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Dissipations in coupled quantum systems

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We investigate the dynamics of a composite quantum system, comprised of coupled subsystems, of which only one is significantly interacting with the environment. The validity of the conventional *ad hoc* approach—assuming that relaxation terms can be extracted directly from the master equation of the subsystem interacting with the reservoir—was examined. We derived the equation of motion for the composite system's reduced density matrix—applying only the factorization approximation, but not the conventional sequence of Markoff, coarse grain, and secular approximations. From our analysis, we concluded that the conventional *ad hoc* approach is applicable to zero-temperature reservoir, but fails for finite temperatures. It is further shown that at finite temperatures, the standard procedure does not even yield a master equation for the composite system, and its dynamics has to be studied by the equations of motion which are developed here. For demonstration we considered a system of a three-level atom, the two excited states are coupled to each other, and only one of them communicates with the ground state via a radiation reservoir.

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I. INTRODUCTION

There is a considerable interest in the physics of quantum coupled systems, especially their dissipations due to the interaction with the environment. In particular, a number of experiments were conducted on microscopic systems, such as two coupled microcavities [1], and an atom interacting with a microcavity, which is described by the Jaynes-Cumming model [2]. In the present paper, we investigate the dissipations in a system composed of two coupled parts, when only one of them is in effective contact with the environment, while the interaction with the environment of the second one is negligible. The environment is modeled as a *heat reservoir*.

The most common tool used for treating dissipation in a system interacting with a reservoir is the *master equation* for the *reduced density matrix* of the system [3]. This equation is derived from the equation of motion for the total density matrix of the combined system and reservoir, by eliminating the reservoir degrees of freedom. This *standard procedure*, which leads from the exact equation of motion for the total density matrix to the master equation, is executed along the following steps.

(a) The differential equation of motion for the total density matrix is formally integrated in time once, and the result is substituted back into the equation, this yields an integro-differential equation.

(b) The trace over the reservoir's degrees of freedom is formally performed.

(c) *The factorization approximation* [4] is carried out, namely, the total density matrix of the combined system reservoir is replaced, within the time integration, by a product of the system reduced density matrix and the reservoir stationary density matrix.

(d) *The Markoff approximation* [5] is invoked, assuming that the reservoir, which contains a huge number of degrees of freedom, has a very short memory, and on the time scale of the reservoir's correlations the system is stationary.

(e) *The coarse-grained rate of variation* [6] is then calculated by taking the time average of the equation for the reduced density matrix over a period of time, much longer than the reservoir's correlation time, but significantly shorter than the relaxation time of the system itself.

(f) Finally, *the secular approximation* [3,7] is applied, and energy-nonconserving terms are eliminated.

This procedure yields the relaxation terms of the master equation, which is a group of *rate equations* [8] for the populations and coherences of the system.

In this paper, we consider a *composite* system S, composed of two coupled parts A and C. The system's Hamiltonian is described by $H_S = H_A + H_C + H_{AC}$, where H_A is the Hamiltonian of subsystem A, H_C is that of subsystem C, and H_{AC} is the coupling Hamiltonian between the two parts. The reservoir R is represented by a Hamiltonian H_R, which is usually modeled as an ensemble of many harmonic oscillators. Only one of the two subsystems, say C, is interacting with the reservoir by a Hamiltonian H_{CR}. A canonical example is the atom-cavity system. Here a two-level atom, A, is interacting with a cavity, C, of a single mode, and the strength of the coupling to the environment of one subsystem is much different from that of the other subsystems. We study here the derivation of the equation of motion for the reduced density matrix of the composite system, and examine the applicability of the standard procedure, which is outlined above, and the role of the different approximations involved.

The conventional *ad hoc* approach to the derivation of the equations of motion for this kind of a composite system is as follows. The relaxation terms are borrowed from the master equation of subsystem C, disregarding its coupling to subsystem A. These terms are obtained by applying the standard procedure to subsystem C, interacting with the reservoir R, employing the Hamiltonian $H_C + H_R + H_{CR}$. We shall argue that this *ad hoc* approach is only justified when the reservoir is at zero temperature. This *ad hoc* approach fails, e.g., when a Rydberg atom is traversing a superconducting resonant cavity [9], since the temperature of the environment can be of the order of the transition frequency. However, when the environment is at finite temperatures, it fails even to produce detailed balance at steady state. This point was noted by Cresser [10], who observed that the usual relaxation terms do not lead to a proper thermal equilibrium. Murato and Shibata [11] have investigated the Jaynes-Cumming model, when the atom is strongly coupled to the electromagnetic single mode of the cavity, and only one component of this composite system of atom and cavity has a considerable dissipation. In our previous paper [12], we have attempted an investigation of the dissipation in a system of two coupled microcavities when only one of them is in contact with a reservoir.

We develop here an alternative approach for investigating a composite system in a reservoir. We implement only the first three steps (a)–(c) of the standard procedure. Thus, only the factorization approximation is invoked, while the other steps (d)–(f) of the scheme, with the assumptions embedded in them, are not executed. Employing the Laplace transform enables us, in principle, to carry on and study the time dependence of the reduced density matrix elements of the system. We demonstrate our *exact* [13] approach on a model of an atom having three levels, where the two excited ones are coupled to each other, while only one of them is communicating with the ground state via a radiation reservoir. The three lowest energy states of hydrogen atom [3], i.e., the ground state $1s_{1/2}$, and the two quasidegenerate excited states $2p_{1/2}$ and the metastable $2s_{1/2}$, can be thought of as a realization of this toy model.

The paper is developed and organized as follows. In Sec. II we introduce the Hamiltonian of the composite system in a reservoir and outline the derivation of the equation of motion for the reduced density matrix executing the first three steps (a)–(c) of the standard procedure. The correlation functions for a reservoir made up of an ensemble of many harmonic oscillators, which play an important role in the dynamics of the system, are introduced. The model of the three-level atom is introduced in Sec. III, and the integrodifferential equations for its reduced density matrix are converted to algebraic ones by the Laplace transform. The steady-state solutions are shown in Sec. IV to yield detailed balance, and proper thermal equilibrium behavior. The conventional *ad hoc* scheme for a composite system is reviewed in Sec. V, and the equations of motion are derived. It is apparent that at finite-temperature environment, these equations of motion do not yield a proper detailed balance. The composite system at zero-temperature reservoir is investigated in Sec. VI. Here, the results of the exact method of Sec. III are compared with those obtained by the conventional scheme. It is demonstrated that at zero temperature, the *ad hoc* equations of motion can be safely used. The effect of a finite-temperature reservoir is examined in Sec. VII, and it is confirmed that the *ad hoc* scheme is not applicable. Zero temperature is understood here as the limit when the temperature of the environment is much smaller than the characteristic energies of the system. The evolution of the composite system in time is obtained by solving the equations of motion of the exact scheme in their Laplace transform version, without invoking the Markoff ansatz and the other approximations of steps (d)–(f). In Sec. VIII, we attempt a derivation of a master equation for the composite system by applying the assumptions of steps (d)–(f) to the the equations of Sec. III. It is established that, in general, at finite temperatures, the standard procedure does not yield a master equation. This is due to the presence of an extra time scale generated by the coupling between the subsystems. The conclusions are discussed in Sec. IX.

II. A COMPOSITE SYSTEM IN A RESERVOIR

For a later reference, we review here the standard procedure [3], for deriving the equation of motion for a system interacting with an environment. We consider a system S, which is composed of two coupled subsystems A and C, when only one of them, C, is coupled to a reservoir R. The Hamiltonian of the composed system is

$$H_S = H_A + H_C + H_{AC}, \tag{1}$$

where H_A is the Hamiltonian of subsystem A , H_C is that of the subsystem C, and H_{AC} is the Hamiltonian of coupling between the subsystems A and C. The reservoir is described by the Hamiltonian H_R, and its interaction Hamiltonian H_{SR} with the system will be eventually described by

$$H_{SR} = H_{CR}, \tag{2}$$

where the Hamiltonian H_{CR} indicates explicitly that only subsystem C interacts with the environment. The total Hamiltonian of the combined system reservoir is

$$H_{tot} = H_S + H_R + H_{SR}. \tag{3}$$

The equation of motion for total density matrix ρ, of the combined system reservoir in the interaction picture, is

$$\frac{\partial}{\partial t}\tilde{\rho}(t) = \frac{1}{i\hbar}[\tilde{H}_{SR}(t), \tilde{\rho}(t)], \tag{4}$$

where

$$\tilde{\rho}(t) = e^{i(H_S+H_R)t/\hbar}\rho(t)e^{-i(H_S+H_R)t/\hbar},$$

$$\tilde{H}_{SR}(t) = e^{i(H_S+H_R)t/\hbar}H_{SR}e^{-i(H_S+H_R)t/\hbar}. \tag{5}$$

By a formal integration of Eq. (4), and resubstituting the result, we have

$$\frac{\partial}{\partial t}\tilde{\rho}(t) = \frac{1}{i\hbar}[\tilde{H}_{SR}(t), \tilde{\rho}(0)] + \left(\frac{1}{i\hbar}\right)^2 \int_0^t dt' [\tilde{H}_{SR}(t), [\tilde{H}_{SR}(t'), \tilde{\rho}(t')]], \tag{6}$$

where $\tilde{\rho}(0)$ is the initial value of $\tilde{\rho}(t)$. Note that even if the *initial state* of the system plus bath is a *product state*, Eq. (6) establishes, at time t, a state of *entanglement* between the system and the bath. In other words, the bath is affected by the system.

