Comment on "Quantitative wave-particle duality in multibeam interferometers"

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In a recent paper, Dürr [Phys. Rev. A **64**, 042113 (2001)] proposed an interesting multibeam generalization of the quantitative formulation of interferometric wave-particle duality, discovered by Englert for two-beam interferometers. The proposed generalization is an inequality that relates a generalized measure of the fringe visibility to certain measures of the maximum amount of which-way knowledge that can be stored in a which-way detector. We construct an explicit example where, with three beams in a pure state, the scheme proposed by Dürr leads to the possibility of an ideal which-way detector that can achieve a better path discrimination at the same time as a better fringe visibility. In our opinion, this seems to be in contrast with the intuitive idea of complementarity, as it is implemented in the two-beams case, where an increase in path discrimination always implies a decrease of fringe visibility, if the beams and the detector are in pure states.

I. INTRODUCTION

As it is well known, the Einstein-Bohr debate on Bohr's *Principle of Complementarity*, has helped to shape the basic concepts of quantum mechanics. The central issue was the possibility of detecting, as proposed by Einstein, "which way" individual quantum systems ("quantons," for short) take, in double-slit interference experiments. However, this early discussion on the duality between fringe visibility and which-way information, as it is called today, was essentially semiclassical in nature. The history of the attempts of formulating such a duality, for the two-beams case, within the full framework of quantum mechanics, has been quite long, perhaps surprisingly long, and has found, it seems fair to say, a satisfactory conclusion in 1996 in a paper by Englert [2]. Following a suggestion present in the pioneering work of Wootters and Zurek [3], Englert was able to establish a complementarity relationship between the *distinguishability* $\mathcal{D}$, which gives a quantitative estimate of the ways, and the *visibility* $\mathcal{V}$, which measures the quality of the interference fringes:

$$\mathcal{D}^2 + \mathcal{V}^2 \leq 1. \tag{1}$$

An important feature of Eq. (1) is that it becomes an *equality* when the beams and the detector are prepared in a pure state; when this is the case, Eq. (1) implies that a larger visibility is necessarily accompanied by a smaller path distinguishability.

It is interesting to explore if an analogous form of interferometric duality can be formulated for more than two beams of interfering quantons. An important step toward the understanding of this question has been made by Dürr [1]; he argued that an appropriate multibeam generalization of the usual concept of fringe visibility $\mathcal{V}$ is provided by the (properly normalized) rms spread V of the fringes intensity from its mean value (Eq. (1.10) of Ref. [1]). By a corresponding

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generalization of the concept of path predictability $\mathcal{P}$, provided by the quantity P defined in Eq. (1.16) of Ref. [1], Dürr was able to derive an inequality analogous to that found by Greenberger and Ya Sin [4] for two beams:

$$P^2 + V^2 \leq 1. \tag{2}$$

Similar to Eq. (1), the above inequality becomes an equality if the beams are in a pure state, which ensures the existence of a general see-saw relation between V and P. Since V and P undoubtedly measure, respectively, wavelike and particle-like attributes of the interfering quantons, we thus think that Eq. (2) can be correctly interpreted as expressing a form of wave-particle duality in the multibeam case.

However interesting, an inequality like Eq. (2) does not convey yet the concept of wave-particle duality, as it is involved, say, in the famous ideal experiment with two moving slits, conceived by Einstein. Indeed, the quantity P above does not represent any real knowledge of the paths followed by individual quantons, but only constitutes some measure of one's *a priori* ability to predict them, based on unequal populations of the beams. The relevant schemes for a discussion of wave-particle duality as conceived by Einstein are those in which one actually tries to obtain which-way knowledge, by placing detectors along the paths of the quantons. In order to measure the amount of which-way information, which can be obtained by measuring the detector's observable W, after the passage of each quanton, Dürr defines the which-way knowledge $K(W)$ as a weighted average of the generalized predictabilities P of the sorted subensembles of quantons, for which a certain result of the measurement is obtained (Eqs. (2.3) and (2.4) of Ref. [1]). (Actually, Dürr also introduces an alternative measure I_{KW} of the which-way information, in Eq. (6.6) of Ref. [1]. For the sake of simplicity, in this Comment, we will refer only to the first one and we address the interested reader to Ref. [5], where an extensive discussion of the problem is given.) Then, the multibeam analog, D of Englert's path distinguishability $\mathcal{D}$ is defined as the maximum value of $K(W)$, over the set of all detector's observables (Eq. (2.11) of Ref. [1]). By using this definition, Dürr is able to prove an inequality analogous to Englert's Eq. (1):

