Energy-band structure and intrinsic coherent properties in two weakly linked Bose-Einstein condensates

Wei-Dong Li,[1] Yunbo Zhang,[1,2] and J.-Q. Liang[1]
[1]*Department of Physics and Institute of Theoretical Physics, Shanxi University, Taiyuan 030006, China*
[2]*Department of Physics, University of Turku, FIN-20014 Turun yliopisto, Finland*
(Received 25 November 2002; published 13 June 2003)

The energy-band structure and energy splitting due to quantum tunneling in two weakly linked Bose-Einstein condensates were calculated by using the instanton method. The intrinsic coherent properties of Bose-Josephson junction (BJJ) were investigated in terms of energy splitting. For $E_C/E_J \ll 1$, the energy splitting is small and the system is globally phase coherent. In the opposite limit, $E_C/E_J \gg 1$, the energy splitting is large and the system becomes phase dissipated. Our results suggest that one should investigate the coherence phenomena of BJJ in proper condition such as $E_C/E_J \sim 1$.

DOI: 10.1103/PhysRevA.67.065601 PACS number(s): 03.75.Lm, 05.30.Jp, 64.60.My, 67.40.Fd

I. INTRODUCTION

Two weakly linked Bose-Einstein condensates (BECs) behave as the superconductor Josephson junction. This remarkable feature has been investigated both theoretically [1–7] and experimentally [8–12]. The existence of a Josephson current through a potential barrier between two superconductors or between two superfluids is a direct manifestation of macroscopic quantum coherence [13]. The experimental realization of Bose-Einstein condensation (BEC) of weakly interacting alkali atoms [8] has provided a route to study neutral superfluids in a controlled and tunable environment [9]. The possibility of loading a BEC in a one-dimensional periodic potential has allowed the observation of quantum phase effects on a macroscopic scale such as quantum interference [10], superfluidity on a local scale [11] and an oscillating atomic current in Josephson-junction arrays [12].

The analogy of the voltage-current characteristic in superconductor Josephson junction was proposed theoretically in Bose-Josephson junction (BJJ) [2–5]. The macroscopic BEC's coherence has been demonstrated by interference experiments [14], and the evidence of the coherent tunneling in an atomic array, related to the ''ac'' Josephson effect, has been reported [10]. A ''dc'' current-biased Bose-Josephson junction can be simulated by a tunneling barrier moving with constant velocity across the trap [3]. At a critical velocity of the barrier (proportional to the critical tunneling current), a sharp transition between the ''dc'' and ''ac'' Josephson regimes was predicted. Thus, two weakly linked condensates exhibit the analog of the resistively shunted superconductor Josephson junctions. The ''secondary'' quantum macroscopic effects in small capacitance Josephson junction had attracted much attention both from theorists and experimentalists for decades [15–18]. These phenomena manifest the quantum behavior of a Josephson junction as a macroscopic object, in contrast with such ''primary'' quantum macroscopic phenomena as the Josephson junction itself [15]. In these cases, quantum fluctuations of the phase difference φ across the junctions become important. This necessitates treating the phase as a quantum operator $\hat{\varphi}$, which is canonically conjugate to the operator $\hat{N}$. Novel macroscopic quantum phenomena, such as ''Bloch'' oscillation, had been reported in current-biased Josephson junctions [15]. To understand these effects and present a general picture of the low-temperature dynamics of Josephson junctions, a simple theory [16–18] has been suggested based on the extended coordinates, $\varphi \in [-\infty, +\infty]$.

More recently, the first experiment of BJJ has been realized in a purely magnetic double-well potential [24]. It will stimulate the further study on the BJJ. Then, can we investigate the secondary quantum macroscopic effects [15] in two weakly linked BECs? It was argued in Refs. [2,3,6] that the dynamics of the phase of this system can be mapped onto the sine-Gordon quantum mechanical Hamiltonian (1). In this paper, we calculated the energy-band structure, Bolch wave and the energy splitting of the Bose-Josephson junction due to the quantum tunneling using the instanton method [21]. As a simple application, we investigate the coherent properties of the Bose-Josephson junction. The results from this method agree exactly with the prediction of Stringari [6].

