Levinson theorem for the Dirac equation in $D+1$ dimensions

Xiao-Yan Gu* and Zhong-Qi Ma†
China Center for Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, People's Republic of China and Institute of High Energy Physics, Beijing 100039, People's Republic of China

Shi-Hai Dong‡
Programa de Ingeniería Molecular, Instituto Mexicano del Petróleo, Lázaro Cárdenas 152, 07730 México, D.F., Mexico

(Received 13 January 2003; published 26 June 2003)

In terms of the generalized Sturm-Liouville theorem, the Levinson theorem for the Dirac equation with a spherically symmetric potential in $D+1$ dimensions is uniformly established as a relation between the total number of bound states and the sum of the phase shifts of the scattering states at $E=\pm M$ with a given angular momentum. The critical case, where the Dirac equation has a half bound state, is analyzed in detail. A half bound state is a zero-momentum solution if its wave function is finite but does not decay fast enough at infinity to be square integrable.

DOI: 10.1103/PhysRevA.67.062715 PACS number(s): 03.65.Pm, 03.65.Ge, 11.80.−m, 73.50.Bk

I. INTRODUCTION

The Levinson theorem [1] is an important theorem in the quantum-scattering theory, which sets up the relation between the number of bound states and the phase shift at zero momentum. It has been generalized [2–9] and has been applied to different fields in modern physics [10–21]. With the interest of higher-dimensional field theory, the Levinson theorem for the Schrödinger equation in arbitrary D dimensions was studied, recently, [22]. However, the Levinson theorem for the Dirac equation in $D+1$ dimensions has not been uniformly studied. The problem is how to derive the radial equation of the Dirac equation in $D+1$ dimensions.

In our recent paper [23], we generalized the Dirac equation with a spherically symmetric potential to arbitrary $D+1$ dimensions, found the eigenfunctions of the total angular momentum, and derived the radial equations. It is worth noticing that the total (or orbital, spinor) angular momentum in D-dimensional space is described by an irreducible representation of the SO(D) group, which is denoted by the highest weight, instead of only one parameter j (or l, s) in three-dimensional space. In this paper, we will uniformly study the Levinson theorem for the Dirac equation in $D+1$ dimensions by the Sturm-Liouville theorem. In Sec. II, we will sketch the derivation of the radial equations for the Dirac equation with a spherically symmetric potential in $D+1$ space time, both for even D and odd D. Then, we will study the generalized Sturm-Liouville theorem in Sec. III. The number of bound states will be calculated in Sec. IV. In Sec. V, the Levinson theorem is established by proving the number of bound states to be equal to the sum of the phase shifts of the scattering states at $E=\pm M$ with the given angular momentum. The critical cases are also analyzed there. Some discussions are given in Sec. VI.

*Electronic address: guxy@mail.ihep.ac.cn
†Electronic address: mazq@sun.ihep.ac.cn
‡Electronic address: dongsh2@yahoo.com

II. RADIAL EQUATIONS

The Dirac equation in $D+1$ dimensions can be expressed as [24]

$$i\sum_{\mu=0}^{D}\gamma^{\mu}(\partial_{\mu}+ieA_{\mu})\Psi(\mathbf{x},t)=M\Psi(\mathbf{x},t), \tag{1}$$

where M is the mass of the particle, and $(D+1)$ matrices γ_{μ} satisfy the anticommutative relations:

$$\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2\eta^{\mu\nu}\mathbf{1}, \tag{2}$$

with the metric tensor $\eta^{\mu\nu}$ satisfying

$$\eta^{\mu\nu}=\eta_{\mu\nu}=\begin{cases}\delta_{\mu\nu} & \text{when} \quad \mu=0\\ -\delta_{\mu\nu} & \text{when} \quad \mu\neq 0.\end{cases} \tag{3}$$

For simplicity, the natural units $\hbar=c=1$ are employed throughout this paper. Discuss the special case where only the zero component of A_{μ} is nonvanishing and spherically symmetric:

$$eA_0=V(r), \quad A_a=0, \quad \text{when} \quad a\neq 0. \tag{4}$$

The Hamiltonian $H(\mathbf{x})$ of the system is expressed as

$$\begin{gathered} i\partial_0\Psi(\mathbf{x},t)=H(\mathbf{x})\Psi(\mathbf{x},t),\\ H(\mathbf{x})=\sum_{a=1}^{D}\gamma^0\gamma^a p_a+V(r)+\gamma^0 M,\\ p_a=-i\partial_a=-i\frac{\partial}{\partial x^a}, \quad 1\leqslant a\leqslant D. \end{gathered} \tag{5}$$

The orbital angular-momentum operators L_{ab}, the spinor operators S_{ab}, and the total angular-momentum operators J_{ab} are defined as follows

$$L_{ab}=-L_{ba}=ix_a\partial_b-ix_b\partial_a, \quad S_{ab}=-S_{ba}=i\gamma_a\gamma_b/2,$$

$$J_{ab}=L_{ab}+S_{ab}, \quad 1\leq a<b\leq D, \tag{6}$$

$$J^2=\sum_{a<b=2}^{D} J_{ab}^2, \quad L^2=\sum_{a<b=2}^{D} L_{ab}^2, \quad S^2=\sum_{a<b=2}^{D} S_{ab}^2.$$

It is easy to show by the standard method [24] that J_{ab} and κ are commutant with the Hamiltonian $H(\mathbf{x})$,

$$\kappa=\gamma^0\left\{\sum_{a<b} i\gamma^a\gamma^b L_{ab}+(D-1)/2\right\}$$
$$=\gamma^0\{J^2-L^2-S^2+(D-1)/2\}. \tag{7}$$

Since the potential $V(r)$ is spherically symmetric, the symmetry group of the system is SO(D). Following Erdelyi [25] and Louck [26,27], we introduce the hyperspherical coordinates in the real D-dimensional space

$$x^1=r\cos\theta_1\sin\theta_2\cdots\sin\theta_{D-1},$$

$$x^2=r\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-1},$$

$$x^b=r\cos\theta_{b-1}\sin\theta_b\cdots\sin\theta_{D-1}, \quad 3\leq b\leq D-1, \tag{8}$$

$$x^D=r\cos\theta_{D-1},$$

$$\sum_{a=1}^{D}(x^a)^2=r^2.$$

The unit vector along $\mathbf{x}$ is usually denoted by $\hat{\mathbf{x}}=\mathbf{x}/r$. The volume element of the configuration space is

$$\prod_{a=1}^{D} dx^a=r^{D-1}drd\Omega, \quad d\Omega=\prod_{a=1}^{D-1}(\sin\theta_a)^{a-1}d\theta_a,$$

$$0\leq r<\infty, \quad -\pi\leq\theta_1\leq\pi, \quad 0\leq\theta_b\leq\pi, \quad 2\leq b\leq D-1. \tag{9}$$

As is well known, the Lie algebras of the SO($2N+1$) group and the SO($2N$) group are B_N and D_N, respectively. Their Chevalley bases with the subscript μ, $1\leq\mu\leq N-1$, are the same:

$$H_\mu(J)=J_{(2\mu-1)(2\mu)}-J_{(2\mu+1)(2\mu+2)},$$

$$E_\mu(J)=(J_{(2\mu)(2\mu+1)}-iJ_{(2\mu-1)(2\mu+1)}-iJ_{(2\mu)(2\mu+2)}$$
$$-J_{(2\mu-1)(2\mu+2)})/2, \tag{10a}$$

$$F_\mu(J)=(J_{(2\mu)(2\mu+1)}+iJ_{(2\mu-1)(2\mu+1)}+iJ_{(2\mu)(2\mu+2)}$$
$$-J_{(2\mu-1)(2\mu+2)})/2.$$

But the bases with the subscript N are different:

$$H_N(J)=2J_{(2N-1)(2N)},$$

$$E_N(J)=J_{(2N)(2N+1)}-iJ_{(2N-1)(2N+1)}, \tag{10b}$$

$$F_N(J)=J_{(2N)(2N+1)}+iJ_{(2N-1)(2N+1)},$$

for SO($2N+1$), and

$$H_N(J)=J_{(2N-3)(2N-2)}+J_{(2N-1)(2N)},$$

$$E_N(J)=(J_{(2N-2)(2N-1)}-iJ_{(2N-3)(2N-1)}+iJ_{(2N-2)(2N)}$$
$$+J_{(2N-3)(2N)})/2, \tag{10c}$$

$$F_N(J)=(J_{(2N-2)(2N-1)}+iJ_{(2N-3)(2N-1)}-iJ_{(2N-2)(2N)}$$
$$+J_{(2N-3)(2N)})/2,$$

for SO($2N$). The operator J_{ab} can be replaced with L_{ab} or S_{ab} depending on the wave functions one is discussing. $H_\mu(J)$ span the Cartan subalgebra, and their eigenvalues m_μ for an eigenstate $|\mathbf{m}\rangle$ in a given irreducible representation are the components of the weight vector $\mathbf{m}=(m_1,\ldots,m_N)$:

$$H_\mu(J)|\mathbf{m}\rangle=m_\mu|\mathbf{m}\rangle, \quad 1\leq\mu\leq N. \tag{11}$$

