

Amplitude and phase representation of quantum invariants for the time-dependent harmonic oscillator

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The correspondence between classical and quantum invariants is established. The Ermakov-Lewis quantum invariant of the time-dependent harmonic oscillator is translated from the coordinate and momentum operators into amplitude and phase operators. In doing so, Turski's phase operator as well as Susskind-Glogower operators are generalized to the time-dependent harmonic-oscillator case. A quantum derivation of the Manley-Rowe relations is shown as an example.

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I. INTRODUCTION

Exact invariants have been extensively used to solve the time-dependent Schrödinger equation [1]. Various related invariants have been obtained for the quantum-mechanical time-dependent harmonic-oscillator (QM-TDHO) equation in one dimension. The Ermakov-Lewis invariant and orthogonal functions invariant are two such constants of motion that have been used to solve the QM-TDHO problem. The Ermakov-Lewis invariant is usually expressed in terms of coordinate and momentum operators although it has also been expressed in terms of raising and lowering operators that lead to number states for wave functions that are eigenstates of the invariant operator [2]. However, the amplitude operator stemming from this procedure does not correspond to the amplitude of the oscillator in the classical limit.

The purpose of this paper is to translate the invariant formalism from the coordinate and momentum operators into an invariant in terms of amplitude and phase operators that reduce to the corresponding variables in the classical limit. In the Sec. II, the solution to the QM-TDHO equation is stated using the square of the orthogonal functions invariant and the Ermakov invariant. In the following section, a second linear Hermitian invariant is introduced and the Ermakov-Lewis invariant is economically obtained from these two constants of motion. Two distinct annihilation and creation operators are presented in Sec. IV and their equations of motion are established. In Sec. V, the quantum phase is defined using the Turski and Susskind and Glogower formalisms. The former definition is shown to yield an amplitude and phase representation that is consistent with the classical limit. In Sec. VI, the Ermakov-Lewis invariant is written in amplitude and phase variables. The energy conservation and photon number relations in nonlinear optical processes are shown as an example.

II. EVOLUTION OPERATORS AND INVARIANTS

Consider the time-dependent Schrödinger equation with $\hbar = 1$,

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle. \quad (1)$$

The solution to this equation for a time-independent Hamiltonian is formally given by $|\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle$, where $\hat{U}(t)$ is the evolution operator $\hat{U}(t) = \exp(-i\hat{H}t)$. For the time-dependent harmonic-oscillator Hamiltonian

$$\hat{H}(t) = \frac{1}{2} [\hat{p}^2 + \Omega^2(t) \hat{q}^2], \quad (2)$$

the solution may be written in terms of a propagator that involves a time-independent operator together with an appropriate transformation of the wave function

$$|\psi(t)\rangle = \hat{U}_I \hat{T}^\dagger \hat{T}(0) |\psi(0)\rangle. \quad (3)$$

The propagator is given by

$$\hat{U}_I = \exp(-i s_\alpha \hat{I}_\alpha), \quad s_\alpha \equiv \int_0^t \frac{dt'}{\alpha^2}, \quad (4)$$

where the function s_α is a time-dependent c number and α satisfies a differential equation appropriate to the invariant being used. The transformation is defined as

$$\begin{aligned} \hat{T} &= \exp\left(i \frac{\ln \alpha}{2} \frac{d\hat{q}^2}{dt}\right) \exp\left(-i \frac{d \ln \alpha}{dt} \frac{\hat{q}^2}{2}\right) \\ &= \exp\left(i \frac{\ln \alpha}{2} (\hat{q}\hat{p} + \hat{p}\hat{q})\right) \exp\left(-i \frac{\dot{\alpha}}{2\alpha} \hat{q}^2\right). \end{aligned} \quad (5)$$

The time-independent operator in the propagator is an invariant that is not unique [3]. On the one hand, it may be proportional to the square of the orthogonal functions invariant operator

$$\hat{I}_u = \frac{1}{2} (u\hat{p} - \dot{u}\hat{q})^2, \quad (6)$$

where the function $\alpha \rightarrow u \in \mathbb{R}$, replaced in the invariant as well as in the transformation expressions, obeys the TDHO equation

$$\ddot{u} + \Omega^2(t)u = 0. \quad (7)$$

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On the other hand, the propagator may be written using the Ermakov-Lewis invariant with $\alpha \rightarrow \rho$, where ρ obeys the Ermakov equation

$$\ddot{\rho} + \Omega^2(t)\rho = \rho^{-3}. \quad (8)$$

In either case, it is seen that the invariant in the time-dependent case enters the propagator expression in an analogous fashion as the Hamiltonian does in the time-independent case.

