctuations in one of the collective spin components, with possible applications in atomic interferometers and high-precision atomic clocks. It is found that spin squeezing is closely related to and implies quantum entanglement [21–23]. As there are various kinds of entanglement, a question naturally arises: what kind of entanglement does spin squeezing imply? Recently, it has been found that, for a two-qubit symmetric state, spin squeezing is equivalent to its bipartite entanglement [24], i.e., spin squeezing implies bipartite entanglement and vice versa. Here, we generalize the above result to the multiqubit case, and study relationships between spin squeezing and quantum entanglement.

Specifically, we first show that spin squeezing implies pairwise entanglement for arbitrary symmetric multiqubit states. If the squeezing parameter $\xi^2 \leqslant 1$ (defined below), we give a quantitative relation between the squeezing parameter and the concurrence [25] for the even and odd states, where the concurrence is a measure of the degree of two-qubit entanglement and even (odd) states refer to those where only even (odd) excitations contribute. We further consider the multiqubit states dynamically generated from the initial state with all qubits being spin down via (i) the one-axis twisting Hamiltonian [1,26], (ii) the one-axis twisting Hamiltonian with an external transverse field [27], and (iii) the two-axis countertwisting Hamiltonian [1]. We prove that the states generated via the first Hamiltonian are spin squeezed if and only if they are pairwise entangled. For the states generated via the second Hamiltonian and third Hamiltonian with even number of qubits, numerical results for the squeezing parameter and concurrence suggest that the spin squeezing implies pairwise entanglement and vice versa.

II. SPIN SQUEEZING AND PAIRWISE ENTANGLEMENT

A collection of N qubits is represented by the collective operators

$$S_\alpha = \sum_{i=1}^{N} \frac{\sigma_{i\alpha}}{2}, \quad \alpha \in \{x,y,z\}, \tag{1}$$

where $\sigma_{i\alpha}$ are the Pauli operators for the ith qubit. The collective operators satisfy the usual angular-momentum commutation relations. Following Kitagawa and Ueda's criterion of spin squeezing, we introduce the spin squeezing parameter [1]

$$\xi^2 = \frac{2(\Delta S_{\vec{n}_\perp})^2}{J} = \frac{4(\Delta S_{\vec{n}_\perp})^2}{N}. \tag{2}$$

Here the subscript $\vec{n}_\perp$ refers to an axis perpendicular to the mean spin $\langle \vec{S} \rangle$, where the minimal value of the variance $(\Delta S)^2$ is obtained, $J = N/2$, and $S_{\vec{n}_\perp} = \vec{S} \cdot \vec{n}_\perp$. The inequality $\xi^2 < 1$ indicates that the system is spin squeezed.

To find the relation between spin squeezing and quantum entanglement, we first give the following lemma.

Lemma 1. For a symmetric separable state of N qubits, the correlation function $\langle \sigma_{i\vec{n}_\perp} \otimes \sigma_{j\vec{n}_\perp} \rangle \geqslant 0$, where i and j can take any values from 1 to N as long as they are different, and $\sigma_{i\vec{n}_\perp} = \vec{\sigma}_i \cdot \vec{n}_\perp$.

Proof. We first note that the expectation values $\langle \sigma_{i\vec{n}_\perp} \rangle$ and the correlation function $\langle \sigma_{i\vec{n}_\perp} \otimes \sigma_{j\vec{n}_\perp} \rangle \; \forall i \neq j$ are independent of indices due to the exchange symmetry. The symmetric separable state is given by

$$\rho_{\text{sep}} = \sum_k p_k \rho^{(k)} \otimes \rho^{(k)} \otimes \cdots \otimes \rho^{(k)} \tag{3}$$

with $\Sigma_k p_k = 1$. The correlation function $\langle \sigma_{i\vec{n}_\perp} \otimes \sigma_{j\vec{n}_\perp} \rangle$ over the separable state can be obtained from the two-qubit reduced density matrix