E_μ are called the raising operators and F_μ are the lowering ones. For an irreducible representation, there is a highest weight $\mathbf{M}$, which is a simple weight and can be used to describe the irreducible representation

$$H_a|\mathbf{M}\rangle=M_a|\mathbf{M}\rangle, \quad E_b|\mathbf{M}\rangle=0. \tag{12}$$

Usually, the irreducible representation is also called the highest weight representation and is directly denoted by $\mathbf{M}$. The eigenvalue of J^2 (or L^2, S^2) is the Casimir invariant $C_2(\mathbf{M})$ in the representation $\mathbf{M}$ to which the total (or orbital, spinor) wave function belongs. The Casimir invariant $C_2(\mathbf{M})$ can be calculated by the formula [e.g., see Eq. (1.131) in Ref. [28]]

$$C_2(\mathbf{M})=\mathbf{M}\cdot(\mathbf{M}+2\boldsymbol{\rho})=\sum_{\mu,\nu=1}^{N} M_\mu d_\mu(A^{-1})_{\mu\nu}(M_\nu+2), \tag{13}$$

where $\boldsymbol{\rho}$ is the half sum of the positive roots in the Lie algebra, A^{-1} is the inverse of the Cartan matrix, and d_μ are the half square lengths of the simple roots.

The orbital wave function in D-dimensional space is usually expressed by the spherical harmonic $Y^{(l)}_{\mathbf{m}}(\hat{\mathbf{x}})$ [25,26,23], which belongs to the weight $\mathbf{m}$ of the highest weight representation $(l)\equiv(l,0,\ldots,0)$. For the highest weight state, $\mathbf{m}=(l)$, we have

$$Y^{(l)}_{(l)}(\hat{\mathbf{x}})=N_{D,l}r^{-l}(x^1+ix^2)^l,$$

$$N_{D,l}=\begin{cases} 2^{-N-l}\left\{\dfrac{(2l+2N-1)!}{\pi^N l!(l+N-1)!}\right\}^{1/2} & \text{when} \quad D=2N+1 \\ \left\{\dfrac{(l+N-1)!}{2\pi^N l!}\right\}^{1/2} & \text{when} \quad D=2N, \end{cases} \tag{14}$$

where $N_{D,l}$ is the normalization factor. The partners $Y^{(l)}_{\mathbf{m}}(\hat{\mathbf{x}})$ can be calculated from $Y^{(l)}_{(l)}(\hat{\mathbf{x}})$ by the lowering operators $F_\mu(L)$. The Casimir invariant for the spherical harmonic $Y^{(l)}_{\mathbf{m}}(\hat{\mathbf{x}})$ is calculated from Eq. (13):

$$L^2 Y^{(l)}_{\mathbf{m}}(\hat{\mathbf{x}})=C_2[(l)]Y^{(l)}_{\mathbf{m}}(\hat{\mathbf{x}}), \quad C_2[(l)]=l(l+D-2). \tag{15}$$

Now, we define 2^N-dimensional matrices β_a satisfying

$$\beta_a\beta_b+\beta_b\beta_a=2\delta_{ab}\mathbf{1}, \quad a,b=1,2,\ldots,(2N+1). \tag{16}$$

We choose the representation such that $\beta_{2N+1}=\sigma_3\times\mathbf{1}_{2^{N-1}}$, where σ_a is the Pauli matrix, and $\mathbf{1}_n$ denotes the n-dimensional unit matrix.

For $D=2N+1$, $N>1$, we have

$$\gamma^0=\sigma_3\times\mathbf{1}, \quad \gamma^a=(i\sigma_2)\times\beta_a, \quad 1\leqslant a\leqslant 2N+1, \tag{17}$$

Thus, the spinor operator S_{ab} and the κ operator become the block matrices

$$S_{ab}=\mathbf{1}\times\overline{S_{ab}}, \quad \overline{S_{ab}}=-i\beta_a\beta_b/2, \tag{18}$$

$$\kappa=\sigma_3\times\overline{\kappa}, \quad \overline{\kappa}=-i\sum_{a<b}\beta_a\beta_b L_{ab}+(D-1)/2. \tag{19}$$

The relation between S_{ab} and $\overline{S_{ab}}$ is similar to that between the spinor operators for the Dirac spinors and for the Pauli spinors. In the level of the Pauli spinors, we define the fundamental spinor $\chi(\mathbf{m})$ belonging to the fundamental spinor representation $(s)\equiv(0,\ldots,0,1)$ with the Casimir invariant $C_2[(s)]=(2N^2+N)/4$. There are two ways to construct the eigenfunctions of the total angular momentum belonging to the representation $(j)\equiv(l,0,\ldots,0,1)$. They have different eigenvalues of $\overline{\kappa}$. Since the system is spherically symmetric, we only need to calculate the highest weight state for the representation (j) in terms of the Clebsch-Gordan coefficients

$$\begin{aligned}\phi_{|K|,(j)}(\hat{\mathbf{x}})&=Y^{(l)}_{(l)}(\hat{\mathbf{x}})\chi[(s)]=N_{D,l}r^{-l}(x^1+ix^2)^l\chi[(s)],\\ C_2[(j)]&-C_2[(l)]-C_2[(s)]+N=l+N=|K|,\end{aligned} \tag{20}$$

$$\begin{aligned}\phi_{-|K|,(j)}(\hat{\mathbf{x}})&=\sum_{\mathbf{m}} Y^{(l+1)}_{\mathbf{m}}(\hat{\mathbf{x}})\chi[(j)-\mathbf{m}]\langle(l+1),\mathbf{m},(s),(j)-\mathbf{m}|(j),(j)\rangle\\ &=N_{D,l}r^{-l-1}(x^1+ix^2)^l\{x^{2N+1}\chi[(s)]+(x^{2N-1}+ix^{2N})\chi[(0,\ldots,0,1,\overline{1})]+(x^{2N-3}+ix^{2N-2})\\ &\quad\times\chi[(0,\ldots,0,1,\overline{1},1)]+\cdots+(x^3+ix^4)\chi[(1,\overline{1},0,\ldots,0,1)]+(x^1+ix^2)\chi[(\overline{1},0,\ldots,0,1)]\},\\ &C_2[(j)]-C_2[(l+1)]-C_2[(s)]+N=-l-N=-|K|.\end{aligned} \tag{21}$$

In the level of the Dirac spinors, the wave function $\Psi_{K,(j)}(\mathbf{x})$ of the total angular momentum belonging to the highest weight state of the irreducible representation (j) can be expressed as

$$\Psi_{K,(j)}(\mathbf{x},t)=r^{-N}e^{-iEt}\begin{pmatrix}F(r)\phi_{K,(j)}(\hat{\mathbf{x}})\\ iG(r)\phi_{-K,(j)}(\hat{\mathbf{x}})\end{pmatrix},$$

$$\kappa\Psi_{K,(j)}(\mathbf{x})=K\Psi_{K,(j)}(\mathbf{x}), \quad K=\pm(l+N). \tag{22}$$

Its partners can be calculated from it by the lowering operators $F_\mu(J)$.

Substituting $\Psi_{K(j)}(\mathbf{x})$ into the Dirac equation (5), we obtain the radial equation [23]

$$\begin{aligned}\frac{dG(r)}{dr}+\frac{K}{r}G(r)&=[E-V(r)-M]F(r),\\ -\frac{dF(r)}{dr}+\frac{K}{r}F(r)&=[E-V(r)+M]G(r).\end{aligned} \tag{23}$$

As is well known, Eq. (23) also holds when $D=3$.