In order to get the equation of motion for σ, the reduced density matrix of the system, we have to eliminate the bath degrees of freedom by taking the trace of ρ with respect to the reservoir, namely, $\sigma=\mathrm{Tr}_R\{\rho\}$. This program can be pursued by specifying the properties of the reservoir. In Eq. (6), which is still an *exact* equation, we now introduce the *factorization approximation*, i.e., we replace $\tilde{\rho}$, within the time integral, by product $\tilde{\sigma}(t)\otimes\sigma_R$, where σ_R is the *stationary* density matrix of the bath, say in thermal equilibrium. This basic assumption is the only approximation made up to this point. The essence of this approximation is that although the bath is responding in time to the motion of the system, it relaxes very fast to equilibrium, and compels the density matrix to become a product. Taking the trace over the bath, we obtain

$$\frac{\partial}{\partial t}\tilde{\sigma}(t)=\left(\frac{1}{i\hbar}\right)^2\int_0^t dt'\,\mathrm{Tr}_R\{[\tilde{H}_{SR}(t),[\tilde{H}_{SR}(t'),\tilde{\sigma}(t')\otimes\sigma_R]]\}. \tag{7}$$

Here, we have assumed that the average value of H_{SR}, with respect to the reservoir, vanishes, that is, $\mathrm{Tr}_R\{\tilde{H}_{SR}(t)\sigma_R\}=0$. Note that $\tilde{\sigma}(t')$ within the time integration of Eq. (7) is not the initial value of $\tilde{\sigma}(t)$ but rather its instantaneous value at the time t'. Since the reservoir is assumed to relax extremely fast to its stationary state, σ_R is both the initial and the instantaneous values of the bath density matrix. If we had used the initial value of $\tilde{\sigma}(t)$ in the right-hand side of Eq. (7), the procedure would be simply the *second-order perturbation expansion* in terms of the interaction Hamiltonian H_{SR}. The factorization assumption is equivalent to the *summation of selective terms*, presumably the dominant ones, in an *infinite order* perturbation procedure. The reservoir is being affected by the dynamics of the system, though it preserves its stationarity on a very short time scale compared to the time scales of the system.

The system-reservoir interaction Hamiltonian is, in general, of the form $H_{SR}=\Sigma_i S_i R_i$, where S_i are dynamic operators of the system S and the bath operators R_i depend only on the reservoir degrees of freedom. Since H_S and H_R commute with each other, in the interaction picture,

$$\tilde{H}_{SR}(t)=\sum_i \tilde{S}_i(t)\tilde{R}_i(t), \tag{8}$$

where

$$\tilde{S}_i(t)=e^{iH_St/\hbar}S_ie^{-iH_St/\hbar},$$

$$\tilde{R}_i(t)=e^{iH_Rt/\hbar}R_ie^{-iH_Rt/\hbar}. \tag{9}$$

Substituting Eq. (8) into Eq. (7) yields that

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{\sigma}(t)=&-\frac{1}{\hbar^2}\int_0^t dt'\sum_{i\neq j}\{g_{ij}(\tau)[\tilde{S}_i(t)\tilde{S}_j(t')\tilde{\sigma}(t')\\&-\tilde{S}_j(t')\tilde{\sigma}(t')\tilde{S}_i(t)]+g_{ji}(-\tau)[\tilde{\sigma}(t')\tilde{S}_j(t')\tilde{S}_i(t)\\&-\tilde{S}_i(t)\tilde{\sigma}(t')\tilde{S}_j(t')]\},\end{aligned} \tag{10}$$

where the trace over the reservoir was separated from the dynamics of the system's operators. Here we define, for a bath's operator O, the expectation value $\langle O\rangle_R=\mathrm{Tr}_R\{O\sigma_R\}$, and introduce the bath correlation functions

$$g_{ij}(\tau)=\langle\tilde{R}_i(t)\tilde{R}_j(t')\rangle_R,$$

$$g_{ji}(-\tau)=\langle\tilde{R}_j(t')\tilde{R}_i(t)\rangle_R, \tag{11}$$

which depend on the time difference $\tau=t-t'$.

We shall consider a simple model for the reservoir, that is, an ensemble of many harmonic oscillators. It is represented by the Hamiltonian

$$H_R=\sum_r \hbar\omega_r(b_r^\dagger b_r+1/2), \tag{12}$$

where ω_r is the frequency of the rth reservoir oscillator, and b_r and $b_r^\dagger$ are the annihilation and creation operators of this oscillator.

The interaction Hamiltonian for the system with the reservoir can be cast into the following form:

$$V=S^+R^-+S^-R^+, \tag{13}$$

where

$$R^-=\sum_r f_rb_r,\quad R^+=\sum_r f_r^*b_r^\dagger. \tag{14}$$

Here, f_r and f_r^* are the coupling parameters of the interaction between the system and the rth harmonic oscillator of the reservoir. For the correlation functions, Eq. (11), we get

$$g_{12}(\tau)=\sum_r |f_r|^2(\langle n_r\rangle+1)e^{-i\omega_r\tau},$$

$$g_{21}(\tau)=\sum_r |f_r|^2\langle n_r\rangle e^{i\omega_r\tau},\quad g_{11}(\tau)=g_{22}(\tau)=0, \tag{15}$$

where $\langle n_r\rangle=\langle b_r^\dagger b_r\rangle$ is the average number of excitation quanta of the rth oscillator of the reservoir.

It is only natural to write Eq. (10) for the reduced density matrix by its components. Introducing the complete set of states $|a\rangle$, which are the energy eigenstates of H_S, namely,

$$H_S|a\rangle=E_a|a\rangle. \tag{16}$$

Now we write Eq. (10) as

$$\frac{\partial}{\partial t}\sigma_{ab}(t)=\frac{1}{i\hbar}(E_a-E_b)\sigma_{ab}(t)-\sum_{c,d}\int_0^t dt'(\sigma_{db}(t')\{S^+_{ac}S^-_{cd}[G(\omega_{bc},\tau)+G_T(\omega_{bc},\tau)]+S^-_{ac}S^+_{cd}G^*_T(\omega_{cb},\tau)\}+\sigma_{ac}(t')$$
$$\times\{S^+_{cd}S^-_{db}[G^*(\omega_{ad},\tau)+G^*_T(\omega_{ad},\tau)]+S^-_{cd}S^+_{db}G_T(\omega_{da},\tau)\}-\sigma_{cd}(t')\{S^-_{ac}S^+_{db}[G^*(\omega_{cb},\tau)+G^*_T(\omega_{cb},\tau)$$
$$+G(\omega_{da},\tau)+G_T(\omega_{da},\tau)]+S^+_{ac}S^-_{db}[G^*_T(\omega_{ad},\tau)+G_T(\omega_{bc},\tau)]\}). \tag{17}$$

Here, we have introduced two kinds of bath correlation functions

$$G(\omega,\tau)=\frac{1}{\hbar^2}\sum_k |f_k|^2 e^{-i(\omega_k-\omega)\tau},$$

$$G_T(\omega,\tau)=\frac{1}{\hbar^2}\sum_k |f_k|^2\langle n_k\rangle e^{-i(\omega_k-\omega)\tau}, \tag{18}$$

and have used $O_{ab}=\langle a|O|b\rangle$. Note that the reduced density matrix components $\sigma_{ab}(t)$ in Eq. (17) are expressed in the Schrödinger picture. We observe that although Eq. (17) is written explicitly in terms the energy eigenstates $|a\rangle$ of H_S, it is a general equation for the atomic density matrix, and the only approximation made is the factorization approximation. We continue the development of the theory *without* applying to the standard Markoff approximation of replacing $\sigma(t')$ in Eq. (17) by $\sigma(t)$. To make the point clear, we shall study a simple model system, which displays the features of a composite system interacting with a bath.

III. A THREE-LEVEL ATOM IN A RESERVOIR

Our model system is an atom with three energy states, where the two excited states, $|2\rangle$ of energy $\hbar\omega_2$ and $|3\rangle$ of energy $\hbar\omega_3$, are coupled directly to each other, while the ground state $|1\rangle$ of energy $\hbar\omega_1=0$ is coupled through a radiation reservoir to the excited state $|2\rangle$. In terms of these *bare* energy states, the Hamiltonian of the atom is expressed, in the form of Eq. (1), as

$$H_A=\hbar\omega_3|3\rangle\langle 3|,$$

$$H_C=\hbar\omega_2|2\rangle\langle 2|,$$

$$H_{AC}=\hbar g(|3\rangle\langle 2|+|2\rangle\langle 3|), \tag{19}$$

where $\hbar g$ is the coupling energy between the excited states, and we choose

$$g\ll\omega_2,\omega_3. \tag{20}$$

The radiation reservoir is given by Eq. (12), and its interaction with the atom is written explicitly as

$$H_{SR}=\hbar\sum_k (f_k S^+ b_k+f^*_k b^\dagger_k S^-), \tag{21}$$

where

$$S^+=|2\rangle\langle 1|,\quad S^-=|1\rangle\langle 2|. \tag{22}$$

To follow the steps of the preceding section, from Eq. (16) onwards, the atom's Hamiltonian of Eq. (19), $H_S=H_A+H_C+H_{AC}$, is diagonalized. Expressed in terms of the original states, we get the following for the diagonal eigenstates:

$$|+\rangle=\frac{1}{\sqrt{L}}[g|2\rangle+\Lambda|3\rangle],$$

$$|-\rangle=\frac{1}{\sqrt{L}}[\Lambda|2\rangle-g|3\rangle],$$

$$|1\rangle=|1\rangle, \tag{23}$$

with the respective eigenenergies

$$E_+=\hbar\omega_+=\frac{\hbar}{2}[\omega_2+\omega_3+\Delta],$$

$$E_-=\hbar\omega_-=\frac{\hbar}{2}[\omega_2+\omega_3-\Delta],$$

$$E_1=\hbar\omega_1=0 \tag{24}$$

and with

$$\delta=\omega_3-\omega_2,\quad \Delta^2=\delta^2+4g^2,$$

$$L=\frac{1}{2}\Delta(\Delta+\delta),\quad \Lambda=\frac{\delta+\Delta}{2}. \tag{25}$$

For the case where the two excited states are *degenerate*, i.e., when

$$\omega_2=\omega_3=\omega_0, \tag{26}$$

Eq. (23) is reduced to

$$|+\rangle=\frac{1}{\sqrt{2}}(|2\rangle+|3\rangle),$$

$$|-\rangle=\frac{1}{\sqrt{2}}(|2\rangle-|3\rangle), \tag{27}$$

and the corresponding eigenenergies are

$$E_+=\hbar\omega_+=\hbar(\omega_0+g),$$

$$E_-=\hbar\omega_-=\hbar(\omega_0-g). \tag{28}$$

The Hamiltonian of the atomic system, in the diagonal representation, is given by

$$H_S=\hbar\omega_+|+\rangle\langle+|+\hbar\omega_-|-\rangle\langle-|. \tag{29}$$

We shall now investigate the equations of motion for this model of three-level atom in a radiation bath. We consider the degenerate case, and use Eq. (17) to express the equations of motion for the populations and coherences of our system in the diagonal representation explicitly as the following.