III. CLASSICAL AND QUANTUM INVARIANTS

The classical orthogonal functions invariant is

$$G = u_1 \dot{u}_2 - u_2 \dot{u}_1, \quad (9)$$

where u_1 and u_2 are real linearly independent solutions of the TDHO equation [4]. The constancy of this quantity is readily obtained from the evaluation of its time derivative $\dot{G} = u_1 \ddot{u}_2 - u_2 \ddot{u}_1 = 0$. The quantum invariant arising from the mapping of u_2 and \dot{u}_2 into the coordinate and momentum operators is

$$\hat{G}_1 = u_1 \hat{p} - \dot{u}_1 \hat{q}. \quad (10)$$

The time derivative of this operator is

$$\frac{d\hat{G}_1}{dt} = \frac{\partial \hat{G}_1}{\partial t} - \frac{i}{\hbar} [\hat{G}_1, \hat{H}(t)] = (\ddot{u} + \Omega^2 u) \hat{q} = 0, \quad (11)$$

thus confirming its invariance. The obtention of a second invariant given a first invariant has been a subject of several communications [5,6]. It is worth remarking that the existence of a second invariant warrants complete integrability for a Hamiltonian-Ermakov system [7]. Within the present formalism, it is straightforward to introduce a second invariant stemming from the mapping of u_1 and \dot{u}_1 into the coordinate and momentum operators,

$$\hat{G}_2 = -u_2 \hat{p} + \dot{u}_2 \hat{q}. \quad (12)$$

These two invariants obey the commutation relation $[\hat{G}_1, \hat{G}_2] = -iG$, where G is the constant given by Eq. (9). From the sum of their squares, we may construct the invariant operator

$$\hat{I} = \frac{1}{2}(\hat{G}_1^2 + \hat{G}_2^2), \quad (13)$$

which in terms of the position and momentum operators is

$$\begin{aligned} \hat{G}_1^2 + \hat{G}_2^2 &= (u_1 \hat{p} - \dot{u}_1 \hat{q})^2 + (-u_2 \hat{p} + \dot{u}_2 \hat{q})^2 \\ &= (u_1^2 + u_2^2) \hat{p}^2 + (\dot{u}_1^2 + \dot{u}_2^2) \hat{q}^2 \\ &\quad - (u_1 \dot{u}_1 + u_2 \dot{u}_2) (\hat{p} \hat{q} + \hat{q} \hat{p}). \end{aligned} \quad (14)$$

We may rewrite this expression as a function of an amplitude function $\rho = \sqrt{u_1^2 + u_2^2}$, by noticing that $\dot{\rho} = u_1 \dot{u}_1 + u_2 \dot{u}_2$ and that the orthogonal functions obey Eq. (9), so that

$$\frac{G^2}{\rho^2} + \dot{\rho}^2 = \frac{(u_1 \dot{u}_2 - u_2 \dot{u}_1)^2 + (u_1 \dot{u}_1 + u_2 \dot{u}_2)^2}{\rho^2} = (\dot{u}_1^2 + \dot{u}_2^2). \quad (15)$$

The operator in terms of ρ is then

$$\hat{I} = \frac{1}{2} \left[\left(\frac{G \hat{q}}{\rho} \right)^2 + (\rho \hat{p} - \dot{\rho} \hat{q})^2 \right] = \hat{I}_\rho, \quad (16)$$

but this is precisely the Ermakov-Lewis invariant where the real constant G is usually normalized to unity but may, in general, be different from one [8]. The above procedure is a simple derivation of the quantum Ermakov-Lewis invariant, which has otherwise been obtained using rather more complex mathematical methods [9]. The non-Hermitian linear invariant \hat{I}_c introduced by Malkin *et al.* written in terms of the orthogonal functions invariants is $\hat{I}_c = \hat{G}_1 - i\hat{G}_2$.