For $D=2N$, $N>2$, we have

$$\gamma^0=\beta_{2N+1}, \quad \gamma^a=\beta_{2N+1}\beta_a, \quad 1\leqslant a\leqslant 2N. \tag{24}$$

As is well known, the spinor representation of SO($2N$) group is reducible and can be reduced to two inequivalent fundamental spinor representations $(+s)\equiv(0,0,\ldots,0,1)$ and $(-s)\equiv(0,0,\ldots,0,1,0)$ with the same Casimir invariant $C_2[(\pm s)]=(2N^2-N)/4$. γ^0 is a diagonal matrix where half of the diagonal elements are equal to $+1$ and the remaining are equal to -1. Because the spinor operator S_{ab} and the operator κ are commutant with γ^0, each of these becomes a direct sum of two matrices, referring to the rows with the eigenvalues $+1$ and -1 of γ^0, respectively. In the level of the Pauli spinors, the fundamental spinors $\chi_{\pm}(\mathbf{m})$ belong to the fundamental spinor representations $(+s)$ and $(-s)$, respectively, and satisfy

$$\gamma^0\chi_{\pm}(\mathbf{m})=\pm\chi_{\pm}(\mathbf{m}). \tag{25}$$

For the total angular momentum, there are two kinds of representations $(j_1)\equiv(l,0,\ldots,0,1)$ and $(j_2)\equiv(l,0,\ldots,0,1,0)$ with the same Casimir invariant:

$$C_2[(j_1)]=C_2[(j_2)]=l(l+2N-1)+(2N^2-N)/4. \tag{26}$$

There are two sets of wave functions belonging to the representation (j_1). The highest weight states are

$$\phi_{|K|,(j_1)}(\hat{\mathbf{x}})=Y^{(l)}_{(l)}(\hat{\mathbf{x}})\chi_+[(+s)]=N_{D,l}r^{-l}(x^1+ix^2)^l\chi_+[(+s)],$$

$$\begin{aligned}\phi_{-|K|,(j_1)}(\hat{\mathbf{x}})=&\sum_{\mathbf{m}} Y^{(l+1)}_{\mathbf{m}}(\hat{\mathbf{x}})\chi_-[(j_1)-\mathbf{m}]\langle(l+1),\mathbf{m},(-s),(j_1)-\mathbf{m}|(j_1),(j_1)\rangle\\ =&N_{D,l}r^{-l-1}(x^1+ix^2)^l\{(x^{2N-1}+ix^{2N})\chi_-[(-s)]+(x^{2N-3}+ix^{2N-2})\chi_-[(0,\ldots,0,1,\overline{1},0)]\\ &+(x^{2N-5}+ix^{2N-4})\chi_-[(0,\ldots,0,1,\overline{1},0,1)]\\ &+\cdots+(x^3+ix^4)\chi_-[(1,\overline{1},0,\ldots,0,1)]+(x^1+ix^2)\chi_-[(\overline{1},0,\ldots,0,1)]\},\end{aligned} \tag{27}$$

For the representation $(j_2)\equiv(l,0,\ldots,0,1,0)$, we have

$$\begin{aligned}\phi_{-|K|,(j_2)}(\hat{\mathbf{x}})=&\sum_{\mathbf{m}} Y^{(l+1)}_{\mathbf{m}}(\hat{\mathbf{x}})\chi_+[(j_2)-\mathbf{m}]\langle(l+1),\mathbf{m},(+s),(j_2)-\mathbf{m}|(j_2),(j_2)\rangle\\ =&N_{D,l}r^{-l-1}(x^1+ix^2)^l\{(x^{2N-1}-ix^{2N})\chi_+[(+s)]+(x^{2N-3}+ix^{2N-2})\chi_+[(0,\ldots,0,1,0,\overline{1})]+(x^{2N-5}+ix^{2N-4})\\ &\times\chi_+[(0,\ldots,0,1,\overline{1},1,0)]+\cdots+(x^3+ix^4)\chi_+[(1,\overline{1},0,\ldots,0,1,0)]+(x^1+ix^2)\chi_+[(\overline{1},0,\ldots,0,1,0)]\},\\ \phi_{|K|,(j_2)}(\hat{\mathbf{x}})=&Y^{(l)}_{(l)}(\hat{\mathbf{x}})\chi_-[(-s)]=N_{D,l}r^{-l}(x^1+ix^2)^l\chi_-[(-s)],\end{aligned} \tag{28}$$

where

$$\begin{aligned}&C_2[(j_1)]-C_2[(l)]-C_2[(+s)]+N-1/2=l+N-1/2=|K|,\\ &C_2[(j_2)]-C_2[(l+1)]-C_2[(+s)]+N-1/2\\ &\quad=-l-N+1/2=-|K|.\end{aligned} \tag{29}$$

In the level of the Dirac spinors, the eigenfunctions $\Psi_{K,(j_\omega)}(\mathbf{x})$ of the total angular momentum belonging to the highest weight state of the irreducible representation (j_ω) can be expressed as

$$\begin{aligned}\Psi_{|K|,(j_1)}(\mathbf{x},t)=&r^{-N+1/2}e^{-iEt}\{F(r)\phi_{|K|,(j_1)}(\hat{\mathbf{x}})\\ &+iG(r)\phi_{-|K|,(j_1)}(\hat{\mathbf{x}})\},\\ \Psi_{-|K|,(j_2)}(\mathbf{x},t)=&r^{-N+1/2}e^{-iEt}\{F(r)\phi_{-|K|,(j_2)}(\hat{\mathbf{x}})\\ &+iG(r)\phi_{|K|,(j_2)}(\hat{\mathbf{x}})\},\\ \kappa\Psi_{K,(j_\omega)}(\mathbf{x})=&K\Psi_{K,(j_\omega)}(\mathbf{x}),\\ K=&\begin{cases}l+N-1/2 & \text{when } \omega=1\\ -l-N+1/2 & \text{when } \omega=2.\end{cases}\end{aligned} \tag{30}$$

Their partners can be calculated from them by the lowering operators $F_\mu(J)$.

Substituting $\Psi_{K(j_\omega)}(\mathbf{x})$ into the Dirac equation (5), we obtain the radial equations, which are in the same forms as Eq. (25) in the $D=2N+1$ case [23]:

$$\begin{aligned}&\frac{dG(r)}{dr}+\frac{K}{r}G(r)=[E-V(r)-M]F(r),\\ &-\frac{dF(r)}{dr}+\frac{K}{r}F(r)=[E-V(r)+M]G(r).\end{aligned} \tag{31}$$

When $D=4$, the SO(4) group is homomorphism to SU(2) $\times$SU(2), and the representations j_1 and j_2 belong to two different SU(2) groups, respectively. When $D=2$, the SO(2) group is an Abelian group and $K=\pm j=\pm 1/2$, $\pm 3/2$, etc. However, Eq. (33) still holds for these cases.

III. THE GENERALIZED STURM-LIOUVILLE THEOREM

The spherically symmetric potential $V(r)$ has to satisfy the boundary condition at the origin for the nice behavior of the wave function

$$\int_0^1 r|V(r)|dr<\infty. \tag{32}$$

For simplicity, we first discuss the case where the potential $V(r)$ is a cutoff one at a sufficiently large radius r_0:

$$V(r)=0, \quad \text{when } r\geqslant r_0. \tag{33}$$

The general case where the potential $V(r)$ has a tail at infinity will be discussed in Sec. VI.

Introduce a parameter λ for the potential $V(r)$:

$$V(r,\lambda)=\lambda V(r), \quad V(r,1)=V(r). \tag{34}$$

As λ increases from 0 to 1, the potential $V(r,\lambda)$ changes from zero to the given potential $V(r)$. If λ changes its sign, the potential $V(r,\lambda)$ changes sign, too.

Although the spherical spinor functions and eigenvalues K are different for the $D=2N+1$ case and the $D=2N$ case, the forms of the radial equations are uniform:

$$\frac{dG_{KE}(r,\lambda)}{dr}+\frac{K}{r}G_{KE}(r,\lambda)=[E-V(r,\lambda)-M]F_{KE}(r,\lambda),$$

$$-\frac{dF_{KE}(r,\lambda)}{dr}+\frac{K}{r}F_{KE}(r,\lambda)=[E-V(r,\lambda)+M]G_{KE}(r,\lambda),$$

$$K=\pm 1/2, \pm 1, \pm 3/2, \ldots. \tag{35}$$

It is easy to see that the solutions with a negative K can be obtained from those with a positive K by interchanging $F_{KE}(r,\lambda)\leftrightarrow G_{-K-E}(r,-\lambda)$, so that in the following we only discuss the solutions with a positive K. The main results for the case with a negative K will be indicated in the text.

