Qutrit quantum computer with trapped ions

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(Received 25 June 2002; revised manuscript received 14 January 2003; published 26 June 2003)

We study the physical implementation of a qutrit quantum computer in the context of trapped ions. Qutrits are defined in terms of electronic levels of trapped ions. We concentrate our attention on a universal two-qutrit gate, which corresponds to a controlled-NOT gate between qutrits. Using this gate and a general gate of an individual qutrit, any gate can be decomposed into a sequence of these gates. In particular, we show how this works for performing the quantum Fourier transform for n qutrits.

DOI: 10.1103/PhysRevA.67.062313 PACS number(s): 03.67.Lx

I. INTRODUCTION

The building blocks of a quantum computer are qubits [1], which are distinguishable two-level physical systems, which are manipulated individually as well as collectively. These operations are performed using the so-called quantum gates in analogy to their classical counterparts. In the simplest case, collective manipulation reduces to bipartite quantum gates, for instance, the controlled-NOT gate [2], also called XOR gate. In this gate one qubit acts as a control and the other as the target, so that if the state of the control qubit is in state $|1\rangle$, the state of the target qubit is flipped, and in other cases the target qubit remains unchanged. The experimental implementation of a two-qubit XOR has been studied in different physical contexts, as for instance, in trapped ions [3,4], nuclear magnetic resonance [5–7], cavity QED systems and quantum heterostructures [8–10].

In recent works, the use of quantum entanglement in higher-dimensional quantum systems has been studied. The notion of entanglement generation and characterization in the case of three-level quantum systems, qutrits, has been considered by authors in Ref. [11]. The use of qutrits instead of qubits has been proven to be more secure against a symmetric attack on a quantum key distribution protocol [12]. These studies require a generalized version of an XOR, which has been given in Ref. [13], which is called the GXOR gate. This gate operates on a tensor product of two qudits, states lying in a d-dimensional Hilbert space. Further studies on qudit systems have been considered, for example, bounds on entanglement between qudits [14], discrimination among the Bell states of qudits [15], entanglement among qudits [16], Greenberger-Horne-Zeilinger paradox for many qudits [17], quantum computing with qudits [18], quantum tomography for qudits states [19], entanglement swapping between multiqudit systems [20]. Recently, a quantum communication complexity protocol with two entangled qutrits has been proposed [21]. Finally, an important step towards the use of higher-dimensional quantum systems has been given with the experimental generation of entangled qutrits using two-photon states from a parametric down-conversion process [22].

In the present work, we study the accomplishment of quantum gates for a qutrit based quantum computer, by exploiting the possibility of coherent manipulation of ions in a linear trap [3]. The main goal in studying quantum computation with qutrits instead of qubits is the exponential increase of the available Hilbert space with the same amount of physical resources. In particular, we focus our attention both to implementing a quantum Fourier transform for one qutrit and to the conditional quantum gate between two qutrits, $\mathrm{XOR}^{(3)}$. We also give a protocol for a quantum Fourier transform among many qutrits. The extension of the physical implementation from qubits to qutrits is, of course, nontrivial because of the coherent operations required in a three-dimensional Hilbert space. As far as we know, there is a proposal for an implementation of a conditional gate for qudits, called GXOR based on two-mode field interactions [13], which assumes the existence of a discrete Fourier transform for an arbitrary field state lying in a D-dimensional subspace of one of the field modes.

Here, conditional gates between two qutrits are conveniently expressed as

$$\mathrm{XOR}_{12}^{(3)}|i\rangle_1|j\rangle_2=|i\rangle_1|j\pm i\rangle_2, \tag{1}$$

where $j\pm i$ denotes the addition (difference) $i\pm j$, modulo 3. As will be shown, this gate is decomposed as $\mathrm{XOR}_{mn}^{(3)}=F_n^{-1}P_{mn}F_n$, where F_n is the quantum Fourier transform for one qutrit, and P_{mn} is a conditional phase-shift gate for qutrits, where m and n are the control and target qutrits, respectively. As indicated above, the $\mathrm{XOR}^{(3)}$ gate is given in two main steps. One is the generation of the discrete quantum Fourier transform, which requires the coherent manipulation of populations in a three-dimensional Hilbert space. The second is the conditional phase-shift gate, which requires the intervention of an ancillary quantum channel between the qutrits.