$$D^2 + V^2 \leq 1. \tag{3}$$

This generalization of Eq. (1) to the multibeam case is an interesting relation, which can be tested, in principle, by experiments. However, there exists a difference between the two beams and the multibeam case. In fact, different from Eqs. (1) and (2), the inequality (3) cannot be saturated in general, even if the beams and the detector are prepared in pure states (in Ref. [5], we actually prove that in the multibeam case the above inequality can be saturated only if the visibility V is either equal to one or equal to zero). Therefore, one may conceive the possibility of designing two which-way detectors D_1 and D_2, such that $V_1 > V_2$, while, at the same time $D_1 > D_2$.

It is the purpose of this Comment to show that this possibility actually occurs, as will be seen in the following section, by an explicit example. In the final considerations that close this Comment, we argue that such a behavior rises doubts on the possibility of interpreting Eq. (3) as a statement of wave-particle duality.

II. A THREE-BEAM EXAMPLE

In this section, the problem announced in the preceding section is presented in an example with three beams of quantons in a pure state. So, we consider a three-beam interferometer with equally populated beams, described by the pure state

$$\rho = \frac{1}{3} \sum_{i,j=1}^{3} |\psi_i\rangle\langle\psi_j|. \tag{4}$$

If a detector, initially prepared in some pure initial state $|\chi_0\rangle$, is placed along the trajectories followed by the quantons, its interaction with the quantons will give rise to an entangled state $\rho_{b\&d}$ of the form

$$\rho_{b\&d} = \frac{1}{3} \sum_{i,j=1}^{3} |\chi_i\rangle\langle\chi_j| \otimes |\psi_i\rangle\langle\psi_j|, \tag{5}$$

where $|\chi_i\rangle$ are normalized, but not necessarily orthogonal, detector's states. Suppose, for simplicity, that the detector's Hilbert space $\mathcal{H}_D$ is two dimensional. In order to further specify the states $|\chi_i\rangle$, it is then convenient to use the Bloch parametrization, to represent rays of $\mathcal{H}_D$ by unit three-vectors, $\hat{n} = (n^x, n^y, n^z)$, via the map

$$\frac{1 + \hat{n} \cdot \vec{\sigma}}{2} = |\chi\rangle\langle\chi|, \tag{6}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is any representation of the Pauli matrices in $\mathcal{H}_D$. We shall denote by $|\hat{n}\rangle\langle\hat{n}|$ the ray corresponding to the vector $\hat{n}$. We require that the directions $\hat{n}_+, \hat{n}_-, \hat{n}_0$, associated with states $|\chi_i\rangle$, are coplanar, and such that $\hat{n}_+$ and $\hat{n}_-$ both form an angle θ with $\hat{n}_0$. We imagine that θ can be varied at will, by acting on the detector. By properly choosing the orientation of the coordinate
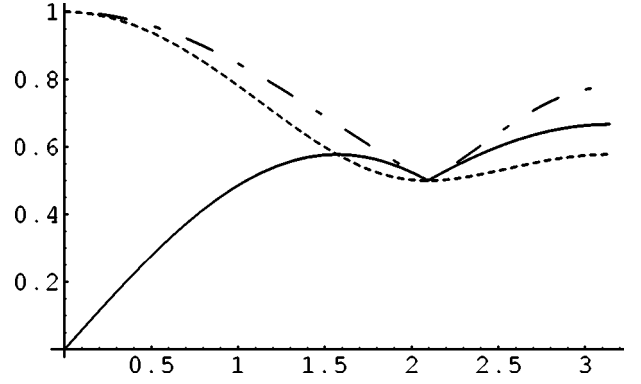