The physically admissible solutions are finite, continuous, vanishing at the origin, and square integrable:

$$F_{KE}(r,\lambda)=G_{KE}(r,\lambda)=0, \quad \text{when } r=0, \tag{36}$$

$$\int_0^\infty dr\{|F_{KE}(r,\lambda)|^2+|G_{KE}(r,\lambda)|^2\}<\infty. \tag{37}$$

The solutions for $|E|>M$ describe the scattering states and those for $|E|\leqslant M$ describe the bound states. We will solve Eq. (37) in two regions, $0\leqslant r<r_0$ and $r_0<r<\infty$, and then match the two solutions at r_0 by the match condition:

$$A_K(E,\lambda)\equiv\left.\frac{F_{KE}(r,\lambda)}{G_{KE}(r,\lambda)}\right|_{r=r_0-}=\left.\frac{F_{KE}(r,\lambda)}{G_{KE}(r,\lambda)}\right|_{r=r_0+}. \tag{38}$$

When r_0 is the zero point of $G_{KE}(r,\lambda)$, the match condition can be replaced by its inverse $G_{KE}(r,\lambda)/F_{KE}(r,\lambda)$ instead. The merit of using this match condition is that we need not consider the normalization factor in the solutions.

The key point for the proof of the Levinson theorem is that the ratio $F_{KE}(r,\lambda)/G_{KE}(r,\lambda)$ is monotonic with respect to the energy E. For simplicity, we briefly denote $F_{KE}(r_0,\lambda)$ and $G_{KE}(r_0,\lambda)$ by F and G, respectively, and those with the energy E_1 by F_1 and G_1. From Eq. (35) we have

$$\frac{d}{dr}(F_1G-G_1F)=-(E_1-E)(F_1F+G_1G). \tag{39}$$

From the boundary condition that both solutions vanish at the origin, we integrate Eq. (39) in the region $0\leqslant r\leqslant r_0$ and obtain

$$(F_1G-G_1F)|_{r=r_0-}=-(E_1-E)\int_0^{r_0}(F_1F+G_1G)dr.$$

Taking the limit as E_1 tends to E, we have

$$\lim_{E_1\to E}\left.\frac{F_1G-G_1F}{E_1-E}\right|_{r=r_0-}$$

$$=\{G_{KE}(r_0,\lambda)\}^2\frac{\partial}{\partial E}A_K(E,\lambda)$$

$$=-\int_0^{r_0}\{F_{KE}^2(r,\lambda)+G_{KE}^2(r,\lambda)\}dr<0. \tag{40}$$

Thus, when $|E|\geqslant M$ we have

$$A_K(E,\lambda)=A_K(M,\lambda)-c_1^2k^2+\cdots,$$

when

$$E>M \quad \text{and} \quad E\sim M,$$

$$A_K(E,\lambda)=A_K(-M,\lambda)+c_2^2k^2+\cdots,$$

when

$$E<-M \quad \text{and} \quad E\sim -M, \tag{41}$$

where c_1^2 and c_2^2 are non-negative numbers, and the momentum k is defined as follows:

$$k=(E^2-M^2)^{1/2}. \tag{42}$$

Similarly, from the boundary condition that the radial functions $F_{KE}(r,\lambda)$ and $G_{KE}(r,\lambda)$ for $|E|\leqslant M$ tend to zero at infinity, we obtain, by integrating Eq. (39) in the region $r_0\leqslant r<\infty$,

$$\{G_{KE}(r_0,\lambda)\}^2\frac{\partial}{\partial E}\left.\left(\frac{F_{KE}(r,\lambda)}{G_{KE}(r,\lambda)}\right)\right|_{r=r_{0_+}}$$

$$=\int_{r_0}^\infty\{F_{KE}^2(r,\lambda)+G_{KE}^2(r,\lambda)\}dr>0. \tag{43}$$

Thus, as the energy E increases, the ratio $F_{KE}(r,\lambda)/G_{KE}(r,\lambda)$ at r_{0_-} decreases monotonically, but the ratio $F_{KE}(r,\lambda)/G_{KE}(r,\lambda)$ at r_{0_+} when $|E|\leqslant M$ increases monotonically. This is called the generalized Sturm-Liouville theorem [29].

IV. THE NUMBER OF BOUND STATES

Now, we solve Eq. (35) for the energy $|E|\leqslant M$. In the region $0\leqslant r<r_0$, when $\lambda=0$, we have

$$F_{KE}(r,0)=e^{-i(K-1/2)\pi/2}\{(M+E)\pi k_1 r/2\}^{1/2}J_{K-1/2}(ik_1 r),$$

$$G_{KE}(r,0)=e^{-i(K-3/2)\pi/2}\{(M-E)\pi k_1 r/2\}^{1/2}J_{K+1/2}(ik_1 r), \tag{44}$$

where $J_m(x)$ is the Bessel function and

$$k_1=(M^2-E^2)^{1/2}. \tag{45}$$

The ratio at $r=r_0-$ when $\lambda=0$ is

$$A_K(E,0)=-i\left(\frac{M+E}{M-E}\right)^{1/2}\frac{J_{K-1/2}(ik_1 r_0)}{J_{K+1/2}(ik_1 r_0)} =\begin{cases} -\dfrac{2M(2K+1)}{k_1^2 r_0}\sim-\infty & \text{when} \quad E\sim M \\ -\dfrac{2K+1}{2Mr_0} & \text{when} \quad E\sim -M. \end{cases} \tag{46}$$

In the region $r_0<r<\infty$, due to the cutoff potential we have $V(r)=0$ and

$$F_{KE}(r,\lambda)=e^{i(K+1/2)\pi/2}\{(M+E)\pi k_1 r/2\}^{1/2}H^{(1)}_{K-1/2}(ik_1 r),$$

$$G_{KE}(r,\lambda)=e^{i(K+3/2)\pi/2}\{(M-E)\pi k_1 r/2\}^{1/2}H^{(1)}_{K+1/2}(ik_1 r), \tag{47}$$

where $H^{(1)}_m(x)$ is the Hankel function of the first kind. The ratio at $r=r_0+$ does not depend on λ and is given as follows:

$$\left.\frac{F_{KE}(r,\lambda)}{G_{KE}(r,\lambda)}\right|_{r=r_0+}=-i\left(\frac{M+E}{M-E}\right)^{1/2}\frac{H^{(1)}_{K-1/2}(ik_1 r_0)}{H^{(1)}_{K+1/2}(ik_1 r_0)}=\begin{cases} \dfrac{2Mr_0}{2K-1} & \text{when} \quad E\sim M \quad \text{and} \quad K\geqslant 1 \\ -2Mr_0\ln(k_1 r_0)\sim\infty & \text{when} \quad E\sim M \quad \text{and} \quad K=1/2 \\ \dfrac{k_1^2 r_0}{2M(2K-1)}\sim 0 & \text{when} \quad E\sim -M, \quad \text{and} \quad K\geqslant 1 \\ \dfrac{-k_1^2 r_0\ln(k_1 r_0)}{2M}\sim 0 & \text{when} \quad E\sim -M, \quad \text{and} \quad K=1/2. \end{cases} \tag{48}$$

It is evident from Eqs. (46) and (48) that as the energy E increases from $-M$ to M, there is no overlap between two variant ranges of the ratio at two sides of r_0 when $\lambda=0$ (no potential), except for $K=1/2$ where there is a half bound state at $E=M$. The half bound state will be discussed in the following section.

As λ increases from 0 to 1, the potential $V(r,\lambda)$ changes from zero to the given potential $V(r)$ and $A_K(E,\lambda)$ changes, too. If $A_K(M,\lambda)$ decreases across the value $2Mr_0/(2K-1)$ as λ increases, an overlap between the variant ranges of the ratios at two sides of r_0 appears. Since the ratio $A_K(E,\lambda)$ of two radial functions at r_0- decreases monotonically as the energy E increases, and the ratio at r_0+ increases monotonically, the overlap means that there must be one and only one energy where the matching condition (38) is satisfied, namely, a bound state appears.

As λ increases, $A_K(M,\lambda)$ may decrease to $-\infty$, jumps to ∞, and then decreases again across the value $2Mr_0/(2K-1)$, so that another bound state appears. Note that when r_0 is a zero point of the wave function $G_{KE}(r,\lambda)$, $A_K(E,\lambda)$ goes to infinity. It is not a singularity.