(i) The equation for $\sigma_{(++)}$ is

$$\begin{aligned}\frac{\partial}{\partial t}\sigma_{(++)}(t)=-\frac{1}{2}\int_0^t dt'\{&[G(\omega_+,\tau)+G^*(\omega_+,\tau)\\&+G_T(\omega_+,\tau)+G_T^*(\omega_+,\tau)]\sigma_{(++)}(t')\\&+[G(\omega_+,\tau)+G_T(\omega_+,\tau)]\sigma_{(-+)}(t')\\&+[G^*(\omega_+,\tau)+G_T^*(\omega_+,\tau)]\sigma_{(+-)}(t')\\&-[G_T(\omega_+,\tau)+G_T^*(\omega_+,\tau)]\sigma_{(11)}(t')\}.\end{aligned} \tag{30}$$

(ii) The equation for $\sigma_{(--)}$ is

$$\begin{aligned}\frac{\partial}{\partial t}\sigma_{(--)}(t)=-\frac{1}{2}\int_0^t dt'\{&[G(\omega_-,\tau)+G^*(\omega_-,\tau)\\&+G_T(\omega_-,\tau)+G_T^*(\omega_-,\tau)]\sigma_{(--)}(t')\\&+[G(\omega_-,\tau)+G_T(\omega_-,\tau)]\sigma_{(+-)}(t')\\&+[G^*(\omega_-,\tau)+G_T^*(\omega_-,\tau)]\sigma_{(-+)}(t')\\&-[G_T(\omega_-,\tau)+G_T^*(\omega_-,\tau)]\sigma_{(11)}(t')\}.\end{aligned} \tag{31}$$

(iii) The equation for $\sigma_{(+-)}$ is

$$\begin{aligned}\frac{\partial}{\partial t}\sigma_{(+-)}(t)=-2ig\sigma_{(+-)}(t)-\frac{1}{2}\int_0^t dt'\{&[G(\omega_-,\tau)\\&+G^*(\omega_+,\tau)+G_T(\omega_-,\tau)\\&+G_T^*(\omega_+,\tau)]\sigma_{(+-)}(t')+[G(\omega_-,\tau)\\&+G_T(\omega_-,\tau)]\sigma_{(--)}(t')+[G^*(\omega_+,\tau)\\&+G_T^*(\omega_+,\tau)]\sigma_{(++)}(t')-[G_T(\omega_-,\tau)\\&+G_T^*(\omega_+,\tau)]\sigma_{(11)}(t')\}.\end{aligned} \tag{32}$$

(iv) Finally, the equation for $\sigma_{(11)}$ is

$$\begin{aligned}\frac{\partial}{\partial t}\sigma_{(11)}(t)=-\frac{1}{2}\int_0^t dt'\{&[G_T(\omega_+,\tau)+G_T^*(\omega_+,\tau)+G_T(\omega_-,\tau)+G_T^*(\omega_-,\tau)]\sigma_{(11)}(t')-[G(\omega_+,\tau)+G^*(\omega_+,\tau)\\&+G_T(\omega_+,\tau)+G_T^*(\omega_+,\tau)]\sigma_{(++)}(t')-[G(\omega_-,\tau)+G^*(\omega_-,\tau)+G_T(\omega_-,\tau)+G_T^*(\omega_-,\tau)]\sigma_{(--)}(t')\\&-[G(\omega_-,\tau)+G^*(\omega_+,\tau)+G_T(\omega_-,\tau)+G_T^*(\omega_+,\tau)]\sigma_{(+-)}(t')-[G(\omega_+,\tau)+G^*(\omega_-,\tau)+G_T(\omega_+,\tau)\\&+G_T^*(\omega_-,\tau)]\sigma_{(-+)}(t')\}.\end{aligned} \tag{33}$$

Here $\tau=t-t'$, the reservoir correlation function G^* is the complex conjugate of G of Eq. (18), and $\sigma_{(-+)}(t)=\sigma^*_{(+-)}(t)$. Note that the coherences, such as $\sigma_{(1+)}(t)$, do not appear in these equations.

We observe that Eqs. (30)–(33) are integrodifferential equations in time. In the standard method, these equations are converted into a set of differential equations by using the Markoff approximation, where the $\sigma(t')$ is replaced by $\sigma(t)$ and is being taken out of the integral. The rational for this replacement is that in the interaction picture, $\sigma(t)$ is slowly varying in time. However, as can clearly be seen from Eq. (32), this is not the case here, since, e.g., $\sigma_{(+-)}(t)$ is oscillatory due to the internal coupling, with frequency $2g$. It is evident that the Markoff procedure should be avoided for a composite system, and we should explore a different approach.

The integrodifferential equations, Eqs. (30)–(33), being of the time-convolution type, call for employing Laplace transform. We introduce the Laplace transform of a function $f(t)$ of the time t by

$$\overline{f}(s)=\int_0^\infty dte^{-st}f(t), \tag{34}$$

and its inverse transform by

$$f(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}dse^{st}\overline{f}(s), \tag{35}$$

with $\gamma>0$ and real. The Laplace transform of Eq. (30) is

$$\begin{aligned}(s+\overline{G}_1+\overline{T}_1)\overline{\sigma}_{(++)}+\left(\frac{\overline{G}_1+\overline{T}_1}{2}\right)\overline{\sigma}_{(+-)}+\left(\frac{\overline{G}_1+\overline{T}_1}{2}\right)\overline{\sigma}_{(-+)}\\-\overline{T}_1\overline{\sigma}_{(11)}=\sigma_{(++)}(0),\end{aligned} \tag{36}$$

that of Eq. (31) is

$$(s+\overline{G}_2+\overline{T}_2)\overline{\sigma}_{(--)}+\left(\frac{\overline{G}_2+\overline{T}_2}{2}\right)\overline{\sigma}_{(+-)}+\left(\frac{\overline{G}_2+\overline{T}_2}{2}\right)\overline{\sigma}_{(-+)} -\overline{T}_2\overline{\sigma}_{(11)}=\sigma_{(--)}(0), \tag{37}$$

that of Eq. (32) is

$$\left(s+2ig+\frac{\overline{G}_1+\overline{T}_1+\overline{G}_2+\overline{T}_2}{2}\right)\overline{\sigma}_{(+-)}+\left(\frac{\overline{G}_1+\overline{T}_1}{2}\right)\overline{\sigma}_{(++)} +\left(\frac{\overline{G}_2+\overline{T}_2}{2}\right)\overline{\sigma}_{(--)}-\left(\frac{\overline{T}_1+\overline{T}_2}{2}\right)\overline{\sigma}_{(11)}=\sigma_{(+-)}(0), \tag{38}$$

and that of Eq. (33) is

$$(s+\overline{T}_1+\overline{T}_2)\overline{\sigma}_{(11)}-(\overline{G}_1+\overline{T}_1)\overline{\sigma}_{(++)}-(\overline{G}_2+\overline{T}_2)\overline{\sigma}_{(--)} -\left(\frac{\overline{G}_1+\overline{T}_1+\overline{G}_2+\overline{T}_2}{2}\right)\overline{\sigma}_{(+-)} -\left(\frac{\overline{G}_1+\overline{T}_1+\overline{G}_2+\overline{T}_2}{2}\right)\overline{\sigma}_{(-+)}=\sigma_{(11)}(0). \tag{39}$$

Here, the $\sigma(0)$ on the right-hand sides are the initial values in time $t=0$, of $\sigma(t)$. Note that this set of equations is algebraic. The Laplace transform for the bath correlation functions of Eq. (18) is simply

$$\overline{\mathcal{G}}(s,\omega)=\frac{1}{\hbar^2}\sum_k |f_k|^2\frac{1}{s+i(\omega_k-\omega)},$$

$$\overline{G}_T(s,\omega)=\frac{1}{\hbar^2}\sum_k |f_k|^2 n_k\frac{1}{s+i(\omega_k-\omega)}. \tag{40}$$

We have introduced two new functions $\overline{G}$ and $\overline{T}$ by

$$\overline{G}(s,\omega)=\Re e\,\overline{\mathcal{G}}(s,\omega)=\frac{1}{\hbar^2}\sum_k |f_k|^2\frac{s}{s^2+(\omega_k-\omega)^2}, \tag{41}$$

and

$$\overline{T}(s,\omega)=\Re e\,\overline{G}_T(s,\omega)=\frac{1}{\hbar^2}\sum_k |f_k|^2 n_k\frac{s}{s^2+(\omega_k-\omega)^2}, \tag{42}$$

where $\Re e$ stands for the real part. We point out that the imaginary parts of the G functions in Eq. (38) have been discarded, since these contribute only to the renormalization of the frequency $2g$. We have also introduced, in Eqs. (36)–(39), the definitions

$$\overline{G}_1=\overline{G}(\omega_+),\quad \overline{G}_2=\overline{G}(\omega_-) \tag{43}$$

and

$$\overline{T}_1=\overline{T}(\omega_+),\quad \overline{T}_2=\overline{T}(\omega_-). \tag{44}$$

Before we continue with the development of the theory, we make a digression, and study an exact solution of our set of equations, i.e., the steady-state behavior of the atomic system. It should be noticed that this can be done without further approximations, such as Markoff's.