$$\rho_{ij} = \text{Tr}_{\{1,2,\ldots,N\} \setminus \{i,j\}}(\rho_{\text{sep}}) = \sum_k p_k \rho^{(k)} \otimes \rho^{(k)}, \tag{4}$$

yielding

$$\langle \sigma_{i\vec{n}_\perp} \otimes \sigma_{j\vec{n}_\perp} \rangle = \sum_k p_k \mathrm{Tr}_{ij}[(\rho^{(k)} \otimes \rho^{(k)})(\sigma_{i\vec{n}_\perp} \otimes \sigma_{j\vec{n}_\perp})]$$

$$= \sum_k p_k \langle \sigma^{(k)}_{i\vec{n}_\perp} \rangle \langle \sigma^{(k)}_{j\vec{n}_\perp} \rangle$$

$$= \sum_k p_k \langle \sigma^{(k)}_{i\vec{n}_\perp} \rangle^2 \geqslant 0. \tag{5}$$

■

From Lemma 1, we immediately have the following proposition.

Proposition 1. For an arbitrary symmetric multiqubit state, spin squeezing implies pairwise entanglement.

Proof. Due to the exchange symmetry, we may write the expectation value $\langle S^2_{\vec{n}_\perp} \rangle$ as

$$\langle S^2_{\vec{n}_\perp} \rangle = \frac{1}{4}[N + N(N-1)\langle \sigma_{i\vec{n}_\perp} \otimes \sigma_{j\vec{n}_\perp} \rangle]. \tag{6}$$

Substituting the above equation into Eq. (2) leads to

$$\xi^2 = \frac{4\langle S^2_{\vec{n}_\perp} \rangle}{N} = 1 + (N-1)\langle \sigma_{i\vec{n}_\perp} \otimes \sigma_{j\vec{n}_\perp} \rangle. \tag{7}$$

The above equation shows that spin squeezing is equivalent to the negative pairwise correlation ($\langle \sigma_{i\vec{n}_\perp} \otimes \sigma_{j\vec{n}_\perp} \rangle < 0$) [24]. This equivalence relation and the above lemma directly leads to the proposition. ■

Having shown the close relation between spin squeezing and pairwise entanglement, we now proceed to give a quantitative relation between the squeezing parameter and the concurrence [25]. We consider an even (odd) pure or mixed state ρ. The even (odd) state refers to the state for which only the Dicke states [28] $|n\rangle_J \equiv |J, -J+n\rangle$ with even (odd) n contribute. For examples, the pure even and odd states are given by

$$|\Psi\rangle_{\mathrm{e}} = \sum_{\text{even n}} c_n |n\rangle_J, \quad |\Psi\rangle_{\mathrm{o}} = \sum_{\text{odd n}} c_n |n\rangle_J, \tag{8}$$

respectively. As we will see in the following section, these states can be dynamically generated via a large class of Hamiltonians, and can also be obtained as a superposition of spin coherent states [20].

For the even and odd states, we immediately have the following property:

$$\langle S_\beta \rangle = \langle S_z S_\beta \rangle = \langle S_\beta S_z \rangle = 0, \quad \beta \in \{x, y\}. \tag{9}$$

Therefore, the mean spin is along the z direction. We assume that the mean spin satisfies $\langle S_z \rangle \neq 0$.

With the mean spin along the z direction, we have $\vec{n}_\perp = (\cos\theta, \sin\theta, 0)$, and thus the operator $S_{\vec{n}_\perp}$ can be written as

$$S_\theta = \vec{S} \cdot \vec{n}_\perp = \cos\theta S_x + \sin\theta S_y. \tag{10}$$