On the other hand, as λ increases, if $A_K(-M,\lambda)$ decreases across zero, an overlap between the variant ranges of the ratios at two sides of r_0 disappears so that a bound state disappears. Therefore, each time $A_K(M,\lambda)$ decreases across the value $2Mr_0/(2K-1)$ as λ increases, a new overlap between the variant ranges of the ratios at two sides of r_0 appears such that a scattering state of a positive energy becomes a bound state, and each time $A_K(-M,\lambda)$ decreases across zero, an overlap between the variant ranges of the ratio at two sides of r_0 disappears such that a bound state becomes a scattering state of a negative energy. Conversely, each time $A_K(M,\lambda)$ increases across the value $2Mr_0/(2K-1)$, an overlap between the variant ranges disappears such that a bound state becomes a scattering state of a positive energy, and each time $A_K(-M,\lambda)$ increases across zero, a new overlap between the variant ranges appears such that a scattering state of a negative energy becomes a bound state. Now, the number n_K of bound states with the parameter K is equal to the sum (or subtraction) of four times as λ increases from 0 to 1: the times that $A_K(M,\lambda)$ decreases across the value $2Mr_0/(2K-1)$, minus the times that $A_K(M,\lambda)$ increases across the value $2Mr_0/(2K-1)$, minus the times that $A_K(-M,\lambda)$ decreases across zero, plus the times that $A_K(-M,\lambda)$ increases across zero.

When $K=1/2$, the value $2Mr_0/(2K-1)$ becomes infinity. We may check the times that $A_K(M,\lambda)^{-1}$ increases (or decreases) across zero to replace the times that $A_K(M,\lambda)$ decreases (or increases) across infinity.

V. THE RELATIVISTIC LEVINSON THEOREM

We turn to discuss the phase shifts of the scattering states. Solving Eq. (35) in the region $r_0<r<\infty$ for the energy $|E|>M$, we have

$$f_{KE}(r,\lambda)=B(E)\left(\frac{\pi kr}{2}\right)^{1/2}\{\cos\delta_K(E,\lambda)J_{K-1/2}(kr)-\sin\delta_K(E,\lambda)N_{K-1/2}(kr)\},$$

$$g_{KE}(r,\lambda)=\left(\frac{\pi kr}{2}\right)^{1/2}\{\cos\delta_K(E,\lambda)J_{K+1/2}(kr)-\sin\delta_K(E,\lambda)N_{K+1/2}(kr)\}, \tag{49}$$

where $N_m(x)$ denotes the Neumann function, the momentum k is given in Eq. (16), and $B(E)$ is defined as

$$B(E)=\begin{cases}\left(\dfrac{E+M}{E-M}\right)^{1/2} & \text{when} \quad E>M\\ -\left(\dfrac{|E|-M}{|E|+M}\right)^{1/2} & \text{when} \quad E<-M.\end{cases} \tag{50}$$

The asymptotic form of solution (49) at $r\to\infty$ is

$$f_{KE}(r,\lambda)\sim B(E)\cos[kr-K\pi/2+\delta_K(E,\lambda)],$$

$$g_{KE}(r,\lambda)\sim\sin[kr-K\pi/2+\delta_K(E,\lambda)]. \tag{51}$$

Substituting Eq. (49) into the match condition (38), we obtain the formula for the phase shift $\delta_K(E,\lambda)$:

$$\tan\delta_K(E,\lambda)=\frac{J_{K+1/2}(kr_0)}{N_{K+1/2}(kr_0)}\,\frac{A_K(E,\lambda)-B(E)J_{K-1/2}(kr_0)/J_{K+1/2}(kr_0)}{A_K(E,\lambda)-B(E)N_{K-1/2}(kr_0)/N_{K+1/2}(kr_0)}$$
$$=\frac{J_{K-1/2}(kr_0)}{N_{K-1/2}(kr_0)}\,\frac{\{A_K(E,\lambda)\}^{-1}-B(E)^{-1}J_{K+1/2}(kr_0)/J_{K-1/2}(kr_0)}{\{A_K(E,\lambda)\}^{-1}-B(E)^{-1}N_{K+1/2}(kr_0)/N_{K-1/2}(kr_0)}. \tag{52}$$

The phase shift $\delta_K(E,\lambda)$ is determined up to a multiple of π due to the period of the tangent function. We use the convention that the phase shifts for the free particles ($V(r)=0$) are vanishing:

$$\delta_K(E,0)=0. \tag{53}$$

Under this convention, the phase shifts $\delta_K(E)$ are determined completely as λ increases from zero to one:

$$\delta_K(E)\equiv\delta_K(E,1). \tag{54}$$

The phase shifts $\delta_K(\pm M,\lambda)$ are the limits of the phase shifts $\delta_K(E,\lambda)$ as E tends to $\pm M$. At the sufficiently small k, $k\ll 1/r_0$, when $E>M$, we have

$$\tan\delta_K(E,\lambda)\sim-\frac{\pi(kr_0/2)^{2K-1}}{(K+1/2)!(K-1/2)!}\,\frac{A_K(M,\lambda)(kr_0/2)^2-Mr_0(K+1/2)}{A_K(M,\lambda)-c_1^2k^2-\dfrac{2Mr_0}{2K-1}\left(1+\dfrac{(kr_0)^2}{(2K-1)(2K-3)}\right)}, \tag{55a}$$

when $K>3/2$,

$$\tan\delta_K(E,\lambda)\sim-\frac{\pi}{2}\left(\frac{kr_0}{2}\right)^2\frac{A_K(M,\lambda)(kr_0/2)^2-2Mr_0}{A_K(M,\lambda)-c_1^2k^2-Mr_0\left(1-\dfrac{(kr_0)^2}{2}\ln(kr_0)\right)}, \tag{55b}$$

when $K=3/2$,

$$\tan\delta_K(E,\lambda)\sim(kr_0)\frac{A_K(M,\lambda)(kr_0/2)^2-3Mr_0/2}{\{A_K(M,\lambda)\}-c_1^2k^2-2Mr_0(1-(kr_0)^2)}, \tag{55c}$$

when $K=1$,

$$\tan\delta_K(E,\lambda)\sim\frac{\pi}{2\ln(kr_0)}\frac{\{A_K(M,\lambda)\}^{-1}+c_1^2k^2-k^2r_0/(4M)}{\{A_K(M,\lambda)\}^{-1}+c_1^2k^2+\{2Mr_0\ln(kr_0)\}^{-1}},\tag{55d}$$

when $K=1/2$. When $E<-M$ we have for a sufficient small k,

$$\tan\delta_K(E,\lambda)\sim-\frac{\pi(kr_0/2)^{2K+1}}{(K+1/2)!(K-1/2)!}\frac{A_K(-M,\lambda)+(2K+1)/(2Mr_0)}{A_K(-M,\lambda)+c_2^2k^2+\dfrac{k^2r_0}{2M(2K-1)}},\tag{56a}$$

when $K\geqslant 1$,

$$\tan\delta_K(E,\lambda)\sim-\pi\left(\frac{kr_0}{2}\right)^2\frac{A_K(-M,\lambda)+1/(Mr_0)}{A_K(-M,\lambda)+c_2^2k^2-k^2r_0\ln(kr_0)/(2M)},\tag{56b}$$

when $K=1/2$. The asymptotic forms (41) have been used in driving Eqs. (55) and (56). In addition to the leading terms, we include the next leading terms in some of Eqs. (55) and (56), which are useful only for the critical case where the leading terms are canceled with each other.

First, from Eqs. (55) and (56) we see that, except for some critical cases, $\tan\delta_K(E,\lambda)$ tends to zero as E goes to $\pm M$, namely, $\delta_K(\pm M,\lambda)$ are always equal to the multiple of π. In other words, if the phase shift $\delta_K(E,\lambda)$ for a sufficiently small k is expressed as a positive or negative acute angle plus $n\pi$, its limit $\delta_K(M,\lambda)$ [or $\delta_K(-M,\lambda)$] is equal to $n\pi$. It means that $\delta_K(M,\lambda)$ [or $\delta_K(-M,\lambda)$] changes discontinuously when $\delta_K(E,\lambda)$ changes through the value $(n+1/2)\pi$.

Second, from Eq. (52) we have

$$\left.\frac{\partial\delta_K(E,\lambda)}{\partial A_K(E,\lambda)}\right|_E=-\left(\frac{E+M}{E-M}\right)^{1/2}\frac{2\{\cos\delta_K(E,\lambda)\}^2}{\pi kr_0\{N_{K+1/2}(kr_0)A_K(E,\lambda)-B(E)N_{K-1/2}(kr_0)\}^2}\leqslant 0,\quad E>M,$$

$$\left.\frac{\partial\delta_K(E,\lambda)}{\partial A_K(E,\lambda)}\right|_E=\left(\frac{|E|-M}{|E|+M}\right)^{1/2}\frac{2\{\cos\delta_K(E,\lambda)\}^2}{\pi kr_0\{N_{K+1/2}(kr_0)A_K(E,\lambda)-B(E)N_{K-1/2}(kr_0)\}^2}\geqslant 0,\quad E<-M.\tag{57}$$

Equation (57) shows that, as the ratio $A_K(E,\lambda)$ decreases, the phase shift $\delta_K(E,\lambda)$ for $E>M$ increases monotonically, but $\delta_K(E,\lambda)$ for $E<-M$ decreases monotonically. In terms of the monotonic properties, we are able to determine the jump of the phase shifts $\delta_K(\pm M,\lambda)$.