IV. STEADY STATE

Now we consider our system of three-level atom in contact with a heat reservoir in thermal equilibrium at temperature *T*. At *steady state*, the atomic system is expected to reach thermal equilibrium, and the reduced density matrix of the system should obey

$$\sigma^{eq}=\frac{e^{-\beta H_S}}{\mathrm{Tr}\{e^{-\beta H_S}\}}, \tag{45}$$

where $\beta=1/k_BT$, and k_B is the Boltzmann constant. It should be emphasized that in Eq. (45), H_S is the entire Hamiltonian of the atom with the intrinsic coupling. In the diagonal representation, for the degenerate case, using Eq. (27), and Eq. (28), the steady-state populations should be

$$\sigma^{eq}_{(11)}=\frac{1}{Z},\quad \sigma^{eq}_{(++)}=\frac{e^{-\beta\hbar(\omega_0+g)}}{Z},\quad \sigma^{eq}_{(--)}=\frac{e^{-\beta\hbar(\omega_0-g)}}{Z}, \tag{46}$$

where the partition function is

$$Z=\mathrm{Tr}\{e^{-\beta H_S}\}=1+2e^{-\beta\hbar\omega_0}\cosh(\beta\hbar g). \tag{47}$$

Now we show that the steady-state solutions of Eqs. (36)–(39) are indeed given by Eqs. (46). It is easy to obtain the steady-state solutions directly from the Laplace transformed version of the equations of motion. The steady-state value of $\sigma(t)$ is found by taking the limit

$$\sigma^{ss}(t)=\lim_{s\to 0^+} s\overline{\sigma}(s), \tag{48}$$

of the proper $\overline{\sigma}(s)$. We multiply each side of these equations by *s*, take the limit $s\to 0^+$, and readily get

$$\sigma^{ss}_{(++)}=\frac{\overline{T}(s\to+0,\omega_+)}{\overline{G}(s\to+0,\omega_+)+\overline{T}(s\to+0,\omega_+)}\sigma^{ss}_{(11)},$$

$$\sigma^{ss}_{(--)}=\frac{\overline{T}(s\to+0,\omega_-)}{\overline{G}(s\to+0,\omega_-)+\overline{T}(s\to+0,\omega_-)}\sigma^{ss}_{(11)},$$

$$\sigma^{ss}_{(+-)}=\sigma^{ss}_{(-+)}=0. \tag{49}$$

As expected, the off-diagonal elements of the equilibrium reduced density matrix in the diagonal representation indeed vanish. To find the steady-state values of the diagonal populations, we use Eqs. (41) and (42) to find

$$\overline{G}(s\to 0,\omega_{\pm})=\Gamma(\omega_{\pm}) \tag{50}$$

and for the thermal functions

$$\overline{T}(s\to 0,\omega_{\pm})=\frac{\Gamma'(\omega_{\pm})}{2}N(\omega_{\pm}). \tag{51}$$

Here, we have introduced the inverse relaxation times—the damping rates—for the system, which are defined by

$$\Gamma(\omega)=\frac{2\pi}{\hbar^2}\sum_k |f_k|^2\delta(\omega_k-\omega) \tag{52}$$

and

$$\Gamma'(\omega)=\frac{2\pi}{\hbar^2}\sum_k |f_k|^2 n_k\delta(\omega_k-\omega)=\Gamma(\omega)N(\omega), \tag{53}$$

and the thermal distribution function for the bath oscillators,

$$N(\omega)=\frac{1}{e^{\beta\hbar\omega}-1}, \tag{54}$$

is the Bose-Einstein distribution.

The *detailed balance* conditions for the populations are simply obtained by substituting Eqs. (50) and (51) into Eq. (49), i.e.,

$$\frac{\sigma^{ss}_{(++)}}{\sigma^{ss}_{(11)}}=e^{-\beta\hbar(\omega_0+g)},\quad \frac{\sigma^{ss}_{(--)}}{\sigma^{ss}_{(11)}}=e^{-\beta\hbar(\omega_0-g)}. \tag{55}$$

Note that this result is independent of the $\Gamma's$. The detailed balance means that in the steady state, the number of transitions from one state to another is compensated by the reverse transitions. Since the normalization dictates that

$$\sigma_{(++)}+\sigma_{(--)}+\sigma_{(11)}=1, \tag{56}$$

we can write Eq. (55) as

$$\sigma^{ss}_{(++)}=\frac{e^{-\beta\hbar(\omega_0+g)}}{Z},\quad \sigma^{ss}_{(--)}=\frac{e^{-\beta\hbar(\omega_0-g)}}{Z},$$

$$\sigma^{ss}_{(11)}=\frac{1}{Z},\quad \overline{\sigma}^{ss}_{(+-)}=\overline{\sigma}^{ss}_{(-+)}=0, \tag{57}$$

where Z is given by Eq. (47), and conclude that indeed, at steady state, $\sigma^{ss}=\sigma^{eq}$ of Eq. (45). To express the steady state in the original representation of the bare states, we invert the transformation as the following:

$$\sigma_{(++)}=\frac{1}{2}(\sigma_{(22)}+\sigma_{(33)}+\sigma_{(32)}+\sigma_{(23)}),$$

$$\sigma_{(--)}=\frac{1}{2}(\sigma_{(22)}+\sigma_{(33)}-\sigma_{(32)}-\sigma_{(23)}),$$

$$\sigma_{(+-)}=\frac{1}{2}(\sigma_{(22)}-\sigma_{(33)}-\sigma_{(32)}+\sigma_{(23)}), \tag{58}$$

and find for the populations

$$\sigma^{eq}_{(33)}=\sigma^{eq}_{(22)}=\frac{e^{-\beta\hbar\omega_0}\cosh(\beta\hbar g)}{Z} \tag{59}$$

and for the coherences

$$\sigma^{eq}_{(32)}=\sigma^{eq}_{(23)}=\frac{-e^{-\beta\hbar\omega_0}\sinh(\beta\hbar g)}{Z}. \tag{60}$$

We observe that in steady state, the coherences do not vanish, and no detailed balance is established between the levels $|3\rangle$ and $|2\rangle$ with the level $|1\rangle$.

V. THE CONVENTIONAL *AD HOC* APPROACH

The goal of the present study is to compare, for the composite system in a reservoir, the outcome of our exact approach, where only the factorization approximation is used, with that of the standard one. Let us first review the *common procedure* which is utilized to obtain the master equation of a quantum system composed of two coupled subsystem, when only one of them is interacting with the reservoir. It is basically assumed that the coupling between the subsystems can be *disregarded* [14], when calculating the relaxation terms of the master equation. More specific, for a system's Hamiltonian of Eq. (1),

$$H_S=H_A+H_C+H_{AC},$$

when only the subsystem C is coupled to the reservoir through the Hamiltonian H_{RC}, the effect of the reservoir is calculated disregarding H_{AC}. The master equation of the composite system is then written as

$$\frac{d}{dt}\sigma(t)=\frac{1}{i\hbar}[H_S,\sigma(t)]+\left(\frac{d}{dt}\sigma(t)\right)_{rel}, \tag{61}$$

where $\sigma(t)$ is the reduced density matrix of the composite system and the relaxation term $[d\sigma(t)/dt]_{rel}$ has been assessed using the Hamiltonian $H_R+H_C+H_{RC}$.

For our model of three-level atom, with the Hamiltonian of Eq. (19), and Eq. (21), this relaxation term is

$$\left(\frac{d}{dt}\sigma(t)\right)_{rel}=\frac{\Gamma'(\omega_0)+\Gamma(\omega_0)}{2}[2S^-\sigma(t)S^+-S^+S^-\sigma(t)-\sigma(t)S^+S^-]+\frac{\Gamma'(\omega_0)}{2}[2S^+\sigma(t)S^--S^-S^+\sigma(t)-\sigma(t)S^-S^+]. \tag{62}$$

Here the atomic operators are related by Eq. (22) only to the $|1\rangle$ and $|2\rangle$ transitions of frequency ω_0, and the damping rates $\Gamma(\omega_0)$ and $\Gamma'(\omega_0)=\Gamma(\omega_0)N(\omega_0)$ are given by Eqs. (52) and (53), where $N(\omega_0)$ is the reservoir's mean photons number of Eq. (54), at frequency ω_0. It should be pointed out that while the relaxation term was calculated for $\sigma_C(t)$, the reduced density matrix of subsystem C only, it is replaced *ad hoc* by $\sigma(t)$ of the composite system. It should be emphasized that Eq. (62) is derived [3] by applying, beside the factorization approximation, the Markoff ansatz, the coarse-grained procedure, and the Secular approximation.

With the relaxation term of Eq. (62) substituted into Eq. (61), the equations of motion for the populations and the coherences in the Schrödinger picture, in the diagonal representation of Eq. (27), can be written explicitly as the following.

(i) For $\sigma_{(++)}(t)$, we have

$$\frac{d}{dt}\sigma_{(++)} = -\frac{\Gamma+\Gamma'}{2}\sigma_{(++)} + \frac{\Gamma'}{2}\sigma_{(11)} - \frac{\Gamma+\Gamma'}{4} \times(\sigma_{(+-)}+\sigma_{(-+)}). \tag{63}$$

(ii) For $\sigma_{(--)}(t)$, we obtain

$$\frac{d}{dt}\sigma_{(--)} = -\frac{\Gamma+\Gamma'}{2}\sigma_{(--)} + \frac{\Gamma'}{2}\sigma_{(11)} - \frac{\Gamma+\Gamma'}{4} \times(\sigma_{(-+)}+\sigma_{(+-)}). \tag{64}$$

(iii) For the coherence $\sigma_{(+-)}(t)$, we find

$$\frac{d}{dt}\sigma_{(+-)} = -i(\omega_+ - \omega_-)\sigma_{(+-)} - \frac{\Gamma+\Gamma'}{2}\sigma_{(+-)} - \frac{\Gamma+\Gamma'}{4}(\sigma_{(++)}+\sigma_{(--)}) + \frac{\Gamma'}{2}\sigma_{(11)}. \tag{65}$$

(iv) For the ground-state population, we get

$$\frac{d}{dt}\sigma_{(11)} = -\Gamma'\sigma_{(11)} + \frac{\Gamma+\Gamma'}{2}(\sigma_{(++)}+\sigma_{(--)} + \sigma_{(+-)}+\sigma_{(-+)}). \tag{66}$$

In these equations, we have written, for short,

$$\Gamma = \Gamma(\omega_0), \quad \Gamma' = \Gamma'(\omega_0), \tag{67}$$

using the Laplace transform, the above equations become the following.