In the classical case, the amplitude and phase representation of the invariant is straightforward. If the complex coordinate variable \tilde{u} is expressed in polar coordinates,

$$\tilde{u} = \rho e^{is_\rho} + \sigma \rho e^{-is_\rho}, \quad (17)$$

where σ is a constant, the orthogonal solutions may be written as $u_1 = -(1 - \sigma)\rho \sin(s_\rho)$ and $u_2 = (1 + \sigma)\rho \cos(s_\rho)$. Substitution of these variables in the classical orthogonal functions expression (9) yields the invariant in terms of amplitude and phase variables,

$$G/(1 - \sigma^2) = \rho^2 \dot{s}_\rho. \quad (18)$$

The constant $G/(1 - \sigma^2)$ may be normalized to 1 and the derivative of the phase written as the frequency $\omega(t) \equiv \dot{s}_\rho$; the squared amplitude times the frequency then obeys the relationship

$$\rho^2(t) \omega(t) = 1. \quad (19)$$

The energy of a time-independent oscillator is proportional to the squares of the momentum and coordinate variables $\mathcal{E} \propto p^2 + \omega_0^2 q^2$, which in terms of the amplitude and phase variables is $\mathcal{E} \propto \rho_0^2 \omega_0^2$. In the adiabatic approximation, this relationship is considered to hold even in the time-dependent case, i.e., $\mathcal{E} \propto \rho^2(t) \omega^2(t)$. The orthogonal functions exact invariant is then proportional to the ratio of the energy over the frequency $G/(1 - \sigma^2) \propto \mathcal{E}(t)/\omega(t)$ thus yielding the well-known adiabatic invariant [10]. Nonetheless, the exact invariant expression (18) does not rely on this approximation.

However, the quantum versions of this invariant produce a linear form in the coordinate and momentum operators as seen in Eqs. (10) and (12). It therefore comes to no surprise that the argument of the propagator is proportional to the

square of the orthogonal functions quantum invariant (6). On the other hand, the classical Ermakov-Lewis invariant in the amplitude and phase representation follows from the substitution $\hat{q} \rightarrow \rho \cos s_\rho$, $\hat{p} \rightarrow d\hat{q}/dt$ [8]:

$$I = \frac{1}{2} \rho^4 s_\rho^2 \quad (20)$$

that implies a quadratic dependence on the energy of the oscillator. Therefore, a quantum invariant with a quadratic dependence on the coordinate and momentum variables should be in correspondence with the classical orthogonal functions invariant.

IV. CREATION AND ANNIHILATION OPERATORS

An operator that can be written as the sum of two squares may be expressed in terms of two adjoint complex quantities. To wit, given an operator $\hat{\beta}$ that can be expressed as

$$\hat{\beta} = \hat{b}_1^2 + \hat{b}_2^2, \quad (21)$$

provided $[\hat{b}_1, \hat{b}_2] = c$, with c a c -number, there exist annihilation and creation operators $\hat{b} = \hat{b}_1 + i\hat{b}_2$, $\hat{b}^\dagger = \hat{b}_1 - i\hat{b}_2$ so that the operator may be written as $\hat{\beta} = \hat{b}^\dagger \hat{b} - i[\hat{b}_1, \hat{b}_2]$. For instance, annihilation and creation operators for the Hamiltonian (2) may be written as [11]

$$\begin{aligned} \hat{B} &= \frac{1}{\sqrt{2}} [\Omega^{1/2}(t) \hat{q} + i \hat{p} / \Omega^{1/2}(t)], \\ \hat{B}^\dagger &= \frac{1}{\sqrt{2}} [\Omega^{1/2}(t) \hat{q} - i \hat{p} / \Omega^{1/2}(t)]. \end{aligned} \quad (22)$$

However, the way in which the $\hat{\beta}$ operator is written as the sum of two squares need not be unique. In fact, for the invariant operator defined in the preceding section, expressions (13) and (16) are two such possibilities. The former leads to annihilation and creation operators of the form

$$\hat{A} = \frac{1}{\sqrt{2}} (\hat{G}_1 - i\hat{G}_2), \quad \hat{A}^\dagger = \frac{1}{\sqrt{2}} (\hat{G}_1 + i\hat{G}_2), \quad (23)$$

where the identifications $\hat{b}_1 \rightarrow \hat{G}_1$ and $\hat{b}_2 \rightarrow -\hat{G}_2$ have been made. These operators may also be obtained from the non-Hermitian linear invariant, which arises from the complex solution of the TDHO equation [12,13]. The sign in the imaginary part of the above expressions is introduced in order to have consistency with the cited results. These annihilation and creation operators are also invariant since they are composed of invariant operators.