We first consider the scattering states of a positive energy with a sufficiently small momentum k. As $A_K(E,\lambda)$ decreases, if $\tan\delta_K(E,\lambda)$ changes sign from positive to negative, the phase shift $\delta_K(M,\lambda)$ jumps by π. Note that in this case, if $\tan\delta_K(E,\lambda)$ changes sign from negative to positive, the phase shift $\delta_K(M,\lambda)$ keeps invariant. Conversely, as $A_K(E,\lambda)$ increases, if $\tan\delta_K(E,\lambda)$ changes sign from negative to positive, the phase shift $\delta_K(M,\lambda)$ jumps by $-\pi$. Therefore, as λ increases from zero to one, each time the $A_K(M,\lambda)$ decreases from near and larger than the value $2Mr_0/(2K-1)$ to smaller than that value, the denominator in Eq. (57) changes sign from positive to negative and the rest factor keeps positive, so that the phase shift $\delta_K(M,\lambda)$ jumps by π. We have shown in the preceding section that each time $A_K(M,\lambda)$ decreases across the value $2Mr_0/(2K-1)$, a scattering state of a positive energy becomes a bound state. Conversely, each time $A_K(M,\lambda)$ increases across that value, the phase shift $\delta_K(M,\lambda)$ jumps by $-\pi$, and a bound state becomes a scattering state of a positive energy.

Then, we consider the scattering states of a negative energy with a sufficiently small k. As $A_K(E,\lambda)$ decreases, if $\tan\delta_K(E,\lambda)$ changes sign from negative to positive, the phase shift $\delta_K(-M,\lambda)$ jumps by $-\pi$. However, in this case, if $\tan\delta_K(E,\lambda)$ changes sign from positive to negative, the phase shift $\delta_K(-M,\lambda)$ keeps invariant. Conversely, as $A_K(E,\lambda)$ increases, if $\tan\delta_K(E,\lambda)$ changes sign from positive to negative, the phase shift $\delta_K(-M,\lambda)$ jumps by π. Therefore, as λ increases from 0 to 1, each time the $A_K(-M,\lambda)$ decreases from a small and positive number to a negative one, the denominator in Eq. (56) changes sign from positive to negative and the rest factor keeps negative, so that the phase shift $\delta_K(-M,\lambda)$ jumps by $-\pi$. In the preceding section we have shown that each time the $A_K(-M,\lambda)$ decreases across zero, a bound state becomes a scattering state of a negative energy. Conversely, each time the $A_K(-M,\lambda)$ increases across zero, the phase shift $\delta_K(-M,\lambda)$ jumps by π, and a scattering state of a negative energy becomes a bound state. Therefore, we obtain the Levinson theorem for the Dirac equation in D dimensions for noncritical cases:

$$\delta_K(M)+\delta_K(-M)=n_K\pi.\tag{58}$$

It is obvious that the Levinson theorem (58) holds for both positive and negative K in the noncritical cases.

For the case of $K=1/2$ and $E\sim M$, where the value $2Mr_0/(2K-1)$ is infinity. Since $\{A_K(E,\lambda)\}^{-1}$ increases as $A_K(E,\lambda)$ decreases, we can study the variance of $\{A_K(E,\lambda)\}^{-1}$ in this case instead. For the energy $E>M$ where the momentum k is sufficiently small, when $\{A_K(M,\lambda)\}^{-1}$ increases from negative to positive as λ increases, both the numerator and denominator in Eq. (55) change signs, but not simultaneously. The numerator changes sign first, and then, the denominator changes. The front factor in Eq. (55) is negative so that $\tan\delta_K(E,\lambda)$ first changes from negative to positive when the numerator changes sign, and then changes from positive to negative when the denominator changes sign. It is in the second step that the phase shift $\delta_K(M,\lambda)$ jumps by π. Similarly, each time $\{A_K(M,\lambda)\}^{-1}$ decreases across zero as λ increases, $\delta_K(M,\lambda)$ jumps by $-\pi$.

For $\lambda=0$ and $K=1/2$, the numerator in Eq. (55) is equal to zero, and the phase shift $\delta_K(M,0)$ is defined to be zero. For this case there is a half bound state at $E=M$ [see Eq. (59)]. If $\{A_K(M,\lambda)\}^{-1}$ increases $[A_K(M,\lambda)$ decreases] as λ increases from zero, the front factor in Eq. (55) is negative, the numerator becomes positive first, and then, the denominator changes sign from negative to positive, such that the phase shift $\delta_K(M,\lambda)$ jumps by π and simultaneously the half bound state becomes a bound state with $E<M$.

Now, we turn to study the critical cases. First, we study the critical case for $E=M$, where the ratio $A_K(M,1)$ is equal to the value $2Mr_0/(2K-1)$. It is easy to obtain the following solution of $E=M$ in the region $r_0<r<\infty$, satisfying the radial equations (35) and the match condition (38) at r_0:

$$f_{KM}(r,1)=2Mr^{-K+1},\quad g_{KM}(r,1)=(2K-1)r^{-K}. \tag{59}$$

It is a bound state when $K>3/2$, but called a half bound state when $K\leqslant 3/2$. A half bound state is not a bound state, because its wave function is finite but not square integrable.

For definiteness, we assume that in the critical case, as λ increases from a number near and less than one and finally reaches one, $A_K(M,\lambda)$ *decreases* and finally reaches, but not across, the value $2Mr_0/(2K-1)$. In this case, when $\lambda=1$, a new bound state of $E=M$ appears for $K>3/2$, but does not appear for $K\leqslant 3/2$. We should check whether or not the phase shift $\delta_K(M,1)$ increases by an additional π as λ increases and reaches one.

It is evident from the next leading terms in the denominator of Eq. (55) that the denominator for $K\geqslant 3/2$ has changed sign from positive to negative as $A_K(M,\lambda)$ decreases and finally reaches the value $2Mr_0/(2K-1)$, namely, the phase shift $\delta_K(M,\lambda)$ jumps by an additional π at $\lambda=1$. Simultaneously, a new bound state of $E=M$ appears for $K>3/2$, but only a half bound state appears for $K=3/2$, so that the Levinson theorem (58) holds for the critical case with $K>3/2$, but it has to be modified for the critical case when a half bound state occurs at $E=M$ and $K=3/2$:

$$\delta_K(M)+\delta_K(-M)=(n_K+1)\pi.$$

For $K=1$, $\tan\delta_K(E,1)$ tends to infinity as $\{A_K(M,\lambda)\}^{-1}$ increases and finally reaches $2Mr_0$, namely, the phase shift $\delta_K(M,\lambda)$ jumps by $\pi/2$. Simultaneously, only a new half bound state of $E=M$ for $K=1$ appears, so that the Levinson theorem (58) has to be modified for the critical case when a half bound state occurs at $E=M$ and $K=1$:

$$\delta_K(M)+\delta_K(-M)=(n_K+1/2)\pi.$$

For $K=1/2$, the next leading term with $\ln(kr_0)$ in the denominator of Eq. (55) dominates, so that the denominator keeps negative (does not change sign!) as $\{A_K(M,\lambda)\}^{-1}$ increases and finally reaches zero, namely, the phase shift $\delta_K(M,\lambda)$ does not jump, no matter whether the rest part in Eq. (55d) keeps positive or has changed to negative. Simultaneously, only a new half bound state of $E=M$ for $K=1/2$ appears, so that the Levinson theorem (58) holds for the critical case with $K=1/2$.

This conclusion holds for the critical case where $A_K(M,\lambda)$ *increases* and finally reaches, but not across, the value $2Mr_0/(2K-1)$. Therefore, for the critical case when a half bound state occurs at $E=M$ and $K\leqslant 3/2$, the Levinson theorem has to be modified as follows:

$$\delta_K(M)+\delta_K(-M)=(n_K+K-1/2)\pi. \tag{60}$$

Second, we study the critical case for $E=-M$, where the ratio $A_K(-M,1)$ is equal to zero. It is easy to obtain the following solution of $E=-M$ in the region $r_0<r<\infty$, satisfying the radial equations (35) and the match condition (38) at r_0:

$$f_{KM}(r,\lambda)=0,\quad g_{KM}(r,\lambda)=r^{-K}. \tag{61}$$

It is a bound state when $K\geqslant 1$, but a half bound state when $K=1/2$.