(i) Equation (63) is converted to

$$\left(s+\frac{\Gamma+\Gamma'}{2}\right)\bar{\sigma}_{(++)} - \frac{\Gamma'}{2}\bar{\sigma}_{(11)} + \frac{\Gamma+\Gamma'}{4}(\bar{\sigma}_{(+-)}+\bar{\sigma}_{(-+)}) = \sigma_{(++)}(0). \tag{68}$$

(ii) The Laplace transform of Eq. (64) is

$$\left(s+\frac{\Gamma+\Gamma'}{2}\right)\bar{\sigma}_{(--)} - \frac{\Gamma'}{2}\bar{\sigma}_{(11)} + \frac{\Gamma+\Gamma'}{4}(\bar{\sigma}_{(-+)}+\bar{\sigma}_{(+-)}) = \sigma_{(--)}(0). \tag{69}$$

(iii) The Laplace transform of Eq. (65) is

$$\left(s+2ig+\frac{\Gamma+\Gamma'}{2}\right)\bar{\sigma}_{(+-)} - \frac{\Gamma'}{2}\bar{\sigma}_{(11)} + \frac{\Gamma+\Gamma'}{4}(\bar{\sigma}_{(++)} + \bar{\sigma}_{(--)}) = \sigma_{(+-)}(0). \tag{70}$$

(iv) Equation (66) is converted to

$$(s+\Gamma')\bar{\sigma}_{(11)} - \frac{\Gamma+\Gamma'}{2}(\bar{\sigma}_{(++)}+\bar{\sigma}_{(--)}+\bar{\sigma}_{(+-)}+\bar{\sigma}_{(-+)}) = \sigma_{(11)}(0). \tag{71}$$

Here, $\sigma_{(\alpha\beta)}(0)$ are the initial conditions.

The steady-state solutions of these equations are calculated, and compared to our results of Eq. (55). For the case of a heat reservoir at finite temperature, applying Eq. (48) for the steady state to the above set of Laplace transforms, we obtain

$$\frac{\sigma^{ss}_{(++)}}{\sigma^{ss}_{(11)}} = \frac{\Gamma'}{\Gamma+\Gamma'} = e^{-\beta\hbar\omega_0},$$

$$\frac{\sigma^{ss}_{(--)}}{\sigma^{ss}_{(11)}} = \frac{\Gamma'}{\Gamma+\Gamma'} = e^{-\beta\hbar\omega_0}. \tag{72}$$

This indicates that the standard procedure does not establish the proper detailed balance in the composite system. Using the normalization of Eq. (56), we get

$$\sigma^{ss}_{(++)} = \sigma^{ss}_{(--)} = \frac{e^{-\beta\hbar\omega_0}}{1+2e^{-\beta\hbar\omega_0}}, \tag{73}$$

$$\sigma^{ss}_{(11)} = \frac{1}{1+2e^{-\beta\hbar\omega_0}}, \quad \sigma^{ss}_{(+-)} = 0,$$

which does not lead to the thermal equilibrium result of Eq. (46). In contrast, we observe that our exact equations of motion (where only the factorization approximation is applied) render the right detailed balance, and the proper populations at thermal equilibrium. We also note that at least one difference stems from the sensitivity of the distribution function $N(\omega)$ to frequency. While in the exact solutions, $N(\omega)$ is related in the diagonal frequencies $\omega_0 \pm g$, in the conventional master equation the function $N(\omega)$ is expressed only at the bare frequency ω_0.

VI. ZERO-TEMPERATURE RESERVOIR

In this section, we compare our exact method with the conventional master equation, when the reservoir is at zero temperature. First, we write the exact set of equations, i.e., Eqs. (36)–(39), as

$$(s+\bar{G}_1)\bar{\sigma}_{(++)} + \frac{\bar{G}_1}{2}(\bar{\sigma}_{(+-)}+\bar{\sigma}_{(-+)}) = \sigma_{(++)}(0),$$

$$(s+\overline{G}_2)\overline{\sigma}_{(--)}+\frac{\overline{G}_2}{2}(\overline{\sigma}_{(+-)}+\overline{\sigma}_{(-+)})=\sigma_{(--)}(0),$$

$$\left(s+2ig+\frac{\overline{G}_1+\overline{G}_2}{2}\right)\overline{\sigma}_{(+-)}+\frac{\overline{G}_1}{2}\overline{\sigma}_{(++)}+\frac{\overline{G}_2}{2}\overline{\sigma}_{(--)}=\sigma_{(+-)}(0), \tag{74}$$

and

$$s\overline{\sigma}_{(11)}-\overline{G}_1\overline{\sigma}_{(++)}-\overline{G}_2\overline{\sigma}_{(--)}-\left(\frac{\overline{G}_1+\overline{G}_2}{2}\right)(\overline{\sigma}_{(+-)}+\overline{\sigma}_{(-+)})=\sigma_{(11)}(0). \tag{75}$$

Note that $\overline{\sigma}_{(11)}$ does not appear in Eqs. (74). Next, we write the set of the conventional schemes, Eqs. (68)–(71), as

$$\left(s+\frac{\Gamma}{2}\right)\overline{\sigma}_{(++)}+\frac{\Gamma}{4}(\overline{\sigma}_{(+-)}+\overline{\sigma}_{(-+)})=\sigma_{(++)}(0),$$

$$\left(s+\frac{\Gamma}{2}\right)\overline{\sigma}_{(--)}+\frac{\Gamma}{4}(\overline{\sigma}_{(+-)}+\overline{\sigma}_{(-+)})=\sigma_{(--)}(0),$$

$$\left(s+2ig+\frac{\Gamma}{2}\right)\overline{\sigma}_{(+-)}+\frac{\Gamma}{4}(\overline{\sigma}_{(++)}+\overline{\sigma}_{(--)})=\sigma_{(+-)}(0), \tag{76}$$

and

$$s\overline{\sigma}_{(11)}-\frac{\Gamma}{2}(\overline{\sigma}_{(++)}+\overline{\sigma}_{(--)}+\overline{\sigma}_{(+-)}+\overline{\sigma}_{(-+)})=\sigma_{(11)}(0). \tag{77}$$

A superficial comparison of these two sets of equations indicates that if we would replace the functions $\overline{G}_1(s)$ and $\overline{G}_2(s)$, in Eqs. (74) and (75) by $\Gamma/2$, which does not depend on s, the two sets will be identical. We shall now argue that under general conditions, this can be justified. Thus, we conclude that at zero temperature, the conventional master equation, i.e., the set of equations (76), and Eq. (77), is applicable.

To show this, we observe that the solutions *in time* of Eqs. (74) are obtained by the inverse Laplace transform of Eq. (35). The latter, in turn, depends on the zeros of the determinant of the fourth-order algebraic equations of the set of equations (74), amended by the equation for $\overline{\sigma}_{(-+)}$. We can readily write the determinant as

$$D(s)=4g^2\{s^2+s(\overline{G}_1+\overline{G}_2)+\overline{G}_1\overline{G}_2\}+\left(s+\frac{\overline{G}_1+\overline{G}_2}{2}\right)s\left\{s^2+s\left(\frac{3\overline{G}_1+3\overline{G}_2}{2}\right)+2\overline{G}_1\overline{G}_2\right\}, \tag{78}$$

and seek for the zeros, $\tilde{s}$, of the equation $D(\tilde{s})=0$. Before attempting a general solution of this equation, we conjecture that all the rates of change in time are much smaller than the scale of the atomic frequency ω_0. So, beside $g\ll\omega_0$, [see Eq. (20)], we expect that

$$\tilde{s}\ll\omega_0. \tag{79}$$

Further, we notice from Eq. (41), and Eq. (43), that

$$\overline{G}_{1,2}(\tilde{s},\omega)=\frac{1}{\hbar^2}\sum_k |f_k|^2\frac{\tilde{s}}{\tilde{s}^2+(\omega_k-\omega_\pm)^2} \tag{80}$$

are of the order of

$$\Gamma(\omega_\pm)=\frac{2\pi}{\hbar^2}\sum_k |f_k|^2\delta(\omega_k-\omega_\pm), \tag{81}$$

of Eq. (52), which should also obey

$$\Gamma(\omega_\pm)\ll\omega_0. \tag{82}$$

To conclude, we observe that there are two frequency scales beside $\tilde{s}$, namely, the coupling frequency g and the damping rate Γ.

A. Strong coupling limit

First, consider the limit when the intrinsic coupling constant g is much larger than the damping rate Γ: $g\gg\Gamma$. In this strong-coupling limit, since $g\gg\overline{G}_1,\overline{G}_2$, we expand the determinant $D(\tilde{s})$ with respect to $\overline{G}_1$ and $\overline{G}_2$. We find that $D(\tilde{s})$ of Eqs. (78) can be simplified into

$$D(\tilde{s})=(\tilde{s}+\overline{G}_2(\tilde{s}))\left(\tilde{s}+2ig+\frac{\overline{G}_1(\tilde{s})+\overline{G}_2(\tilde{s})}{2}\right)\times(\tilde{s}+\overline{G}_1(\tilde{s}))\left(\tilde{s}-2ig+\frac{\overline{G}_1(\tilde{s})+\overline{G}_2(\tilde{s})}{2}\right), \tag{83}$$

which yields the following four solutions:

$$\tilde{s}_{1,2}=-\overline{G}_{1,2}(0)=-\frac{\Gamma(\omega_\pm)}{2}, \tag{84}$$

$$\tilde{s}_3=-2ig-\frac{\overline{G}_1(-2ig)+\overline{G}_2(-2ig)}{2}=-2ig-\frac{\Gamma(\omega_+-2ig)+\Gamma(\omega_--2ig)}{4}, \tag{85}$$

and

$$\tilde{s}_4=2ig-\frac{\Gamma(\omega_++2ig)+\Gamma(\omega_-+2ig)}{4}. \tag{86}$$

Since for a reservoir of harmonic oscillators the coupling factor $|f_k|^2$ is a smooth function of ω_k, Eq. (81) leads to

$$\Gamma(\omega_\pm),\Gamma(\omega_\pm\pm2ig)\simeq\Gamma(\omega_0)=\Gamma. \tag{87}$$