FIG. 1. Plots of the quantities D (solid line), V (dotted line), and $D^2 + V^2$ (dashed line), as functions of θ, in an ideal three-beam interference experiment.

axis, we can make the vector $\hat{n}_0$ coincide with the z axis and the vectors $\hat{n}_\pm$ lie in the x-z plane, such that

$$\hat{n}_0 \equiv (0,0,1), \quad \hat{n}_\pm = (\pm \sin\theta, 0, \cos\theta). \tag{7}$$

Upon using the well-known formula $|\langle\chi_i|\chi_j\rangle|^2 = (1 + \hat{n}_i \cdot \hat{n}_j)/2$ in Dürr's definition for the generalized fringe visibility V, Eq. (1.12) of Ref. [1], one gets the following expression for V, as a function of θ:

$$V(\theta) = \sqrt{\frac{1}{6} \sum_i \sum_{j \neq i} (1 + \hat{n}_i \cdot \hat{n}_j)} = \sqrt{\frac{1 + \cos\theta + \cos^2\theta}{3}}. \tag{8}$$

We notice that the value of the visibility is equal to one, for $\theta = 0$, and gradually decreases when θ is increased, until it reaches its minimum for $\theta = 2\pi/3$. Afterwards, it starts increasing and keeps on increasing until $\theta = \pi$ (see Fig. 1).

The next step is to evaluate the generalized path distinguishability D as a function of θ. This requires that we determine the observable W_{opt} in $\mathcal{H}_D$ that maximizes the multibeam generalization of the which-way knowledge $K(W)$. We briefly recall the definition of $K(W)$ proposed in Ref. [1]. Consider any detector's observable W, and let Π_l ($l = +, -$) the projector onto the subspace of $\mathcal{H}_D$, relative to the eigenvalue w_l. For any W, we let $\hat{m} \equiv (\sin\beta\cos\gamma, \sin\beta\sin\gamma, \cos\beta)$ being the unique unit three-vector such that

$$\Pi_+ = \frac{1 + \hat{m} \cdot \vec{\sigma}}{2}, \quad \Pi_- = \frac{1 - \hat{m} \cdot \vec{\sigma}}{2}. \tag{9}$$

Now we let $p_{i|l}$ the conditioned probability to find a quanton in beam i, provided that the measurement of W on the which-way detector gave the outcome w_l. According to Bayes' formula

$$p_{i|l} = \frac{\zeta_i q_{i|l}}{p_l}, \tag{10}$$

where $q_{i|l}$ is the probability of getting the outcome w_l, when the quanton occupies with certainty the beam i, while ζ_i are the populations of the beams, and p_l is the total *a priori* probability for obtaining the result w_l, $p_l = \Sigma_i \zeta_i q_{i|l}$. Recall that, in the above equation, we have to set $\zeta_i = 1/3$, because we are considering three equally populated beams. According to Ref. [1], the which-way knowledge $K(W)$ delivered by W is the weighted average $K(W) = \Sigma_l p_l K_l$ of the partial predictabilities K_l for the sorted subensembles of quantons:

$$K_l = \sqrt{\frac{n}{n-1} \sum_i \left(p_{i|l} - \frac{1}{n} \right)^2}. \qquad (11)$$

Now, using the well known formula

$$q_{i|\pm} = \langle \chi_i | \Pi_\pm | \chi_i \rangle = \frac{1 \pm \hat{m} \cdot \hat{n}_i}{2}, \qquad (12)$$

it is easy to verify that

$$K^2 = \frac{4}{9} \left[\cos^2\beta \sin^2\left(\frac{\theta}{2}\right) + 3\sin^2\beta \cos^2\gamma \cos^2\left(\frac{\theta}{2}\right) \right] \sin^2\left(\frac{\theta}{2}\right). \qquad (13)$$