II. ARBITRARY ONE-QUTRIT GATE

Here we consider the achievement of a general gate for a single qutrit. An arbitrary unitary operation on a qutrit state,

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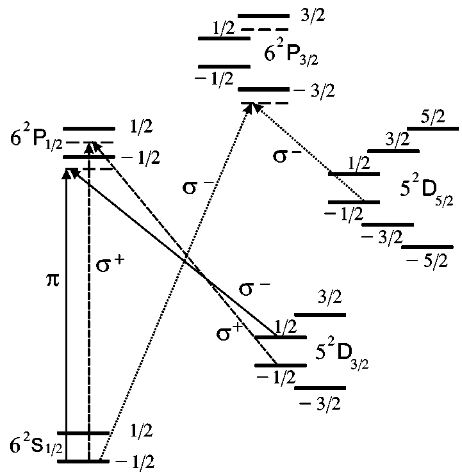

FIG. 1. Raman configurations for defining the logical states of a qutrit.

U(3) operation, is split up into a sequence of SU(2) operations. A physical setup for this is obtained in a linear array of trapped ions, where qutrit states are defined using Zeeman's level structure, in a $^{138}Ba^{+}$ ion in a Paul trap. A representation of this level structure is depicted in Fig. 1. Two Raman configurations are independently implemented between the levels $6S_{1/2}(m=-1/2)$, $6P_{1/2}(m=-1/2)$, $5D_{3/2}(m=-1/2)$ and levels $6S_{1/2}(m=-1/2)$, $6P_{3/2}(m=-3/2)$, $5D_{3/2}(m=-3/2)$. The first and second Raman configurations require a σ^{+} polarized light and a σ^{-} polarized light, respectively. Diode lasers [23] are used in order to achieve the interactions. In this case, typical values of magnetic fields for the Zeeman splitting are of the order of 10 G, and the corresponding energy differences between two consecutive Zeeman sublevels are $u=0.14|\vec{B}|$ (MHz/G) for level $S_{1/2}$, $u=\frac{2}{3}0.14|\vec{B}|$ (MHz/G) for level $P_{1/2}$, and $u=\frac{4}{5}0.14|\vec{B}|$ (MHz/G) for $D_{3/2}$. From these values, one finds that a typical energy difference is of few MHz. The transition frequency between levels $S\Leftrightarrow P$ and $D\Leftrightarrow P$ is typically hundreds of MHz, and the trap oscillation frequency reported is of the order of tens of MHz [1,24].

An extra single-ion operation is needed for implementing the controlled quantum gate operation, due to the presence of a phase if only two Raman configurations are used. This phase is removed with an additional Raman configuration, in this case $6S_{1/2}(m=-1/2)$, $6P_{1/2}(m=-1/2)$, $5D_{3/2}(m=-1/2)$. This transition $S\Leftrightarrow P$ is achieved with a π polarization, and the $S\Leftrightarrow D$ transition using a σ^{-} polarization for the electromagnetic field. A similar physical implementation is performed using Ca^{+}. However, using Ca^{+} extra operations are done via a quadrupole direct transition between the levels $6S_{1/2}$ and $5D_{5/2}$, which are forbidden in dipole couplings. All of these physical parameters make possible the implementation of a qutrit quantum computer with barium trapped ions using the fine structure [25].