We observe that the zeros of the determinant of the set, Eqs. (74), i.e.,

$$\tilde{s}_{1,2}=\frac{\Gamma}{2}, \quad \tilde{s}_{3,4}=\pm 2ig-\frac{\Gamma}{2}, \tag{88}$$

are the same as those of the set of equations (76). It is easy then to invert the Laplace transform and find the solutions in time. For example, if initially the atom was in the state $|3\rangle$, we have

$$\sigma_{(++)}(t)=\sigma_{(--)}(t)=\frac{1}{2}e^{-\Gamma t/2}\left[1-\frac{\Gamma}{4g}\sinh(2gt)\right],$$

$$\sigma_{(+-)}(t)=\frac{1}{2}e^{-\Gamma t/2}\left[\frac{\Gamma}{4ig}(1-e^{2igt})-e^{2igt}\right], \tag{89}$$

which yield also for the bare states

$$\sigma_{(33)}(t)=\frac{1}{2}e^{-\Gamma t/2}\left\{1+\cos(2gt)+\frac{\Gamma}{g}\sin(2gt)\right\},$$

$$\sigma_{(22)}(t)=\frac{1}{2}e^{-\Gamma t/2}\{1-\cos(2gt)\}. \tag{90}$$

B. Weak-coupling limit

Next, we consider the limit when the intrinsic coupling constant g is much smaller than the damping rate, Γ: $g \ll \Gamma$. In this weak-coupling limit, since $g \ll \bar{G}_1, \bar{G}_2$, we expand the determinant $D(\tilde{s})$ with respect to g^2, to get

$$D(\tilde{s})=\left(\tilde{s}+\frac{4g^2}{\bar{G}_1+\bar{G}_2}\right)\left(\tilde{s}+\frac{3(\bar{G}_1+\bar{G}_2)-\Delta}{4}\right) \times\left(\tilde{s}+\frac{3(\bar{G}_1+\bar{G}_2)+\Delta}{4}\right)\left(\tilde{s}+\frac{\bar{G}_1+\bar{G}_2}{2}\right), \tag{91}$$

where

$$\Delta^2=9\bar{G}_1^2+9\bar{G}_2^2-14\bar{G}_1\bar{G}_2. \tag{92}$$

Repeating the same arguments of the strong-coupling limit, we find the following zeros of the determinant of the set Eqs. (74), i.e.,

$$\tilde{s}_{1,2}=-\frac{\Gamma}{2}, \quad \tilde{s}_3=-\Gamma, \quad \tilde{s}_4=-\frac{4g^2}{\Gamma}, \tag{93}$$

which are also the zeros of the set, Eqs. (76), in the weak-coupling case. Here, for the same initial conditions as previously, the solutions in time are

$$\sigma_{(++)}(t)=\sigma_{(--)}(t) =\frac{1}{2}e^{-4g^2t/\Gamma}+\frac{4g^2}{\Gamma}(e^{-4g^2t/\Gamma}+e^{-\Gamma t}-2e^{-\Gamma t/2}),$$

$$\sigma_{(+-)}(t)=\frac{2ig}{\Gamma}(e^{-\Gamma t/2}-e^{-4g^2t/\Gamma})-\frac{1}{2}e^{-4g^2t/\Gamma}, \tag{94}$$

and also

$$\sigma_{(33)}(t)=e^{-4g^2t/\Gamma}+\frac{4g^2}{\Gamma^2}\{\ e^{-4g^2t/\Gamma}+e^{-\Gamma t}-2e^{-\Gamma t/2}\},$$

$$\sigma_{(22)}(t)=\frac{4g^2}{\Gamma^2}\{\ e^{-4g^2t/\Gamma}+e^{-\Gamma t}-2e^{-\Gamma t/2}\}. \tag{95}$$

In passing, for completeness of presentation, we compare these two results for the dependences of the populations $\sigma_{(33)}(t)$ and $\sigma_{(22)}(t)$ of Eq. (90), when $g \gg \Gamma$, as depicted in Fig. 1, with that of Eqs. (95), when $g \ll \Gamma$, as depicted in Fig. 2. We observe, as it is well known [3], that when the coupling g, between the degenerate states, is much larger than the effective width of the line Γ, which is induced by the reservoir, the state $|3\rangle$, though not directly coupled to the radiation bath, decays on the same time scale $T_R=2/\Gamma$, as the state $|2\rangle$ does, while the population oscillate, with frequency $2g$, between the two excited states. On the other hand, when the state $|3\rangle$ is weakly coupled to state $|2\rangle$, it decays on a time scale much longer than T_R, by a factor of $(1/8)(\Gamma/g)^2$, and the two levels decay directly without any oscillations.

C. General solution

Now, we show that it is legitimate to interpolate between these two extreme limits of $g \gg \Gamma$, and $g \ll \Gamma$. We observe that since in general, the zeros $\tilde{s}$ of determinant $D(\tilde{s})$ of Eq. (78) are of the order of g or Γ, we can replace both $\bar{G}_1(\tilde{s})$ and $\bar{G}_2(\tilde{s})$ by $\Gamma/2$. We then find that the zeros of the determinant of the set, Eqs. (74), are

$$\tilde{s}_{1,2}=-\frac{\Gamma}{2}, \quad \tilde{s}_{3,4}=\frac{-\Gamma\pm\Omega}{2}, \tag{96}$$

where $\Omega=\sqrt{\Gamma^2-(4g)^2}$. The solutions in time for the previous initial conditions are

$$\sigma_{(-+)}(t)=e^{-\Gamma t/2}\left[\left(\frac{2ig\Gamma}{\Omega^2}-\frac{1}{2}\right)\cosh\left(\frac{\Omega t}{2}\right) +\left(\frac{2ig}{\Omega}-\frac{\Gamma}{2\Omega}\right)\sinh\left(\frac{\Omega t}{2}\right)-\frac{2ig\Gamma}{\Omega^2}\right],$$

$$\sigma_{(++)}(t)=\sigma_{(--)}(t) =e^{-\Gamma t/2}\left[\frac{\Gamma^2}{2\Omega^2}\cosh\left(\frac{\Omega t}{2}\right)+\frac{\Gamma}{2\Omega}\sinh\left(\frac{\Omega t}{2}\right)-\frac{8g^2}{\Omega^2}\right], \tag{97}$$

and

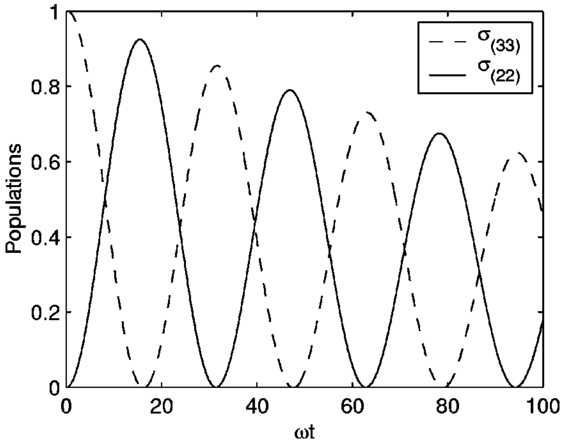

FIG. 1. Original states populations $\sigma_{(33)}$ and $\sigma_{(22)}$ vs scaled time ωt at a zero-temperature reservoir in the limit of strong coupling. The parameters g/ω and Γ/ω are chosen to be 0.1 and 0.01, respectively.

$$\sigma_{(33)}(t)=e^{-\Gamma t/2}\left\{\frac{1}{2}[\cosh(\Omega t/2)+1]+\frac{\Gamma^2}{2\Omega^2}[\cosh(\Omega t/2)-1]+\frac{\Gamma}{\Omega}\sinh(\Omega t/2)\right\},$$

$$\sigma_{(22)}(t)=\frac{8g^2}{\Omega^2}e^{-\Gamma t/2}[\cosh(\Omega t/2)-1]. \tag{98}$$

It is easy to show that these general results lead to the limiting cases of strong and week couplings. These solutions are identical to those obtained from the conventional set of equations, Eqs. (76). We conclude that in the case of zero-temperature reservoir, whose $\Gamma(\omega)$ of Eq. (81) is a flat function of frequency near ω_0, our "exact" derivation of the equations of motion for a composite system supports the *ad hoc* insertion of the relaxation term, Eq. (62), into Eq. (61).

VII. FINITE-TEMPERATURE RESERVOIR

The general case when our composite system is in contact with a finite-temperature reservoir is studied here. Before executing this program we compare the exact set of equations, Eqs. (36)–(39), with the *ad hoc* set, Eqs. (68)–(71). Our suspicion that the *ad hoc* scheme is not justified in this case is emanated from its failure to produce the proper steady state, and the detailed balance. The crucial differences seem to come due to the presence of the thermal functions $\overline{T}(s,\omega_\pm)$ of Eq. (42) in the exact equations, while the thermal function $\Gamma'(\omega_0)$ of Eq. (53) appears in the *ad hoc* equations. These are absent at zero-temperature limit. While at zero temperature, for a flat frequency reservoir, we had $\overline{G}(s\ll\omega_0,\omega_\pm)\simeq\Gamma(\omega_\pm)\simeq\Gamma(\omega_0)$, at finite temperatures the distribution functions $N(\omega)$ appear in the equations, and their dependence on the frequency cannot be discarded. We conclude that the *ad hoc* scheme *cannot* be applied to a composite system at finite temperatures.

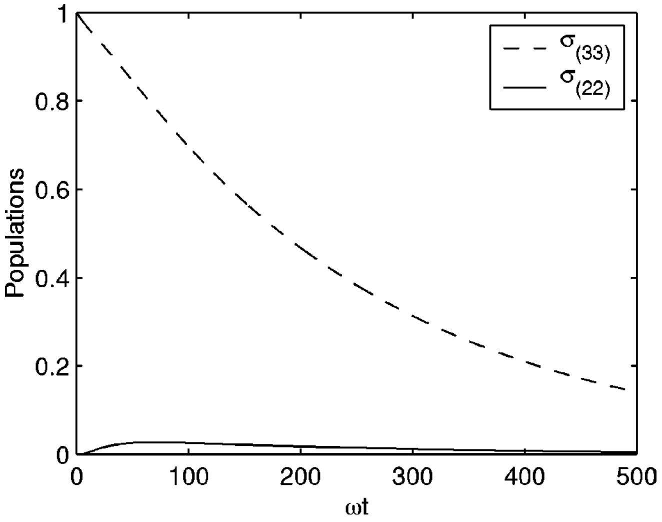

FIG. 2. Original states populations $\sigma_{(33)}$ and $\sigma_{(22)}$ vs scaled time ωt at a zero-temperature reservoir in the limit of weak coupling. The parameters g/ω and Γ/ω are chosen to be 0.01 and 0.1, respectively.