For all values of θ, the which-way information is maximum if $\cos\gamma = \pm 1$, i.e., if the vector $\hat{m}_1$ lies in the same plane as the vectors $\hat{n}_i$. As for the optimal value of β, it depends on θ. For $0 \le \theta < 2\pi/3$, the best choice is $\beta = \pm \pi/2$, and so for the optimal observable W_{opt} we can take any operator such that

$$\Pi_\pm = \frac{1 \pm \sigma_x}{2} \quad \text{for} \quad 0 \le \theta < 2\pi/3, \qquad (14)$$

which delivers an amount of which-way knowledge D equal to

$$D(\theta) = \frac{1}{\sqrt{3}} \sin\theta \quad \text{for} \quad 0 \le \theta < 2/3 \, \pi. \qquad (15)$$

For larger values of θ, the maximum information is reached for $\beta = 0$ and then the optimal operators are those for which

$$\Pi_\pm = \frac{1 \pm \sigma_z}{2} \quad \text{for} \quad 2\pi/3 < \theta \le \pi, \qquad (16)$$

which deliver an amount $D(\theta)$ of which-way information equal to

$$D(\theta) = \frac{2}{3}\sin^2\left(\frac{\theta}{2}\right) \quad \text{for} \quad 2/3 \, \pi < \theta \le \pi. \qquad (17)$$

A plot of the quantities V, D, and $D^2 + V^2$ is shown in the Fig. 1. We see that something unexpected happens; while in the interval $0 \le \theta < \pi/2$, V decreases and D increases, as expected from the wave-particle duality, we see that in the interval $\pi/2 \le \theta \le \pi$, V and D decrease and increase simultaneously If we pick two values θ_1 and θ_2 in this region, we obtain two which-way detectors, which precisely realize the situation described at the end of the Introduction. It can also be seen from the Fig. 1 (the dashed line) that the sum $D^2 + V^2$ is significantly less than one, for most values of θ. We have checked that these problems persist if, rather than D, one uses the alternative measure of which-way information I_D, provided by Eq. (6.16) of Ref. [1], since it turns out that the optimal observable for I_D coincides with that relative to D, in the interval $0 \le \theta < 2/3 \, \pi$.

In the literature on the quantum detection problem, it has been argued that it is sometimes possible to achieve a larger amount of information on an unknown quantum state, by including an auxiliary quantum system, called ancilla, in the read out apparatus of a quantum detector [6,7]. This question has a negative answer in the example above, but we do not touch upon this problem here and we refer the interested reader to Ref. [5] for details.

III. DISCUSSION

In conclusion, the inequalities discovered by Dürr in his analysis of multibeam interferometers are very interesting, because they represent a set of testable relations between measurable quantities, which follow directly from the first principles of quantum mechanics. However, there is an important difference between the two-beam relation, Eq. (1), and its multibeam generalization, Eq. (3). As we pointed out above, the two-beam relation becomes an equality whenever the beams and the detector are prepared in pure states, and this entails the existence of a see-saw relation between $\mathcal{D}$ and $\mathcal{V}$. We think that this behavior expresses the intuitive idea of wave-particle duality, according to which " ... the more clearly we wish to observe the wave nature ... the more information we must give up about ... particle properties" [3]. In other words Eq. (1) conveys the basic idea of interferometric duality, for which, in an ideal interference experiment (namely, one involving pure states) in quantum mechanics, $\mathcal{D}$ and $\mathcal{V}$ exhibit a dual behavior. Any departure from this behavior, occurring for mixed states in the two-beam case, may be attributed to the presence of extra sources of uncertainty, in addition to the unavoidable one entailed by quantum mechanics.

In contrast, the inequality, Eq. (3), is almost never saturated, even for pure states [5]. So, while Eq. (3) sets an upper bound for either quantity, when the other takes a fixed value, it is not strong enough to prevent the behavior exhibited in the example presented in the preceding section. According to it, even in an ideal experiment with pure states, one can easily have cases when D and V both increase or decrease at the same time. In the light of this, it seems to us difficult to regard Eq. (3), as a statement of interferometric duality, similar to Englert's inequality for the two-beam case. It is our opinion that the issue of giving a complete quantum-mechanical formulation of the interferometric duality in multibeam experiments deserves further analysis.

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