So, the squeezing parameter becomes

$$\xi^2 = \frac{4}{N} \min_\theta \langle S^2_\theta \rangle$$

$$= \frac{2}{N} \min_\theta [\langle S_x^2 + S_y^2 \rangle + \cos(2\theta)\langle S_x^2 - S_y^2 \rangle$$

$$+ \sin(2\theta)\langle [S_x, S_y]_+ \rangle]$$

$$= \frac{2}{N}[\langle S_x^2 + S_y^2 \rangle - \sqrt{\langle S_x^2 - S_y^2 \rangle^2 + \langle [S_x, S_y]_+ \rangle^2}]$$

$$= 1 + \frac{N}{2} - \frac{2}{N}[\langle S_z^2 \rangle + |\langle S_+^2 \rangle|], \tag{11}$$

where $S_\pm = S_x \pm i S_y$ are the ladder operators, and $[A,B]_+ = AB + BA$ is the anticommutator for operators A and B.

From Eq. (11), we see that the squeezing parameter is determined by a sum of two expectation values $\langle S_z^2 \rangle$ and $\langle S_+^2 \rangle$, and hence the calculations are greatly simplified. The larger the sum the deeper the spin squeezing. We also see that the squeezing parameter is invariant under the rotation along the z direction, i.e., the squeezing parameter for ρ is the same as that for $e^{-i\theta S_z} \rho e^{i\theta S_z}$.

Since the inequality $\langle S_z^2 \rangle \leqslant N^2/4$ always holds, we obtain a lower bound for the squeezing parameter

$$\xi^2 \geqslant 1 - \frac{2}{N} |\langle S_+^2 \rangle|. \tag{12}$$

From the above equation, we read that if $|\langle S_+^2 \rangle| = 0$, then the squeezing parameter $\xi^2 \geqslant 1$, which implies that a necessary condition for spin squeezing of the even and odd states is $|\langle S_+^2 \rangle| \neq 0$. A direct consequence of this necessary condition is that the Dicke state $|n\rangle_J$ exhibits no spin squeezing since $|\langle S_+^2 \rangle| = 0$. The associated squeezing parameter is given by

$$\xi^2 = 1 + \frac{2n(N-n)}{N} \geqslant 1. \tag{13}$$

However, the Dicke states can be pairwise entangled [29] even though they are not spin squeezed.

Spin squeezing is related to pairwise correlations, and negative pairwise correlation is equivalent to spin squeezing [24]. Then, for our even and odd states, we have the following proposition.

Proposition 2. A necessary and sufficient condition for spin squeezing of even and odd states is given by

$$|u| - y = |\langle \sigma_{i+} \otimes \sigma_{j+} \rangle| + \frac{\langle \sigma_{iz} \otimes \sigma_{jz} \rangle}{4} - \frac{1}{4} > 0, \tag{14}$$

where

$$u = \langle \sigma_{i+} \otimes \sigma_{j+} \rangle, \quad y = \frac{1}{4}(1 - \langle \sigma_{iz} \otimes \sigma_{jz} \rangle). \tag{15}$$

Proof. By considering the exchange symmetry, we have

$$\langle S_+^2 \rangle = N(N-1)u, \quad \langle S_z^2 \rangle = \frac{N^2}{4} - N(N-1)y. \tag{16}$$

Substituting the above equation into Eq. (11), we rewrite the squeezing parameter as

$$\xi^2=1-2(N-1)(|u|-y)=1-2(N-1)\left[|\langle\sigma_{i+}\otimes\sigma_{j+}\rangle| + \frac{\langle\sigma_{iz}\otimes\sigma_{jz}\rangle}{4}-\frac{1}{4}\right]. \tag{17}$$

We see that spin squeezing is determined by the two correlation functions $\langle\sigma_{iz}\otimes\sigma_{jz}\rangle$ and $\langle\sigma_{i+}\otimes\sigma_{j+}\rangle$. From Eq. (17), we obtain the proposition. ■

The two correlation functions $\langle\sigma_{iz}\otimes\sigma_{jz}\rangle$ and $\langle\sigma_{i+}\otimes\sigma_{j+}\rangle$ can be obtained from the reduced density matrix $\rho_{ij}=\mathrm{Tr}_{\{1,2,\dots,N\}\setminus\{i,j\}}(\rho)$. The reduced density matrix with the exchange symmetry is given by [29]

$$\rho_{ij}=\begin{pmatrix} v_+ & x_+^* & x_+^* & u^* \\ x_+ & y & y & x_-^* \\ x_+ & y & y & x_-^* \\ u & x_- & x_- & v_- \end{pmatrix} \tag{18}$$

in the standard basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$. The following lemma on the reduced density matrix is useful for later discussions.