We turn now to the exact set of equations, at finite temperatures, namely, Eqs. (36)–(39), and examine the determinant of these equations. We express this fifth-order polynomial in s as

$$D(s)=s\{s(s+L_1)(s^2+sL_2+L_3)+4g^2(s^2+sL_4+L_5)\}, \tag{99}$$

where we have introduced

$$L_1=\frac{\overline{G}_1+\overline{T}_1+\overline{G}_2+\overline{T}_2}{2},$$

$$L_2=\frac{3\overline{G}_1+5\overline{T}_1+3\overline{G}_2+5\overline{T}_2}{2},$$

$$L_3=2\overline{G}_1\overline{G}_2+3\overline{G}_1\overline{T}_2+3\overline{G}_2\overline{T}_1+4\overline{T}_1\overline{T}_2,$$

$$L_4=\overline{G}_1+\overline{G}_2+2\overline{T}_1+2\overline{T}_2,$$

$$L_5=\overline{G}_1\overline{G}_2+2\overline{G}_1\overline{T}_2+2\overline{G}_2\overline{T}_1+3\overline{T}_1\overline{T}_2, \tag{100}$$

and seek again the zeros $\tilde{s}$ of the equation $D(\tilde{s})=0$. One zero is readily obtained by inspection, namely,

$$\tilde{s}_1=0, \tag{101}$$

which is responsible to the steady-state solutions in time. However, we cannot find the other zeros in the general case, and we will attempt solutions in the previous two limits: (i) the strong coupling and (ii) the weak coupling.

A. Strong-coupling limit

In this case, we consider the intrinsic coupling constant g to be much larger than all the damping rates of the system.

Since here zeros $\tilde{s}$, beside being extremely smaller relative to ω_0, are expected to be of the order of g, or the damping rates of the system, which are related to $\bar{G}_{1,2}(\tilde{s})$, and $\bar{T}_{1,2}(\tilde{s})$. The strong-coupling limit implies that $g \gg \bar{G}_1, \bar{G}_2, \bar{T}_1, \bar{T}_2$. Expanding $D(\tilde{s})$ with respect to $\bar{G}_1, \bar{G}_2, \bar{T}_1, \bar{T}_2$, we get

$$\tilde{s}_2 = -\left(\frac{\bar{G}_1(s)+\bar{G}_2(s)+2\bar{T}_1(s)+2\bar{T}_2(s)-\Delta(s)}{2}\right)_{(s\to 0)},$$

$$\tilde{s}_3 = -\left(\frac{\bar{G}_1(s)+\bar{G}_2(s)+2\bar{T}_1(s)+2\bar{T}_2(s)+\Delta(s)}{2}\right)_{(s\to 0)},$$

$$\tilde{s}_4 = 2ig-\left(\frac{\bar{G}_1(s)+\bar{G}_2(s)+\bar{T}_1(s)+\bar{T}_2(s)}{2}\right)_{(s\to 2ig)},$$

$$\tilde{s}_5 = -2ig-\left(\frac{\bar{G}_1(s)+\bar{G}_2(s)+\bar{T}_1(s)+\bar{T}_2(s)}{2}\right)_{(s\to -2ig)}, \tag{102}$$

where

$$\Delta^2 = [\bar{G}_1(s)-\bar{G}_2(s)+2\bar{T}_1(s)-2\bar{T}_2(s)]^2+4\bar{T}_1(s)\bar{T}_2(s). \tag{103}$$

Since $\Gamma(\omega)$ is not sensitive to a relevant range of frequencies around ω_0, we can replace the G functions with a constant, i.e.,

$$\bar{G}_1(\tilde{s}) = \bar{G}_2(\tilde{s}) = \Gamma/2. \tag{104}$$

This is not so for the T functions, which are sensitive to these frequencies through their dependence on $N(\omega)$. We have

$$\bar{T}_1(\tilde{s}) = \Gamma N_1/2, \quad \bar{T}_2(\tilde{s}) = \Gamma N_2/2, \tag{105}$$

where

$$N_1 = N(\omega_+), \quad N_2 = N(\omega_-), \tag{106}$$

are the mean photon numbers in the reservoir at the frequencies $\omega_+ = \omega + g$ and $\omega_- = \omega - g$, respectively. Having obtain all the zeros of the determinant, which are also the locations of the poles for the contour integrations in the inverse Laplace transformations, we can express the solutions for the populations and the coherences in time. We demonstrate some typical cases as depicted in Figs. 3 and 4, when initially only the state $|3\rangle$ is occupied. We observe that since in the strong coupling case g is dominant, oscillations of the populations in time occur at the beginning. We further notice that after long time, the populations are settled at their thermal equilibrium values.

B. Weak coupling limit

In a similar manner, we tackle the weak-coupling limit. Here, the damping rates are much larger than the intrinsic coupling g, or $g \ll \bar{G}_1, \bar{G}_2, \bar{T}_1, \bar{T}_2$. We expand $D(\tilde{s})$ in Eq. (99) with respect to g^2. First, we can be convinced that the $\bar{G}'s$ and $\bar{T}'s$ can be replaced as in Eqs. (104) and (105). The zeros are then expressed, in terms of the photon distribution of Eq. (106), as

$$\tilde{s}_2 = -\frac{16g^2(1+2N_1+2N_2+3N_2N_1)}{\Gamma(2+N_2+N_1)(2+3N_1+3N_2+4N_2N_1)},$$

$$\tilde{s}_3 = -\frac{\Gamma}{4}(2+N_1+N_2),$$

$$\tilde{s}_4 = -\frac{\Gamma}{8}(6+5N_1+5N_2+F),$$

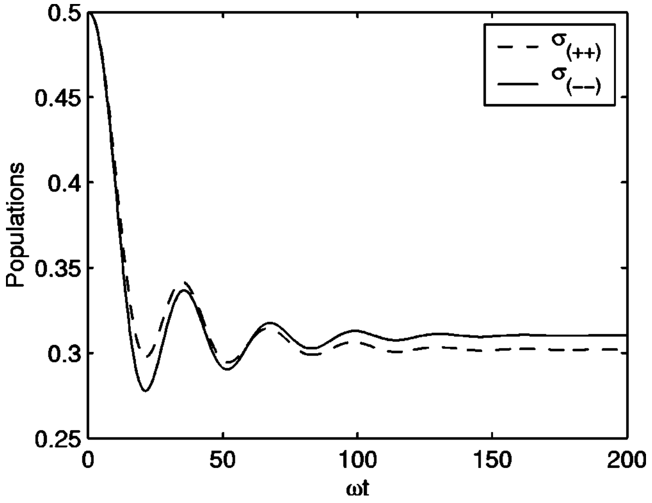

FIG. 3. Diagonal states populations $\sigma_{(++)}$ and $\sigma_{(--)}$ vs scaled time ωt at a finite-temperature reservoir in the limit of strong coupling. The mean photon numbers in the reservoir are $N_1 = 5$ and $N_2 = 6$. The parameters g/ω and Γ/ω are chosen to be 0.1 and 0.01, respectively.

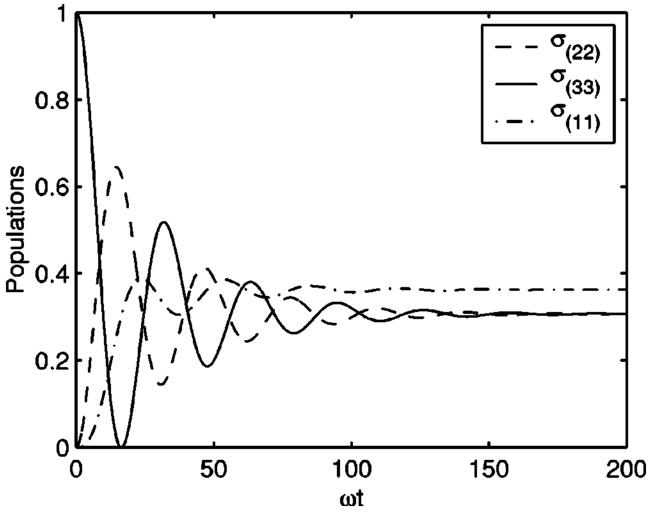

FIG. 4. Original states populations $\sigma_{(33)}$, $\sigma_{(22)}$, and $\sigma_{(11)}$ vs scaled time ωt at a finite-temperature reservoir in the limit of strong coupling. The mean photon numbers in the reservoir are $N_1 = 5$ and $N_2 = 6$. The parameters g/ω and Γ/ω are chosen to be 0.1 and 0.01, respectively.

$$\tilde{s}_5 = -\frac{\Gamma}{8}(6+5N_1+5N_2-F), \tag{107}$$

where

$$F^2 = 4+25N_1^2+25N_2^2+12N_1+12N_2-14N_2N_1. \tag{108}$$

We demonstrate some typical cases as depicted in Figs. 5 and 6. Since the coupling g is small, no oscillations of the populations are developed, and the latter relax directly to their equilibrium values.

VIII. MASTER EQUATION FOR A COMPOSITE SYSTEM

In the previous sections, we have studied the dynamics of a composite system in a reservoir, invoking only the factorization approximation, and making use of the Laplace transform, without using the Markoff ansatz, the coarse-grained procedure, and the secular approximation. The latter are the building blocks of the standard master equation. Instead of continuing from the integrodifferential equations of motion in time of Eq. (17), or rather the set, Eqs. (30)–(33), and invoking the approximations of the standard procedure, we have converted these equations into algebraic ones, i.e., Eqs. (36)–(39), by Laplace transform, and investigated the latter set. In the present section, we shall attempt to follow the standard procedure, as developed, e.g., in Ref. [3], in order to produce a master equation for the composite system, namely, to convert the integrodifferential equation into an approximate differential equation in time for the reduced density matrix $\sigma(t)$ of the coupled subsystems.