For definiteness, we again assume that in the critical case, as λ increases from a number near and less than 1 and finally reaches 1, $A_K(-M,\lambda)$ *decreases* and finally reaches zero, so that when $\lambda=1$ the energy of a bound state decreases to $E=-M$ for $K\geqslant 1$, but a bound state becomes a half bound state for $K=1/2$. We should check whether or not the phase shift $\delta_K(-M,1)$ decreases by π as λ increases and reaches one.

For the energy $E<-M$ where the momentum k is sufficiently small, one can see from the next leading terms in the denominator of Eq. (56) that the denominator does not change sign as $A_K(-M,\lambda)$ decreases and finally reaches zero, namely, the phase shift $\delta_K(-M,\lambda)$ does not jump by an additional $-\pi$ at $\lambda=1$. Simultaneously, the energy of a bound state decreases to $E=-M$ for $K\geqslant 1$, but a bound state becomes a half bound state for $K=1/2$, so that the Levinson theorem (58) holds for the critical case with $K\geqslant 1$, but it has to be modified when a half bound state occurs at $E=-M$ and $K=1/2$:

$$\delta_K(M)+\delta_K(-M)=(n_K+1)\pi. \tag{62}$$

Combining Eqs. (58), (60), and (62) and their corresponding forms for the negative K, we obtain the relativistic Levinson theorem in D dimensions.

VI. DISCUSSIONS

Now, we discuss the general case where the potential $V(r)$ has a tail at $r \geqslant r_0$. Let r_0 be so large that only the leading term in $V(r)$ is concerned:

$$V(r) \sim b r^{-n}, \quad r \geqslant r_0, \tag{63}$$

where b is a nonvanishing constant and n is a positive constant, not necessary to be an integer. Substituting it into Eq. (35) and changing the variable r to ξ:

$$\xi = \begin{cases} kr = r\sqrt{E^2-M^2}, & \text{when } |E|>M \\ \kappa r = r\sqrt{M^2-E^2}, & \text{when } |E| \leqslant M, \end{cases} \tag{64}$$

we obtain the radial equations in the region $r_0 \leqslant r < \infty$:

$$\begin{aligned} \frac{d}{d\xi} g_{KE}(\xi) + \frac{K}{\xi} g_{KE}(\xi) &= \left(\frac{E}{|E|}\sqrt{\frac{E-M}{E+M}} - \frac{b}{\xi^n} k^{n-1} \right) f_{KE}(\xi), \\ -\frac{d}{d\xi} f_{KE}(\xi) + \frac{K}{\xi} f_{KE}(\xi) &= \left(\frac{E}{|E|}\sqrt{\frac{E+M}{E-M}} - \frac{b}{\xi^n} k^{n-1} \right) g_{KE}(\xi), \end{aligned} \tag{65}$$

for $|E|>M$, and

$$\begin{aligned} \frac{d}{d\xi} g_{KE}(\xi) + \frac{K}{\xi} g_{KE}(\xi) &= \left(-\sqrt{\frac{M-E}{M+E}} - \frac{b}{\xi^n} \kappa^{n-1} \right) f_{KE}(\xi), \\ -\frac{d}{d\xi} f_{KE}(\xi) + \frac{K}{\xi} f_{KE}(\xi) &= \left(\sqrt{\frac{M+E}{M-E}} - \frac{b}{\xi^n} \kappa^{n-1} \right) g_{KE}(\xi), \end{aligned} \tag{66}$$

for $|E| \leqslant M$. As far as the Levinson theorem is concerned, we are only interested in the solutions with the sufficiently small k and κ. If $n \geqslant 3$, in comparison with the first term on the right-hand side of Eq. (65) or Eq. (66), the potential term with a factor of k^{n-1} (or κ^{n-1}) is too small to affect the phase shift at the sufficiently small k and the variant range of the ratio $f_{jE}(r,\lambda)/g_{jE}(r,\lambda)$ at r_{0_+}. Therefore, the proof given in the previous sections is effective for those potentials with a tail, so that the Levinson theorem (58) holds.

When $n=2$ and $b \neq 0$, we will only keep the leading terms for the small parameter k (or κ) in solving Eq. (65) [or Eq. (66)]. First, we calculate the solutions with the energy $E \sim M$. Let

$$\alpha = (K^2 - K + 2Mb + 1/4)^{1/2} \neq K - 1/2. \tag{67}$$

If $\alpha^2 < 0$, there is an infinite number of bound states. We will not discuss this case as well as the case with $\alpha = 0$ here. When $\alpha^2 > 0$, we take $\alpha > 0$ for convenience. Some formulas given in the previous sections will be changed.

When $E \leqslant M$, we have

$$\begin{aligned} f_{KE}(r,\lambda) &= e^{i(\alpha+1)\pi/2} 2M(\pi\kappa r/2)^{1/2} H_\alpha^{(1)}(i\kappa r), \\ g_{KE}(r,\lambda) &= e^{i(\alpha+1)\pi/2} \kappa(\pi\kappa r/2)^{1/2} \left\{ -\frac{d}{d(\kappa r)} H_\alpha^{(1)}(i\kappa r) + \frac{K-1/2}{\kappa r} H_\alpha^{(1)}(i\kappa r) \right\}. \end{aligned} \tag{68}$$

Hence, the ratio at $r = r_0+$ for $E=M$ is

$$\left. \frac{f_{KE}(r,\lambda)}{g_{KE}(r,\lambda)} \right|_{r=r_0+} = \frac{2Mr_0}{K+\alpha-1/2}, \quad E=M. \tag{69}$$

When $E>M$, we have

$$\begin{aligned} f_{KE}(r,\lambda) &= 2M(\pi kr/2)^{1/2}\{\cos\eta_\alpha(E,\lambda) J_\alpha(kr) - \sin \eta_\alpha(E,\lambda) N_\alpha(kr)\}, \\ g_{KE}(r,\lambda) &= k(\pi kr/2)^{1/2}\left\{ \cos\eta_\alpha(E,\lambda)\left(-\frac{d}{d(kr)} J_\alpha(kr) + \frac{K-1/2}{kr} J_\alpha(kr) \right) \right. \\ &\quad \left. - \sin\eta_\alpha(E,\lambda)\left(-\frac{d}{d(kr)} N_\alpha(kr) + \frac{K-1/2}{kr} N_\alpha(kr) \right) \right\}, \end{aligned} \tag{70}$$

When kr tends to infinity, the asymptotic form of the solution is

$$f_{KE}(r,\lambda) \sim 2M \cos[kr - \alpha\pi/2 - \pi/4 + \eta_\alpha(E,\lambda)],$$

$$g_{KE}(r,\lambda) \sim k \sin[kr - \alpha\pi/2 - \pi/4 + \eta_\alpha(E,\lambda)].$$

In comparison with solution (51), we obtain the phase shift $\delta_K(E,\lambda)$ for $E>M$:

$$\delta_K(E,\lambda) = \eta_\alpha(E,\lambda) + (K - \alpha - 1/2)\pi/2, \quad E>M. \tag{71}$$

From the match condition (38), for the sufficiently small k we obtain

$$\tan \eta_\alpha(E,\lambda) \sim \frac{-\pi(kr_0/2)^{2\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha)} \left(\frac{K-\alpha-1/2}{K+\alpha-1/2} \right) \frac{A_K(M,\lambda) - 2Mr_0/(K-\alpha-1/2)}{A_K(M,\lambda) - 2Mr_0/(K+\alpha-1/2)}. \tag{72}$$

Therefore, as λ increases from 0 to 1, each time the $A_K(M,\lambda)$ decreases from near and larger than the value $2Mr_0/(K+\alpha-1/2)$ to smaller than that value, the denominator in Eq. (72) changes sign from positive to negative and the rest factor keeps positive, so that $\eta_\alpha(M,\lambda)$ jumps by π. Simultaneously, from Eq. (69) a new overlap between the variant ranges of the ratio at two sides of r_0 appears such that a scattering state of a positive energy becomes a bound state. Conversely, each time the $A_K(M,\lambda)$ increases across that value, $\eta_\alpha(M,\lambda)$ jumps by $-\pi$, and a bound state becomes a scattering state of a positive energy.

Second, we calculate the solutions with the energy $E\sim -M$. Let

$$\beta=(K^2+K-2Mb+1/4)^{1/2}\neq K+1/2. \tag{73}$$

Similarly, we only discuss the cases with $\beta^2>0$, and take $\beta>0$.