Lemma 2. The matrix elements of ρ_{ij} can be determined by

$$v_\pm=\frac{N^2-2N+4\langle S_z^2\rangle\pm4\langle S_z\rangle(N-1)}{4N(N-1)},$$

$$x_\pm=\frac{(N-1)\langle S_+\rangle\pm\langle[S_+,S_z]_+\rangle}{2N(N-1)},$$

$$y=\frac{N^2-4\langle S_z^2\rangle}{4N(N-1)},\quad u=\frac{\langle S_+^2\rangle}{N(N-1)}. \tag{19}$$

Proof. The matrix elements can be represented by the expectation values of Pauli spin operators of the two qubits. $v_\pm$ and $x_\pm$ are given by

$$v_\pm=\frac{1}{4}(1\pm2\langle\sigma_{iz}\rangle+\langle\sigma_{iz}\otimes\sigma_{jz}\rangle),$$

$$x_\pm=\frac{1}{2}(\langle\sigma_{i+}\rangle\pm\langle\sigma_{i+}\otimes\sigma_{jz}\rangle), \tag{20}$$

and u and y are given by Eq. (15).

Due to the exchange symmetry, we have

$$\langle\sigma_{i\alpha}\rangle=\frac{2\langle S_\alpha\rangle}{N},\quad \langle\sigma_{i+}\rangle=\frac{\langle S_+\rangle}{N},\quad \langle\sigma_{i\alpha}\sigma_{j\alpha}\rangle=\frac{4\langle S_\alpha^2\rangle-N}{N(N-1)},$$

$$\langle\sigma_{ix}\sigma_{jy}\rangle=\frac{2\langle[S_x,S_y]_+\rangle}{N(N-1)},\quad \langle\sigma_{i+}\sigma_{jz}\rangle=\frac{\langle[S_+,S_z]_+\rangle}{N(N-1)}. \tag{21}$$

From Eqs. (20) and (21), we may thus express the matrix elements of ρ_{12} in terms of the expectation values of the collective operators. ■

The concurrence quantifying the entanglement of a pair of qubits can be calculated from the reduced density matrix. It is defined as [25]

$$\mathcal{C}=\lambda_1-\lambda_2-\lambda_3-\lambda_4, \tag{22}$$

where the quantities λ_i are the square roots of the eigenvalues in descending order of the matrix product

$$\varrho_{12}=\rho_{12}(\sigma_{1y}\otimes\sigma_{2y})\rho_{12}^*(\sigma_{1y}\otimes\sigma_{2y}). \tag{23}$$

In Eq. (23), ρ_{12}^* denotes the complex conjugate of ρ_{12}. Note that we did not use the max function in the above definition of the concurrence [25]. Therefore, the negative concurrence implies no entanglement here.

Both the squeezing parameter and the concurrence are determined by some correlation functions. So, they may be related to each other. The quantitative relation is given by

Proposition 3. If $\xi^2\leqslant1$ ($|u|\geqslant y$) for even and odd states, then

$$\xi^2=1-(N-1)\mathcal{C}. \tag{24}$$

Proof. For our state ρ, from Eq. (9) and Lemma 2, it is found that $x_\pm=0$. Therefore, the reduced density matrix becomes

$$\rho_{ij}=\begin{pmatrix} v_+ & 0 & 0 & u^* \\ 0 & y & y & 0 \\ 0 & y & y & 0 \\ u & 0 & 0 & v_- \end{pmatrix}. \tag{25}$$