II. MODEL AND ITS INSTANTONS SOLUTION

All quantities describing the junction should be considered as operators rather than the classical variables. The operators corresponding to the main variables, the phase difference φ and the particle number N of the Bose-Josephson junction, satisfy the commutation relation: $[\varphi, N] = i$, so that φ and N are related by the Heisenberg uncertainty relation which was disregarded by the ''classical'' theory of the Bose-Josephson junction [2].

The idealization of two weakly linked BECs can be described as a two-mode bosonic system [2,22]. In the ''phase'' representation, the relevant quantum observables are the difference of phases and number of atoms between the two condensates in each trap. The Hamiltonian can be written in terms of a ''quantum pendulum'' (Mathieu) equation

$$\hat{H} = -\frac{E_C}{2}\frac{\partial^2}{\partial\phi^2} + E_J \cos\phi. \tag{1}$$

The ''charging energy'' E_C and the ''Josephson coupling energy'' E_J can be calculated as overlap integrals [2,3],

$$E_C=2g\int dr\,\Phi_1^4(r)=2g\int dr\,\Phi_2^4(r),$$

$$E_J=N\int dr\,\Phi_1^*(r)\bigg[-\frac{\hbar^2}{2m}\nabla^2+V_{ext}+\frac{gN}{2}[\Phi_1^2(r)+\Phi_2^2(r)]\bigg]\Phi_2(r), \quad (2)$$

with the one-body wave functions $\Phi_1(r),\Phi_2(r)$ localized in the trap 1,2; and $\int dr\,\Phi_1^*(r)\Phi_2(r)=0$, $\int dr\,\Phi_{1,2}^*(r)\Phi_{1,2}(r)=1$, $g=(4\pi\hbar^2a/m)$; a is the scattering length and m the atomic mass; $N=\hat{a}_1^\dagger\hat{a}_1+\hat{a}_2^\dagger\hat{a}_2$ is the total number of atoms.

For convenience, we modify the potential as $U_J(\phi)=E_J(1+\cos\phi)$. Then, the Hamiltonian $\hat{H}$ in Eq. (1) can be considered as one-dimensional quantum particles with mass $m_\phi=\hbar^2/E_C$ moving along the ϕ axis in sine-Gordon potential $U_J(\phi)$. Due to the translational symmetry $(\phi\to\phi+2\pi)$ of Hamiltonian (1), the set of the eigenfunctions should include the Bloch wave functions

$$\psi_n(\phi)=u_{n,q}(\phi)\exp(iq\phi),\quad n=0,1,2,\ldots,$$

$$u_{n,q}(\phi)=u_{n,q}(\phi+2\pi),\quad -\infty<q<\infty, \quad (3)$$

where q is an arbitrary (real) constant vector. Substitution of the wave function (3) in the Schrödinger equation leads immediately to the picture of band energy spectrum and relative well-known effects in solid-state theory. From Refs. [17,18], this is nothing but Mathieu equation, so that these functions can be readily calculated. Some of their asymptotic properties (in tight-binding limit and weak-binding limit) can be expressed analytically.

The energy-band structure and energy splitting for the sine-Gordon potential can be calculated alternatively by the instanton method [19,20]. The advantage of this nonperturbative method, as presented here, is that it gives not only a more accurate description of the tunneling phenomena but also a comprehensive physical understanding in the context of quantum-field theory.

The effective Lagrangian is

$$L=\frac{1}{2}m_\phi\left(\frac{d\phi}{dt}\right)^2-U_J(\phi). \quad (4)$$

The classical solution that extremizes the action is seen to satisfy the equation of motion

$$\frac{1}{2}m_\phi\left(\frac{d\phi_c}{d\tau}\right)^2+U_J(\phi_c)=-E_{cl}, \quad (5)$$

where the Wick rotation $\tau=it$ has taken the system into Euclidean time. Equation (5) can be regarded as the equation of motion of a pseudoparticle with the classical energy $E_{cl}\geqslant 0$, which is a constant of integration. With E_{cl} being confined to a region $0\leqslant E_{cl}\leqslant E_J$, the configuration ϕ_c becomes periodic such that $\phi_c(\tau+\mathcal{T})=\phi_c(\tau)$, which now corresponds to the periodic boundary condition in the space coordinate [19]. The classical solution is

$$\phi_c=2\arcsin[k\,\mathrm{sn}(\omega_0(\tau+\tau_0))], \quad (6)$$

where $\omega_0=\sqrt{E_JE_c}/\hbar$ is the classical plasma frequency [6], sn is the Jacobian elliptic function with modulus $k=\sqrt{1-E_{cl}/2E_J}$. The elliptic function $\mathrm{sn}(\omega_0(\tau+\tau_0))$ has a period $4\mathcal{K}(k)$ with $\mathcal{K}(k)$ the complete elliptic integral of first kind. For zero energy $E_{cl}=0$ $(k\to 1)$, the periodic solution reduces to the vacuum instanton configuration $\phi_c\to 2\arcsin\{\tanh[\omega_0(\tau+\tau_0)]\}$.