On the other hand, the operators arising from Eq. (16) yield

$$\hat{a}(t) = \frac{1}{\sqrt{2}} \left(\frac{\hat{q}}{\rho} + i(\rho \hat{p} - \dot{\rho} \hat{q}) \right),$$

$$\hat{a}^\dagger(t) = \frac{1}{\sqrt{2}} \left(\frac{\hat{q}}{\rho} - i(\rho \hat{p} - \dot{\rho} \hat{q}) \right). \quad (24)$$

These time-dependent annihilation and creation operators were originally introduced by Lewis [2]. The Ermakov invariant in terms of these operators is

$$\hat{I} = \hat{a}^\dagger(t) \hat{a}(t) + \frac{1}{2} = \hat{A}^\dagger \hat{A} + \frac{1}{2}, \quad (25)$$

where the second equality follows from the definition of this invariant in terms of the orthogonal functions quantum invariants (13). In order to obtain the transformation between the distinct annihilation and creation operators, evaluate

$$e^{is_\rho \hat{A}} \hat{A} e^{-is_\rho \hat{A}} = \hat{A} e^{-is_\rho} = \frac{1}{\sqrt{2\rho}} (\hat{G}_1 - i\hat{G}_2) (u_2 + iu_1), \quad (26)$$

where $u_1 = -\rho \sin s_\rho$, $u_2 = \rho \cos s_\rho$. The relationship between the orthogonal functions and their trigonometric representation is not unique. This choice represents the function u_1 leading u_2 by $\pi/2$ as s_ρ increases [10]. Replacing the definitions of the invariants yields

$$\begin{aligned} \hat{A} e^{-is_\rho} &= \frac{1}{\sqrt{2}} \left[\left(\frac{u_1 \dot{u}_2 - \dot{u}_1 u_2}{\rho} \right) \hat{q} + i \left(\frac{u_1^2 + u_2^2}{\rho} \right) \hat{p} \right. \\ &\quad \left. - i \left(\frac{u_1 \dot{u}_1 + u_2 \dot{u}_2}{\rho} \right) \hat{q} \right], \end{aligned} \quad (27)$$

which simplifies to

$$\hat{A} e^{-is_\rho} = \frac{1}{\sqrt{2}} \left(\frac{G\hat{q}}{\rho} + i(\rho \hat{p} - \dot{\rho} \hat{q}) \right) = \hat{a}. \quad (28)$$

Therefore the time-dependent annihilation (creation) operators may be written as the product of the time-independent annihilation (creation) operators times a phase that involves only a c -number function. This expression may be written as a unitary transformation of a phase shift,

$$\hat{a} = \exp(is_\rho \hat{I}) \hat{A} \exp(-is_\rho \hat{I}). \quad (29)$$

The equation of motion of this operator is then

$$\frac{d\hat{a}}{dt} = i\omega(t) [\hat{I}, \hat{a}]. \quad (30)$$

It is thus seen that the operator $\omega(t) \hat{I}$ in the QM-TDHO equation again plays the role that the Hamiltonian does in a time-independent harmonic-oscillator case. This assertion is consistent with the transformation that relates the invariant and the time-dependent Hamiltonian [3]:

$$\omega(t) \hat{I} = \hat{H}(t) - i \frac{\partial \hat{T}^\dagger}{\partial t} \hat{T}. \quad (31)$$

V. PHASE OPERATORS FOR THE TIME-DEPENDENT OSCILLATOR

As it is well known, different operators can be used to define the phase in quantum optics [14]. The invariant formalism will be applied here to the phase operator given by Turski and the exponential phase operators of Susskind and Glogower. In particular, the former operator will allow an appropriate translation of the classical amplitude-phase invariant into the quantum one.