We now continue the review of the standard procedure, starting from Eq. (17), and then apply it to our three-level atom. First, Eq. (17) is written in the interaction picture where the reduced density matrix $\tilde{\sigma}(t)$ is expected to vary on relaxation time states $T_R \sim 1/\Gamma$, much larger than that of the reservoir correlation times τ_c. Then, a coarse-grained version of Eq. (17) is generated by taking the average in time of this equation, over an arbitrary time step, Δt which is confined by

$$\tau_c \ll \Delta t \ll T_R. \tag{109}$$

At this stage, the Markoff ansatz is invoked, and $\tilde{\sigma}(t')$, within the time integrations, are replaced by $\tilde{\sigma}(t)$, and can be taken out of the time integrals. We end up with the coarse-grained $\tilde{\sigma}(t)$ obeying the following differential equation:

$$\begin{aligned}\frac{d}{dt}\tilde{\sigma}_{ab}(t) = -\sum_{c,d} \int_0^\infty d\tau \, \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \big(\tilde{\sigma}_{db}(t)e^{i\omega_{ad}t'}\{S_{ac}^+S_{cd}^-[G(\omega_{bc},\tau)+G_T(\omega_{bc},\tau)]+S_{ac}^-S_{cd}^+G_T^*(\omega_{cb},\tau)\} \\ +\tilde{\sigma}_{ac}(t)e^{i\omega_{cb}t'}\{S_{cd}^+S_{db}^-[G^*(\omega_{ad},\tau)+G_T^*(\omega_{ad},\tau)]+S_{cd}^-S_{db}^+G_T(\omega_{da},\tau)\}-\tilde{\sigma}_{cd}(t)e^{i(\omega_{ac}+\omega_{db})t'} \\ \times\{S_{ac}^-S_{db}^+[G^*(\omega_{cb},\tau)+G_T^*(\omega_{cb},\tau)+G(\omega_{da},\tau)+G_T(\omega_{da},\tau)]+S_{ac}^+S_{db}^-[G_T^*(\omega_{ad},\tau)+G_T(\omega_{bc},\tau)]\}\big),\end{aligned} \tag{110}$$

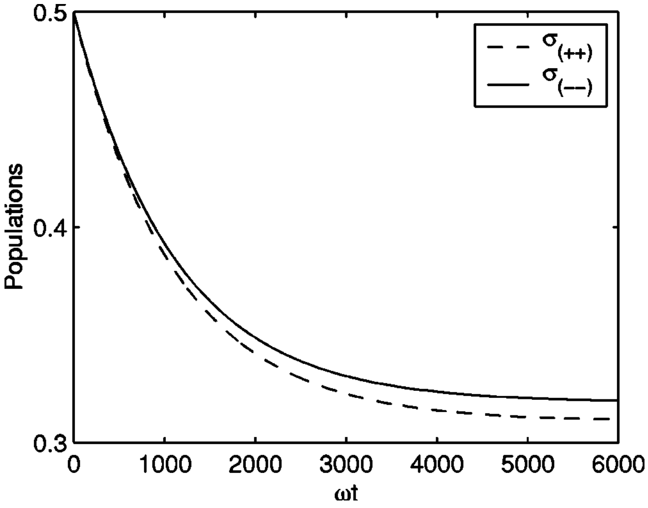

FIG. 5. Diagonal states populations $\sigma_{(++)}$ and $\sigma_{(--)}$ vs scaled time ωt at a finite-temperature reservoir in the limit of weak coupling. The mean photon numbers in the reservoir are $N_1=5$ and $N_2=6$. The parameters g/ω and Γ/ω are chosen to be 0.01 and 0.1, respectively.

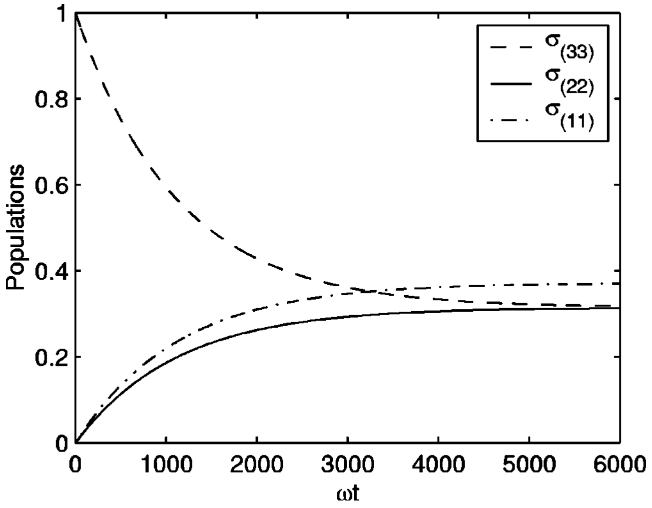

FIG. 6. Original states populations $\sigma_{(33)}$, $\sigma_{(22)}$, and $\sigma_{(11)}$ vs scaled time ωt at a finite-temperature reservoir in the limit of weak coupling. The mean photon numbers in the reservoir are $N_1=5$ and $N_2=6$. The parameters g/ω and Γ/ω are chosen to be 0.01 and 0.1, respectively.

where the correlation functions are given by Eq. (18). Only three time scales were involved in the derivation up to here, as stated by Eq. (109). However, in a composite system, with the coupling Hamiltonian of Eq. (1) between the components of the system, there is another, an intrinsic time scale of the order of $\sim\hbar/\langle H_{AC}\rangle$, presents in the system. This time scale is bound to introduce a dilemma of how to relate it to the other ones, and, in particular, to the coarse-grained time Δt. It may further cast a doubt on the applicability of the standard procedure to a composite system.

We return now to our toy model of the three-level atom, and apply the coarse-grained procedure, and the Markoff ansatz, to the set, Eqs. (30)–(33). We obtain the following.

(i) From Eq. (30), we have

$$\frac{d}{dt}\tilde{\sigma}_{(++)}(t)=-\frac{1}{2}[\Gamma(\omega_+)+\Gamma'(\omega_+)]\tilde{\sigma}_{(++)}(t)+\frac{1}{2}\Gamma'(\omega_+)\tilde{\sigma}_{(11)}(t)-\frac{1}{\Delta t}\int_t^{t+\Delta t}dt'\frac{1}{4}\times[\Gamma(\omega_-)+\Gamma'(\omega_-)][e^{2igt'}\tilde{\sigma}_{(-+)}(t)+e^{-2igt'}\tilde{\sigma}_{(+-)}(t)]. \tag{111}$$

(ii) Equation (31) is turned into

$$\frac{d}{dt}\tilde{\sigma}_{(--)}(t)=-\frac{1}{2}[\Gamma(\omega_-)+\Gamma'(\omega_-)]\tilde{\sigma}_{(--)}(t)+\frac{1}{2}\Gamma'(\omega_-)\tilde{\sigma}_{(11)}(t)-\frac{1}{\Delta t}\int_t^{t+\Delta t}dt'\frac{1}{4}\times[\Gamma(\omega_+)+\Gamma'(\omega_+)][e^{2igt'}\tilde{\sigma}_{(-+)}(t)+e^{-2igt'}\tilde{\sigma}_{(+-)}(t)]. \tag{112}$$

(iii) Equation (32) is expressed as

$$\frac{d}{dt}\tilde{\sigma}_{(+-)}(t)=-\frac{1}{4}[\Gamma(\omega_-)+\Gamma(\omega_+)+\Gamma'(\omega_-)+\Gamma'(\omega_+)]\tilde{\sigma}_{(+-)}(t)-\frac{1}{\Delta t}\int_t^{t+\Delta t}dt'\frac{1}{4}e^{2igt'}\{[\Gamma(\omega_-)+\Gamma'(\omega_-)]\tilde{\sigma}_{(--)}(t)+[\Gamma(\omega_+)+\Gamma'(\omega_+)]\tilde{\sigma}_{(++)}(t)-[\Gamma'(\omega_+)+\Gamma'(\omega_-)]\sigma_{(11)}(t)\}. \tag{113}$$

(iv) Finally, Eq. (33) is converted into

$$\frac{d}{dt}\tilde{\sigma}_{(11)}(t)=-\frac{1}{2}[\Gamma'(\omega_+)+\Gamma'(\omega_-)]\tilde{\sigma}_{(11)}(t)+\frac{1}{2}[\Gamma(\omega_+)+\Gamma'(\omega_+)]\tilde{\sigma}_{(++)}(t)+\frac{1}{2}[\Gamma(\omega_-)+\Gamma'(\omega_-)]\tilde{\sigma}_{(--)}(t)+\frac{1}{\Delta t}\int_t^{t+\Delta t}dt'\frac{1}{4}[\Gamma(\omega_-)+\Gamma(\omega_+)+\Gamma'(\omega_-)+\Gamma'(\omega_+)][e^{2igt'}\tilde{\sigma}_{(-+)}(t)+e^{-2igt'}\tilde{\sigma}_{(+-)}(t)]. \tag{114}$$

Here $\Gamma(\omega)$ is defined in Eq. (52), and $\Gamma'(\omega)=\Gamma(\omega)N(\omega)$. We note that the coarse-grained averaging is boiled down to integration in time t' over the oscillatory term of frequency $2g$, i.e.,

$$I=\frac{1}{\Delta t}\int_t^{t+\Delta t}dt'\,e^{2igt'}=e^{2igt}e^{ig\Delta t}\frac{\sin(g\Delta t)}{g\Delta t}, \tag{115}$$

and this, in general, is not defined. The reason is that the arbitrary time Δt is prescribed to be bound by $\tau_c\ll\Delta t\ll T_R$, see Eq. (109), and it is unrelated to the intrinsic time scale $1/g$. Only in the weak-coupling case, i.e., when $g\ll\Gamma$, since then $T_R\ll 1/g$ we have $g\Delta t\ll 1$, and I of Eq. (115), if is well defined, and replaced by e^{2igt}. In this case, the coarse-grained equations, with the Markoff ansatz, conform with the exact equations in the weak-coupling limit. However, this is as far as we can go, and when g and Γ are not related by $g\ll\Gamma$, the standard procedure is at a dead end, I of Eq. (115), is *not* well defined, and the procedure does not produce a master equation for the composite system. Note that we cannot use to the secular approximation to save the scheme.

IX. CONCLUSION

The present paper has dealt with the dynamics of a composite small system, made up of two coupled subsystems, when only one of them is interacting with the environment. A set of equations of motion, which we coin for short the exact set, for the reduced density matrix of the composite system has been derived using only the factorization approximation, without employing the Markoff ansatz, the coarse-grained averaging, and the secular approximation. The latter are the building blocks of the standard procedure, which leads to the master equation. The conventional *ad hoc* approach, which assumes that the relaxation terms of the equations of motion of the composite system can be borrowed from the master equation of the subsystem interacting with the reservoir, has been compared with the exact approach. It has been demonstrated that when the reservoir is at zero temperature, the *ad hoc* scheme is applicable, however, at finite temperatures it fails to predict the dynamics of the system. It has been further shown that at finite temperatures, the standard procedure does not yield a master equation for the composite system, and one has to appeal to the exact scheme, with the Laplace transforms, to study the dynamics of the system.

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[13] For convenience, we use the word *exact* though the factorization aproximation is invoked.

[14] See, e.g., *Atom-Photon Interactions* (Ref. [3]), pp. 427–428.