III. ENERGY-BAND STRUCTURE AND THE TRANSITION AMPLITUDE FOR QUANTUM TUNNELING

The sine-Gordon potential has an infinite number of degenerate vacua. Quantum tunneling between neighboring vacua leads to the level splitting, while the levels extend to bands due to the translational symmetry expressed by $U_J(\phi+2\pi)=U_J(\phi)$. In the narrow-band approximation one finds, for the energy, the expression

$$E=E_i+\sum_n J(R_m-R_n)e^{iq(R_m-R_n)}, \quad (7)$$

where E_i denotes the ith eigenvalue of the energy in each well for the harmonic-oscillator potential $U(\phi)=\frac{1}{2}(\phi-R_n)^2$, $R_n=2n\pi$ is the position of the nth minimum, $J(R_{n'}-R_n)\equiv\int\psi_i^*(\phi-R_{n'})[U_J(\phi)-U(\phi)]\psi_i(\phi-R_n)d\phi$ is the overlap integral, ψ_i is the eigenfunction corresponding to eigenvalue E_i, and q is the Floquet parameter associated with the Bloch wave function. If only the contribution from the nearest neighbors is taken into account, i.e., $J(R_{n'}-R_n)$ for $|n'-n|>1$ is taken to be zero, the energy-band formula reduces to

$$E=E_i+2J\cos(2\pi q). \quad (8)$$

The parameter J is just the level splitting resulting from quantum tunneling (the wave functions are periodic for $q=0$). We will consider the case of potential wells surrounded by very high potential barriers with correspondingly small tunneling contribution to the eigenvalues. They are almost those of degenerate harmonic oscillators, and in this asymptotic case, we are not concerned with the entire bands but only with their edges that correspond to alternately even and odd states. Then, we suppose $|i\rangle_R,|i\rangle_L$ are degenerate eigenstates in neighboring wells, respectively, with the same energy eigenvalue E_i such that $H^0|i\rangle_{R,L}=E_i|i\rangle_{R,L}$, where H^0 is the Hamiltonian of the harmonic oscillator as the zero-order approximation of the system. The degeneracy will be removed by the small tunneling effect which leads to the level splitting. The eigenstates of the Hamiltonian H become

$$|i\rangle_o=\frac{1}{\sqrt{2}}(|i\rangle_R-|i\rangle_L),\quad |i\rangle_e=\frac{1}{\sqrt{2}}(|i\rangle_R+|i\rangle_L), \quad (9)$$

with eigenvalues $E_i\pm\Delta E_i$, respectively. ΔE_i denotes the shift of one oscillator level. It is obvious that ${}_R\langle i|H-H^0|i\rangle_L=2J=\Delta E_i$. In the following, we calculate this energy shift ΔE_i as resulting from periodic instantons and instanton-antiinstanton pairs.

The amplitude for a transition from one well to its neighboring well at the energy E_i due to instanton tunneling can be written as

$$A_{+,-} = {}_{+}\langle E_i | e^{-2(\hat{H}T/\hbar)} | E_i \rangle_{-} \simeq \pm e^{-2(E_i/\hbar)T} \sinh\left(2\frac{\Delta E_i T}{\hbar} \right), \tag{10}$$

where we neglect overlap of the wave functions that dominate over either well. The amplitude (10) can also be calculated with the help of the path-integral method, $A_{+,-} = \int \psi^*_{E,+}(\phi_f) K(\phi_f, \tau_f; \phi_i, \tau_i) \psi_{E,-}(\phi_i) d\phi_f d\phi_i$, where the Feynman kernel is defined as usual by

$$K(\phi_f, \tau_f; \phi_i, \tau_i) = \int_{\phi_i}^{\phi_f} \mathcal{D}[\phi] e^{-S/\hbar}, \tag{11}$$