For this reduced density matrix (25), the associated concurrence is given by [29]

$$\mathcal{C}=\begin{cases} 2(|u|-y) & \text{if}\quad 2y\leqslant\sqrt{v_+v_-}+|u|, \\ 2(y-\sqrt{v_+v_-}) & \text{if}\quad 2y>\sqrt{v_+v_-}+|u|. \end{cases} \tag{26}$$

If $|u|\geqslant y$, we have

$$2y\leqslant2|u|\leqslant|u|+\sqrt{v_+v_-}, \tag{27}$$

where we have used the fact

$$v_+v_-\geqslant|u|^2,\quad v_\pm\geqslant0. \tag{28}$$

Then, the concurrence (26) simplifies to

$$\mathcal{C}=2(|u|-y). \tag{29}$$

By comparing Eqs. (17) and (29), we obtain the proposition. ■

According to Proposition 3, we have

$$\mathcal{C}=\begin{cases} 0 & \text{if} \quad \xi^2=1 \\ \dfrac{1-\xi^2}{N-1}>0 & \text{if} \quad \xi^2<1, \end{cases} \tag{30}$$

from which we read that (i) if the squeezing parameter $\xi^2=1$ (no squeezing) for even and odd states, then the concurrence is zero (no entanglement); and (ii) if $\xi^2<1$, there is squeezing, then we have a one-to-one relation between the spin squeezing and the pairwise entanglement. However, for the case of $\xi^2>1$, the concurrence can be positive, and we cannot have $\mathcal{C}<0$ as exemplified earlier by the Dicke states (Dicke states are simplest cases of even and odd states). Although the squeezing parameter $\xi^2>1$ implies $\mathcal{C}<0$ is not valid in general, in the following section we will observe that for some even and odd states the squeezing parameter $\xi^2>1$ does imply $\mathcal{C}<0$, thereby establishing an equivalence between the pairwise entanglement and the spin squeezing.

III. HAMILTONIAN EVOLUTION

Now we consider a class of states dynamically generated from $|0\rangle_J$ via the following Hamiltonian ($\hbar=1$):

$$H=\mu S_x^2+\chi S_y^2+\gamma(S_xS_y+S_yS_x)+f(S_z), \tag{31}$$

with f being a function of S_z. When

$$\chi=\gamma=f(S_z)=0 \tag{32}$$

and

$$\mu=\chi=f(S_z)=0, \tag{33}$$

the Hamiltonian reduces to the one-axis twisting Hamiltonian [1,26] and the two-axis countertwisting Hamiltonian [1], respectively. When

$$\chi=\gamma=0, \quad f(S_z)=\Omega S_z, \tag{34}$$

Hamiltonian H reduces to the one considered in Refs. [27,30–34], namely, the one-axis twisting Hamiltonian with a transverse field. The one-axis twisting Hamiltonian [1] may be realized in various quantum systems including quantum optical systems [26], ion traps [35], quantum dots [36], cavity quantum electromagnetic dynamics [37], liquid-state nuclear magnetic resonance system [38], and Bose-Einstein condensates [11,21]. Experimentally, it has been implemented to produce four-qubit maximally entangled states in an ion trap [39].

The Hamiltonian exhibits a parity symmetry,

$$[e^{i\pi S_z},H]=[(-1)^{\mathcal{N}},H]=0, \tag{35}$$

where $\mathcal{N}=S_z+J$ is the "number operator" of the system and $(-1)^{\mathcal{N}}$ is the parity operator. In other words, the Hamiltonian is invariant under π rotation about the z axis. The symmetry can be easily seen from the transformation

$$e^{i\pi S_z}(S_x,S_y,S_z)e^{-i\pi S_z}=(-S_x,-S_y,S_z). \tag{36}$$

We assume that the initial density operator is chosen to be

$$\rho(0)=|0\rangle_J\langle 0|, \tag{37}$$

where $|0\rangle_J=|1\rangle\otimes|1\rangle\otimes\cdots\otimes|1\rangle$ and state $|1\rangle$ denotes the ground state of a qubit. The density operator at time t is then formally written as