A. Turski phase operator

By using the annihilation operator (24) the displacement operator can be written as $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, $\alpha = r\exp(i\theta)$. The vacuum state may then be displaced to obtain a coherent state $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$ and the phase operator introduced by Turski [15] is then generalized to the time-dependent case,

$$\hat{\Phi} = \int \theta|\alpha\rangle\langle\alpha|d^2\alpha. \quad (32)$$

This operator obeys the commutation relation $[\hat{\Phi}, \hat{I}] = -i$. In order to evaluate the time evolution of $\hat{\Phi}$, this operator can be written in terms of the invariant annihilation and creation operators using Eq. (29),

$$\hat{\Phi} = e^{is\rho\hat{I}} \left(\int \theta \hat{D}_A(\alpha) e^{-is\rho\hat{I}} |0\rangle\langle 0| e^{is\rho\hat{I}} \hat{D}_A^\dagger(\alpha) d^2\alpha \right) e^{-is\rho\hat{I}}, \quad (33)$$

where $\hat{D}_A(\alpha) = \exp(\alpha\hat{A}^\dagger - \alpha^*\hat{A})$. The invariant acting over the vacuum state is $\hat{I}|0\rangle = \frac{1}{2}|0\rangle$ and the phase is then

$$\hat{\Phi} = e^{is\rho\hat{I}} \left(\int \theta \hat{D}_A(\alpha) |0\rangle\langle 0| \hat{D}_A^\dagger(\alpha) d^2\alpha \right) e^{-is\rho\hat{I}}, \quad (34)$$

the time derivative of this expression yields the equation of motion for $\hat{\Phi}$:

$$\frac{d\hat{\Phi}}{dt} = i\omega(t)[\hat{I}, \hat{\Phi}] = -\omega(t). \quad (35)$$

The operator $\omega(t)\hat{I}$ once again takes the role of the Hamiltonian.

B. Susskind-Glogower operators

The generalization of the phase to the time-dependent case is also applicable using other formalisms. Consider, for example, the Susskind-Glogower operators [16] given by (see, for instance, Ref. [17])

$$\begin{aligned} \hat{V} &= \frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}} \hat{a} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|, \\ \hat{V}^\dagger &= \hat{a}^\dagger \frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}} = \sum_{n=0}^{\infty} |n+1\rangle\langle n|, \end{aligned} \quad (36)$$

where $|n\rangle$ is a number state, eigenstate of the invariant \hat{I} . The transformation $\hat{V}\hat{I}\hat{V}^\dagger = \hat{I} + 1$, although having discrete nature works as a shifter in the same way as \hat{q} and \hat{p} do: $\exp(i\alpha\hat{p})\hat{q}\exp(-i\alpha\hat{p}) = \hat{q} + \alpha$. The sine and cosine operators for the Susskind-Glogower operators,

$$\hat{C} = \frac{\hat{V} + \hat{V}^\dagger}{2}, \quad \hat{S} = \frac{\hat{V} - \hat{V}^\dagger}{2i}, \quad (37)$$

give the commutation relations $[\hat{I}, \hat{C}] = -i\hat{S}$, $[\hat{I}, \hat{S}] = i\hat{C}$. Following the same treatment as above, i.e., expressing operators that depend on \hat{a} and \hat{a}^\dagger in terms of the invariants \hat{A} , \hat{A}^\dagger , and \hat{I} , the equations of motion for the sine and cosine operators are

$$\frac{d\hat{C}}{dt} = i\omega(t)[\hat{I}, \hat{C}] = \omega(t)\hat{S}, \quad \frac{d\hat{S}}{dt} = i\omega(t)[\hat{I}, \hat{S}] = -\omega(t)\hat{C}. \quad (38)$$

VI. AMPLITUDE AND PHASE REPRESENTATION OF INVARIANT

The coordinate operator from Eq. (24) is

$$\hat{q} = \frac{1}{\sqrt{2\omega(t)}}(\hat{a} + \hat{a}^\dagger), \quad (39)$$

and following Dirac [18] the creation and annihilation operators may be written as

$$\hat{a} = \sqrt{\hat{I}}e^{-i\hat{\Phi}}, \quad \hat{a}^\dagger = e^{i\hat{\Phi}}\sqrt{\hat{I}}. \quad (40)$$

The coordinate operator (39) in the form of amplitude and phase variables is then

$$\hat{q} = \sqrt{\frac{\hat{I}}{2\omega(t)}}e^{-i\hat{\Phi}} + e^{i\hat{\Phi}}\sqrt{\frac{\hat{I}}{2\omega(t)}}, \quad (41)$$

where the amplitude ρ and phase s_ρ are identified as

$$\rho \rightarrow \sqrt{\frac{\hat{I}}{\omega(t)}}, \quad s_\rho \rightarrow \hat{\Phi}. \quad (42)$$