with $\phi_f \equiv \phi(\tau_f)$, $\phi_i \equiv \phi(\tau_i)$, and $\tau_f - \tau_i = 2T$. What we are interested in is this expression in the limits $\phi_i \to -a$, $\phi_f \to a$ ($\pm a$ are the turning points), namely, the tunneling propagator through one of the barriers. $S = \int_{\tau_i}^{\tau_f} [\frac{1}{2} m_\phi (d\phi/d\tau)^2 + U_J(\phi)] d\tau$ is the Euclidean action of the pseudoparticle and $\psi_{E_i,+}(\phi_i)$ [$\psi_{E_i,-}(\phi_i)$] is the wave function of the right- (left-) hand wells.

The functional integral $K(\phi_f, \tau_f; \phi_i, \tau_i)$ can be evaluated with the stationary method by expanding $\phi(\tau)$ about the classical trajectory $\phi_c(\tau)$ and thus we set $\phi(\tau) = \phi_c(\tau) + \chi(\tau)$, where $\chi(\tau)$ is the small fluctuation with boundary conditions $\chi(\tau_i) = \chi(\tau_f) = 0$. Substitution of $\phi(\tau)$ into Eq. (11) and keeping only terms containing $\chi(\tau)$ up to the one-loop approximation yields $K(\phi_f, \tau_f; \phi_i, \tau_i) = \exp[-S_c(\tau)/\hbar] I$, where $I = \int_{\chi(\tau_i)=0}^{\chi(\tau_f)=0} \mathcal{D}[\chi] e^{-\delta S/\hbar}$ is the fluctuation function integral with the fluctuation action $\delta S = \int_{\tau_i}^{\tau_f} \chi M \chi d\tau$, where $M = -m_\phi d^2/d\tau^2 + V''(\phi_c(\tau))$ is the second variational operator of the action. The classical action $S_c(\tau)$ is evaluated along the trajectory $\phi_c(\tau)$ so that

$$S_c(\tau) = \int_{\tau_i}^{\tau_f} \left[\frac{m_\phi}{2} \left(\frac{d\phi_c(\tau)}{d\tau} \right)^2 + V(\phi_c(\tau)) \right] d\tau$$

$$= 2ET + \frac{8E_J}{\omega_0} [\mathcal{E}(k) - k'^2 \mathcal{K}(k)], \tag{12}$$

where $\mathcal{E}(k)$ denotes the complete elliptic integral of the second kind and $k'^2 = 1 - k^2$. Following the standard procedure of the periodic instanton calculation in Refs. [19,20], the functional intergal I can be written as

$$I = \frac{1}{\sqrt{2\pi}} \left[N(\tau_i) N(\tau_f) \int_{\tau_i}^{\tau_f} \frac{d\tau}{N^2(\tau)} \right]^{-1/2},$$

where $N(\tau) = d\phi_c(\tau)/d\tau$, is the zero eigenmode of M. To obtain the desired results, the contributions from the infinite number of the instantons and antiinstanton pairs have to be taken into account. The total amplitude is found to be

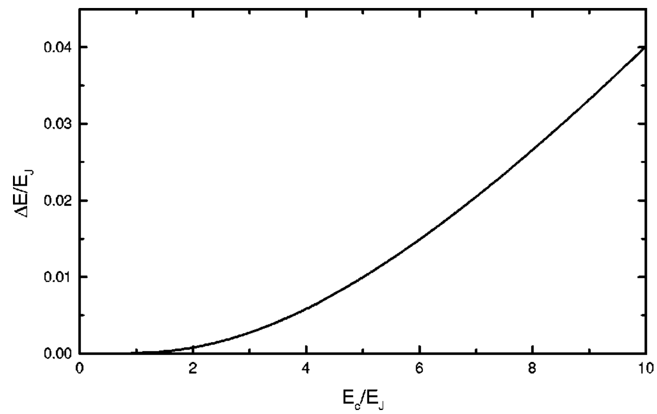

FIG. 1. Ground-state energy splitting as a function of the ratio E_C/E_J.