When $E\geqslant -M$ we have

$$f_{KE}(r,\lambda)=-e^{i(\beta+1)\pi/2}\kappa(\pi\kappa r/2)^{1/2}\left\{\frac{d}{d(\kappa r)}H_\beta^{(1)}(i\kappa r)+\frac{K+1/2}{\kappa r}H_\beta^{(1)}(i\kappa r)\right\},$$

$$g_{KE}(r,\lambda)=e^{i(\beta+1)\pi/2}2M(\pi\kappa r/2)^{1/2}H_\beta^{(1)}(i\kappa r). \tag{74}$$

Hence, the ratio at $r=r_0+$ for $E=-M$ is

$$\left.\frac{f_{KE}(r,\lambda)}{g_{KE}(r,\lambda)}\right|_{r=r_0+}=-\frac{K-\beta+1/2}{2Mr_0},\quad E=-M. \tag{75}$$

When $E<-M$ we have

$$f_{KE}(r,\lambda)=-k(\pi kr/2)^{1/2}\left\{\cos\eta_\beta(E,\lambda)\left(\frac{d}{d(kr)}J_\beta(kr)+\frac{K+1/2}{kr}J_\beta(kr)\right)-\sin\eta_\beta(E,\lambda)\left(\frac{d}{d(kr)}N_\beta(kr)+\frac{K+1/2}{kr}N_\beta(kr)\right)\right\},$$

$$g_{KE}(r,\lambda)=2M(\pi kr/2)^{1/2}\{\cos\eta_\beta(E,\lambda)J_\beta(kr)-\sin\eta_\beta(E,\lambda)N_\beta(kr)\}, \tag{76}$$

When kr tends to infinity, the asymptotic form for the solution is

$$f_{KE}(r,\lambda)\sim k\sin[kr-\beta\pi/2-\pi/4+\eta_\beta(E,\lambda)],$$

$$g_{KE}(r,\lambda)\sim 2M\cos[kr-\beta\pi/2-\pi/4+\eta_\beta(E,\lambda)].$$

In comparison with solution (51), we obtain the phase shift $\delta_K(E,\lambda)$ for $E<-M$:

$$\delta_K(E,\lambda)=\eta_\beta(E,\lambda)+(K-\beta+1/2)\pi/2,\quad E<-M. \tag{77}$$

From the match condition (38), for the sufficiently small k, we obtain

$$\tan\eta_\alpha(E,\lambda)\sim\frac{-\pi(kr_0/2)^{2\beta}}{\Gamma(\beta+1)\Gamma(\beta)}\frac{A_K(-M,\lambda)+(K+\beta+1/2)/(2Mr_0)}{A_K(-M,\lambda)+(K-\beta+1/2)/(2Mr_0)}. \tag{78}$$

Therefore, as λ increases from 0 to 1, each time the $A_K(-M,\lambda)$ decreases from near and larger than the value $-(K-\beta+1/2)/(2Mr_0)$ to smaller than that value, the denominator in Eq. (78) changes sign from positive to negative and the rest factor keeps negative, so that $\eta_\beta(-M,\lambda)$ jumps by $-\pi$. Simultaneously, from Eq. (75), an overlap between the variant ranges of the ratio at two sides of r_0 disappears such that a bound state becomes a scattering state of a negative energy. Conversely, each time the $A_K(-M,\lambda)$ increases across that value, $\eta_\beta(-M,\lambda)$ jumps by π, and a scattering state of a negative energy becomes a bound state.

In summary, we obtain the modified relativistic Levinson theorem for noncritical cases when the potential has a tail (63) with $n=2$ at infinity:

$$\delta_K(M)+\delta_K(-M)=n_K\pi+(2K-\alpha-\beta)\pi/2. \tag{79}$$

We will not discuss the critical cases in detail. In fact, the modified relativistic Levinson theorem (79) holds for the critical cases of $\alpha>1$ and $\beta>1$. When $0<\alpha<1$ or $0<\beta<1$, $\eta_\alpha(M,1)$ or $\eta_\beta(-M,1)$ in the critical case will not be a multiple of π, respectively, so that Eq. (79) is violated for those critical cases.

Furthermore, for potential (63) with a tail at infinity, when $n>2$, even if it contains a logarithm factor, for any arbitrarily small positive ϵ, one can always find a sufficiently large r_0 such that $|V(r)|<\epsilon/r^2$ in the region $r_0<r<\infty$. Thus, from Eqs. (67) and (73) we have for the sufficiently small ϵ,

$$\alpha=(K^2-K\pm 2M\epsilon+1/4)^{1/2}\sim K-1/2,$$

$$\beta=(K^2+K\mp 2M\epsilon+1/4)^{1/2}\sim K+1/2.$$

Hence, Eq. (79) coincides with Eq. (58). In this case, the Levinson theorem (58) holds for the noncritical case.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China.

[1] N. Levinson, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **25**(9), 1 (1949).

[2] R.G. Newton, J. Math. Phys. **1**, 319 (1960); **18**, 1348, 1582 (1977); R.G. Newton, *Scattering Theory of Waves and Particles*, 2nd ed. (Springer-Verlag, New York, 1982), and references therein.

[3] J.M. Jauch, Helv. Phys. Acta **30**, 143 (1957).

[4] A. Martin, Nuovo Cimento **7**, 607 (1958).

[5] G.J. Ni, Phys. Energ. Fortis Phys. Nucl. **3**, 432 (1979); Z.Q. Ma and G.J. Ni, Phys. Rev. D **31**, 1482 (1985).

[6] Z.Q. Ma, J. Math. Phys. **26**, 1995 (1985).

[7] Z.Q. Ma, Phys. Rev. D **32**, 2203 (1985).

[8] Z.R. Iwinski, L. Rosenberg, and L. Spruch, Phys. Rev. A **31**, 1229 (1985).

[9] N. Poliatzky, Phys. Rev. Lett. **70**, 2507 (1993); R.G. Newton, Helv. Phys. Acta **67**, 20 (1994); Z.Q. Ma, Phys. Rev. Lett. **76**, 3654 (1996).

[10] Z.R. Iwinski, L. Rosenberg, and L. Spruch, Phys. Rev. A **33**, 946 (1986); L. Rosenberg and L. Spruch, *ibid.* **54**, 4985 (1996).

[11] R. Blankenbecler and D. Boyanovsky, Physica D **18**, 367 (1986).

[12] A.J. Niemi and G.W. Semenoff, Phys. Rev. D **32**, 471 (1985).

[13] F. Vidal and J. Letourneux, Phys. Rev. C **45**, 418 (1992).

[14] K.A. Kiers and W. van Dijk, J. Math. Phys. **37**, 6033 (1996).

[15] M.S. de Bianchi, J. Math. Phys. **35**, 2719 (1994).

[16] P.A. Martin and M.S. de Bianchi, Europhys. Lett. **34**, 639 (1996).

[17] M.E. Portnoi and I. Galbraith, Solid State Commun. **103**, 325 (1997).

[18] D. Bollé, F. Gesztesy, C. Danneels, and S.F.J. Wilk, Phys. Rev. Lett. **56**, 900 (1986).

[19] Q.G. Lin, Phys. Rev. A **56**, 1938 (1997).

[20] Shi-Hai Dong, Xi-Wen Hou, and Zhong-Qi Ma, Phys. Rev. A **58**, 2790 (1998).

[21] Shi-Hai Dong, Xi-Wen Hou, and Zhong-Qi Ma, Phys. Rev. A **58**, 2160 (1998).

[22] Shi-Hai Dong and Zhong-Qi Ma, Phys. Rev. A **65**, 042717 (2002).

[23] Xiao-Yan Gu, Zhong-Qi Ma, and Shi-Hai Dong, Int. J. Mod. Phys. E **11**, 335 (2002).

[24] J.D. Bjorken and S.D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[25] A. Erdelyi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 2, p. 232.

[26] J.D. Louck, J. Mol. Spectrosc. **4**, 298 (1960).

[27] A. Chatterjee, Phys. Rep. **186**, 249 (1990).

[28] Zhong-Qi Ma, *Yang-Baxter Equation and Quantum Enveloping Algebras*, edited by Dai Yuan-Ben, Hao Bai-Lin, and Su Zhao-Bin, Advanced Series on Theoretical Physical Science (World Scientific, Singapore, 1993), Vol. 1, Chap. 1.

[29] C.N. Yang, in *Monopoles in Quantum Field Theory*, Proceedings of the Monopole Meeting, Trieste, Italy, 1981, edited by N.S. Craigie, P. Goddard, and W. Nahm (World Scientific, Singapore, 1982), p. 237.