The invariant with the aid of Eq. (35) is given in amplitude and phase operators as

$$\hat{I} = -\frac{\hat{a}^\dagger\hat{a} + \frac{1}{2}}{\omega(t)} \frac{d\hat{\Phi}}{dt}, \quad (43)$$

which has the same structure of the orthogonal functions classical invariant written in amplitude and phase variables (18). The number operator is then identified with

$$\hat{n}(t) = \frac{\hat{a}^\dagger\hat{a}}{\omega(t)}. \quad (44)$$

It should be recalled that the number operator is identified with the product of the annihilation and creation pair when these operators arise from the Hamiltonian operator; namely, $\hat{B}^\dagger\hat{B}$ given by Eq. (22). However, the identification of the

number operator in terms of the annihilation-creation pair arising from the invariant is obtained from the correspondence with the orthogonal functions classical invariant. This association may seem dimensionally awkward but it should be remembered that the invariant initial value was normalized to 1 [Eq. (19)]. The explicit introduction of the normalization factor $\rho_0^2 \omega_0$ makes, of course, a dimensionless photon number $\rho^2(t) = \rho_0^2 \omega_0 / \omega(t)$ for a dimensionless amplitude (usually set to unity, $\rho_0 = 1$). The invariant is then

$$\hat{I} = - \left(\hat{n}(t) + \frac{1}{2} \frac{\omega_0}{\omega(t)} \right) \frac{d\hat{\Phi}}{dt}(t). \quad (45)$$

Since $\hat{a}^\dagger \hat{a}$ is invariant from Eq. (25), if the frequency is constant the number of excitations is then also constant. Nonetheless, in the time-dependent case, the number of excitations is inversely proportional to the time-dependent frequency in correspondence with the intensity dependence obtained in the classical limit.

The energy of the excitation at a given time t_s is given by $\mathcal{E} = \hat{n}(t_s) \omega(t_s)$ (with $\hbar = 1$) but this is precisely the quantum invariant value above the vacuum state $\hat{I} = -\hat{n}(t_s)(d\hat{\Phi}/dt) \times (t_s)$. Therefore, the invariant represents the energy conservation of the closed system. In contrast, the time-dependent Hamiltonian (2) is no longer a constant of motion whose eigenvalue is necessarily related to an open system. Consider, as an example of this formalism, the number of excitations to represent the photon number. Allow for a nonlinear process where the field experiences second-harmonic generation. Let the photon number at time t_1 be \hat{n}_1 when the frequency mode is ω_1 and allow it to evolve at a time t_2 to the mode $\omega_2 = 2\omega_1$. The number of photons in the mode ω_2 is then from the invariant relationship (45),

$$\hat{n}_2 = \hat{n}_1 \frac{\omega_1}{\omega_2} = \frac{1}{2} \hat{n}_1. \quad (46)$$

This scheme corresponds to a Lagrangian hydrodynamic framework where a given volume is being followed along its propagating path. The power density, defined as the energy per unit time $\mathcal{W} = \mathcal{E}/\tau = \mathcal{E}\omega$, at these two times is then given by

$$\frac{\mathcal{W}_1}{\omega_1} = \frac{\mathcal{W}_2}{\omega_2}. \quad (47)$$

This reasoning may be extended to an arbitrary number of modes leading to other nonlinear processes such as parametric amplification or frequency difference. These type of equations are known in nonlinear optics as Manley-Rowe relations [19]. They are usually derived in semiclassical theory through a rather cumbersome procedure that relies on the particular nonlinearity being described together with the symmetries that they involve (Kleinmann's condition) when no absorption is present [20]. These semiclassical results are often interpreted in terms of photon numbers participating in each mode [20,21]. This interpretation is naturally embodied in the present quantum treatment.

VII. CONCLUSIONS

An economical derivation of the quantum Ermakov-Lewis invariant has been presented. This invariant may be used in an equivalent fashion as the Hamiltonian is used in the time-independent case. Namely, to obtain evolution operators, to cast the equations of motion of different operators in commutative expressions, and to produce a phase shift with its exponential form. The invariant and time-dependent definitions for annihilation and creation operators have been used to generalize the quantum phase to the time-dependent case. Following the classical form of the orthogonal functions invariant, the quantum Ermakov-Lewis invariant has been expressed in amplitude and phase variables in accordance with the correspondence principle. A quantum derivation of the Manley-Rowe relations has been presented as a particular application of this representation.

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