$$\rho(t)=e^{-iHt}\rho(0)e^{iHt}. \tag{38}$$

The parity symmetry of H in Eq. (35) leads to the useful property given by Eq. (9). For example,

$$\begin{aligned}\langle S_x\rangle &= \mathrm{Tr}[S_xe^{-iHt}\rho(0)e^{iHt}] \\ &= \mathrm{Tr}[S_xe^{-iHt}e^{-i\pi S_z}\rho(0)e^{i\pi S_z}e^{iHt}] \\ &= \mathrm{Tr}[e^{i\pi S_z}S_xe^{-i\pi S_z}\rho(t)] \\ &= -\langle S_x\rangle. \end{aligned} \tag{39}$$

From another point of view, the state $\rho(t)$ is an even state since the Hamiltonian is quadratic in generators S_x and S_y and the initial state is an even state. Then, Eq. (9) follows directly. Since state $\rho(t)$ is an even state, we may apply the results in the preceding section. Next, we consider three representative model Hamiltonians for generating spin squeezing, which are special cases of Hamiltonian H.

A. One-axis twisting Hamiltonian

We first examine the well-established one-axis twisting model [1,26],

$$H_1=\mu S_x^2, \tag{40}$$

for which we have the following lemma.

Lemma 3. For the state dynamically generated from $|0\rangle_J$ via the one-axis twisting Hamiltonian, we always have $\xi^2\leqslant 1$.

Proof. From the results of Refs. [1,29], we have the following expectation values ($\overline{\mu}=2\mu t$):

$$\begin{aligned}\langle S_x^2\rangle &= N/4, \\ \langle S_y^2\rangle &= \frac{1}{8}[N^2+N-N(N-1)\cos^{N-2}\overline{\mu}], \\ \langle S_z^2\rangle &= \frac{1}{8}[N^2+N+N(N-1)\cos^{N-2}\overline{\mu}]. \end{aligned} \tag{41}$$

Then, we obtain a useful relation for density operator $\rho(t)$ at any time t,

$$\langle S_x^2-S_y^2\rangle=\langle S_z^2\rangle-N^2/4=-N(N-1)y, \tag{42}$$

where we have used Eq. (19). From the above equation, we obtain

$$|u|^2 = \frac{1}{N^2(N-1)^2}(\langle S_x^2 - S_y^2\rangle^2 + \langle [S_x, S_y]_+\rangle^2) \geq \frac{\langle S_x^2 - S_y^2\rangle^2}{N^2(N-1)^2} = y^2, \tag{43}$$

which implies $|u| \geq y$ at any time (note that $y \geq 0$). Therefore, the squeezing parameter always satisfies $\xi^2 \leq 1$. ■

Then, from Proposition 3 and Lemma 3, we obtain the following proposition.

Proposition 4. For the state dynamically generated from $|0\rangle_J$ via the one-axis twisting Hamiltonian, it is spin squeezed if and only if it is pairwise entangled. Hence spin squeezing and pairwise entanglement are equivalent for such a state.

At times for which $\mathcal{C}=0$, the state vector is either a product state or an N-partite ($N \geq 3$) maximally entangled state [35,39] which has no pairwise entanglement, and thus no spin squeezing.

B. One-axis twisting Hamiltonian with a transverse field

We consider the one-axis twisting model with an external transverse field described by the Hamiltonian

$$H_2 = \mu S_x^2 + \Omega S_z, \tag{44}$$

where $\Omega > 0$ is the strength of the transverse field. In general, this model cannot be solved analytically. Numerical results show that the squeezing parameter $\xi^2 \leq 1$ for the dynamically generated state $\exp(-iH_2t)|0\rangle_J$ [27]. We perform numerical calculations for N from 2 to 100 qubits, for different values of Ω and $\mu=1$, which indeed display the inequality $\xi^2 \leq 1$. Therefore, according to Proposition 3, these numerical results suggest that spin squeezing implies pairwise entanglement and vice versa for states generated from $|0\rangle_J$ via Hamitonian H_2. In the limit of $\Omega \to 0$, the result of this section, of course, reduces to that of the preceding one.