$$A_{+,-} = e^{-2(E_i/\hbar)T} \sinh\left(\frac{\omega_0 T}{2\mathcal{K}(k')} e^{-W/\hbar} \right), \tag{13}$$

with $W = (8E_J/\omega_0)[\mathcal{E}(k) - k'^2 \mathcal{K}(k)]$. Comparing Eqs. (13) and (10) leads to $\Delta E = [\hbar\omega_0/4\mathcal{K}(k')] \exp(-W/\hbar)$. Rescaling this formula in the unit of E_J, we arrive at our final result,

$$\frac{\Delta E}{E_J} = \sqrt{\frac{E_C}{E_J}} \frac{1}{4\mathcal{K}(k')} \exp\left[-8\sqrt{\frac{E_J}{E_C}} [\mathcal{E}(k) - k'^2 \mathcal{K}(k)] \right]. \tag{14}$$

Formula (14) shows that the widths of the energy bands are very sensitive to the dimensionless parameter of E_C/E_J, which is a critical parameter characterizing the dynamical properties of the system such as the coherence [6] and the transition [15]. A direct application of this energy splitting is to investigate the coherence properties of BJJ. Quantum coherence requires that the relative phase of the order parameter should be preserved over times of the order of Γ^{-1}, which is just the tunneling amplitude [23]. From the above calculation, we know that the energy splitting ΔE is proportional to Γ. Therefore, the energy splitting describes the coherent properties of the system.

Figure 1 shows the energy splitting $\Delta E/E_J$ as a function of the ratio E_C/E_J in the ground state ($E_{cl} = 0$). The figure shows that for values of E_J smaller than E_C the energy splitting is significantly increased, indicating the occurrence of a continuous transition to the phase dissipation (the number squeezed regime) [6,7]. In the limit $E_C/E_J \ll 1$, the system undergoes small oscillations around the equilibrium. In this limit, the correlation between the neighboring wells is small. One can regard the systems as a globally coherent object described by a unique order parameter. On the contrary $E_C/E_J \gg 1$, the behavior of the system is very different. The quantum fluctuation is enhanced due to the increasing of the tunneling between the neighboring wells in phase space, showing that the relative phase between two condensates is distributed in a random way. At the same time, the fluctuation of the relative number of atoms in two traps becomes smaller and smaller.

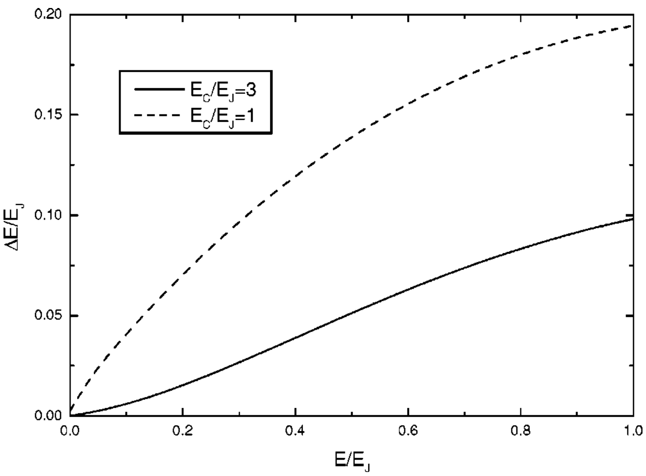

FIG. 2. Energy splitting as a function of the ratio E/E_J for $E_C/E_J=1$ (solid line), $E_C/E_J=3$ (dashed line).

It is interesting to investigate the properties of the systems under the excited states ($E_{cl}\neq 0$) which may be excited by the thermal fluctuations or other reasons. We plotted the energy splitting as a function of the dimensionless parameter E_{cl}/E_J for two different values of E_C/E_J in Fig. 2. One clearly observes that even if quantum effects are small the decoherence due to higher excited states may become important. It is interesting that the response of the system to the decoherence performs a total different behavior. States with small quantum effects are more sensitive to the decoherence fluctuations than those with relatively large vaules of E_C/E_J. The system with little higher E_C/E_J state should be a good candidate to investigate the quantum coherence phenomena, in $E_C/E_J\sim 1$. Because in this case, the system with a good coherence will preserve its coherence for a relative large range of the energy perturbation.

Energy spectrum and energy splitting due to quantum tunneling in BJJ have been calculated by means of instanton method. Based on this energy splitting formula, we also investigated the coherence property of BJJ. Our results agree exactly with that in Ref. [6]. This analysis makes it possible to investigate the secondary quantum phenomena in BJJ (see, for instance, Ref. [18]) and presents a general picture of low-temperature dynamics of Josephson junctions in BEC.