C. Two-axis countertwisting Hamiltonian

Finally, we examine the two-axis countertwisting model described by Hamiltonian

$$H_3 = \frac{\gamma}{2i}(S_+^2 - S_-^2). \tag{45}$$

For the state generated from $|0\rangle_J$ via Hamiltonian H_3, the squeezing parameter can be larger than 1. The numerical results for N from 2 to 100 and $\gamma=1$ suggest that the relation (24) holds for even N,

$$\mathcal{C} = \begin{cases} 0 & \text{if} \quad \xi^2 = 1, \\ \dfrac{1-\xi^2}{N-1} > 0 & \text{if} \quad \xi^2 < 1, \\ \dfrac{1-\xi^2}{N-1} < 0 & \text{if} \quad \xi^2 > 1. \end{cases} \tag{46}$$

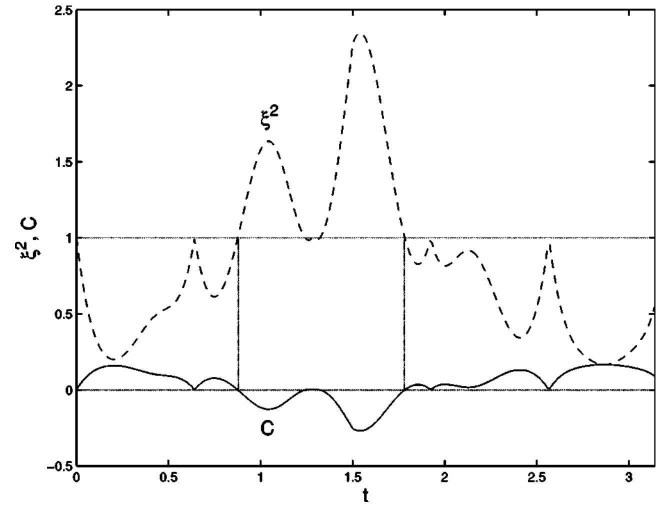

FIG. 1. The spin squeezing parameter and the concurrence against time t for six qubits. The parameter γ is chosen to be 1.

The above equation displays an equivalence relation between spin squeezing and pairwise entanglement for states generated from $|0\rangle_J$ via Hamiltonian H_3 with even N. The case of $N=6$ is demonstrated in Fig. 1, where the plots of the spin squeezing parameter and the concurrence against time t are shown. We make a conjecture that the spin squeezing and pairwise entanglement are equivalent for the states generated via the one-axis twisting Hamiltonian with an external transverse field for any number $N \geq 2$ or via the two-axis countertwisting Hamiltonian for any even number of qubits.

IV. CONCLUSIONS

In conclusion, we have shown that spin squeezing implies pairwise entanglement for arbitrary symmetric multiqubit states. We have identified a large class of multiqubit states, i.e., the even and odd states, for which the quantitative relation of the spin squeezing parameter and the concurrence is given. We have proved that spin squeezing implies pairwise entanglement and vice versa for the states generated from $|0\rangle_J$ via the one-axis twisting Hamiltonian. For the states dynamically generated from $|0\rangle_J$ via the one-axis twisting Hamiltonian with a transverse field for any $N \geq 2$ and the two-axis countertwisting Hamiltonian with any even N, numerical results suggest that spin squeezing implies pairwise entanglement and vice versa. As these three model Hamiltonians have been realized in many physical systems, the close relations between the spin squeezing and pairwise entanglement are meaningful and help to understand quantum correlations in these systems.

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