ACKNOWLEDGMENTS

It is a great pleasure to thank A. Smerzi and L. P. Pitaeviskii for many fruitful discussions. This work was supported by NSF of China (Grants Nos. 10175039 and 90203007), Science Funding (Grant No. 20001003), and Youth Funding (Grant No. 20011002) from Shanxi Province of China. Y.Z. was also supported by the Academy of Finland (Project No. 50314).

[1] J.M. Martinis *et al.*, Phys. Rev. Lett. **55**, 1543 (1985).
[2] A. Smerzi *et al.*, Phys. Rev. Lett. **79**, 4950 (1997); S. Raghavan *et al.*, Phys. Rev. A **59**, 620 (1999); I. Marino *et al.*, *ibid.* **60**, 487 (1999).
[3] S. Giovanazzi *et al.*, Phys. Rev. Lett. **84**, 4521 (2000).
[4] D. Choi and Q. Niu, Phys. Rev. Lett. **82**, 2022 (1999); O. Morsch *et al.*, *ibid.* **87**, 140402 (2001).
[5] J.R. Anglin *et al.*, Phys. Rev. A **64**, 063605 (2001).
[6] L. Pitaevskii *et al.*, Phys. Rev. Lett. **87**, 180402 (2001).
[7] M. Kramer, L. Pitaevskii, and S. Stringari, Phys. Rev. Lett. **88**, 180404 (2002).
[8] *Bose-Einstein Condensates in Atomic Lasers*, edited by M. Inguscio *et al.* (IOS Press, Amsterdam, 1999).
[9] F. Dalfovo *et al.*, Rev. Mod. Phys. **71**, 463 (1999); A.J. Leggett, *ibid.* **73**, 307 (2001).
[10] B.P. Anderson *et al.*, Science (Washington, D.C., U.S.) **282**, 1686 (1998); W.M. Liu *et al.*, Phys. Rev. Lett. **84**, 2294 (2000); W.D. Li *et al.*, Phys. Rev. A **64**, 015602 (2001).
[11] S. Burger *et al.*, Phys. Rev. Lett. **86**, 4447 (2001).
[12] F.S. Cataliotti *et al.*, Science (Washington, D.C., U.S.) **293**, 843 (2001).
[13] P.W. Anderson, *Basic Notions of Condensed Matter Physics* (Benjamin-Cummings, Menlo Park, 1984).
[14] M.R. Andrews *et al.*, Science (Washington, D.C., U.S.) **275**, 637 (1997); D.S. Hall *et al.*, Phys. Rev. Lett. **81**, 1539 (1998); **81**, 1543 (1998).
[15] G. Schon and A.D. Zaikin, Phys. Rep. **198**, 237 (1990).
[16] W. Zwerger, A.T. Dorsey, and Matthew P.A. Fisher, Phys. Rev. B **34**, 6518 (1986); S.M. Apenko, Phys. Lett. A **142**, 277 (1989).
[17] K.K. Likharev *et al.*, J. Low Temp. Phys. **59**, 347 (1985).
[18] D.V. Averin and K.K. Likharev, *Mesoscopic Phenomena in Solids*, edited by B.L. Altshuler *et al.* (Elsevier Science, Amsterdam, B. V., 1991).
[19] J.Q. Liang *et al.*, Phys. Lett. B **282**, 105 (1992).
[20] J.Q. Liang *et al.*, Phys. Rev. D **51**, 718 (1995).
[21] S. Coleman, Phys. Rev. D **15**, 2929 (1977); M.T. Grisaru, *ibid.* **16**, 1962 (1977); Nucl. Phys. B **298**, 178 (1988).
[22] G.J. Milburn *et al.*, Phys. Rev. A **55**, 4318 (1997); I. Zapata *et al.*, *ibid.* **57**, R28 (1998); A. Smerzi *et al.*, *ibid.* **61**, 63601 (2000).
[23] A.J. Leggett, Suppl. Prog. Theor. Phys. **69**, 80 (1980).
[24] T.G. Tiecke, M. Kemmann, Ch. Buggle, I. Shvarchuck, W. von Klitzing, and J.T.M. Walraven, e-print cond-mat/0211604.