For the sake of simplicity, here we consider a simplified level structure of ions, only depicting the relevant electronic transitions of ions, as is shown in Fig. 2. Levels $\{|0\rangle,|1\rangle,|2\rangle\}$ are the logical states of a single qutrit. The level $|0'\rangle$ is an auxiliary level necessary when defining the

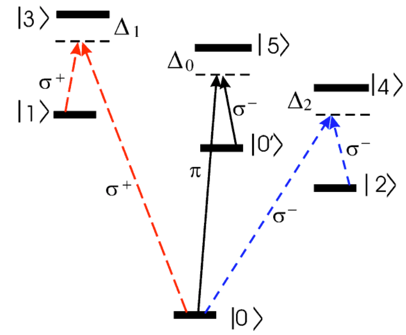

FIG. 2. Electronic level structure of trapped ion. Quantum information of qutrits is stored in levels $|0\rangle$, $|1\rangle$, and $|2\rangle$. The transitions involving effective interactions between levels $|0\rangle\rightarrow|1\rangle$ and $|0\rangle\rightarrow|2\rangle$ are driven by classical fields with different polarizations.

conditional two-qutrit gate. A key ingredient is the existence of electric dipole forbidden transitions $|0\rangle\rightarrow|1\rangle$ and $|0\rangle\rightarrow|2\rangle$. These transitions are addressed via Raman transitions through the independent channels associated with orthogonal polarizations, driven by classical fields Ω_{03}, Ω_{13}, Ω_{04}, and Ω_{24}.

In this system, the ion level populations are manipulated by selecting the desired coherent operation. For instance, we can independently operate with transitions $|0\rangle\rightarrow|1\rangle$, $|0\rangle\rightarrow|2\rangle$, $|1\rangle\rightarrow|2\rangle$ by adjusting the parameters. The Hamiltonian describing this effective system, under the standard dipole and rotating wave approximations, is given by

$$H=\sum_{j}\hbar\omega_j|j\rangle\langle j|+\hbar\{e^{-i\nu_2 t}(\Omega_{04}|4\rangle\langle 0|+\Omega_{03}|3\rangle\langle 0|)+e^{-i\nu_1 t}(\Omega_{13}|3\rangle\langle 1|+\Omega_{24}|4\rangle\langle 2|)+\text{H.c.}\},\tag{2}$$

where $j=0,1,2,3,4$. In the case of single-qutrit gates, only the carrier transition in the ion is considered, so that no explicit effects on the center-of-mass motion of the ion are included. Thus, the spatial dependences of Raman fields have been included as phase factors. Assuming the following conditions: $\Delta=(\omega_4-\omega_0)-\nu_2=(\omega_3-\omega_0)-\nu_2=(\omega_4-\omega_2)-\nu_1=(\omega_3-\omega_1)-\nu_1$ and $\Delta\gg\Omega_{04}$, Ω_{03}, Ω_{31}, Ω_{42}, rapidly decaying upper levels $|3\rangle$ and $|4\rangle$ are adiabatically eliminated, leading to an effective Hamiltonian:

$$\frac{H}{\hbar}=-\frac{|\Omega_{31}|^2}{\Delta}|1\rangle\langle 1|-\frac{|\Omega_{42}|^2}{\Delta}|2\rangle\langle 2|-\frac{|\Omega_{30}|^2+|\Omega_{40}|^2}{\Delta}|0\rangle\langle 0|-\left\{\frac{\Omega_{31}\Omega_{30}^*}{\Delta}|0\rangle\langle 1|+\frac{\Omega_{42}\Omega_{40}^*}{\Delta}|0\rangle\langle 2|+\text{H.c.}\right\}.\tag{3}$$

Assuming the additional condition

$$\frac{|\Omega_{31}|^2}{\Delta}=\frac{|\Omega_{42}|^2}{\Delta}=\frac{|\Omega_{30}|^2+|\Omega_{40}|^2}{\Delta},\tag{4}$$

we can find the evolution for the system. After some calculations, the evolution operator in the restricted three-dimensional space $\{|2\rangle,|1\rangle,|0\rangle\}$ is given by

$$U(\varphi)=\begin{pmatrix} 1+|g|^2C(\varphi) & gg'^*C(\varphi) & -ig\sin\varphi \\ g'g^*C(\varphi) & 1+|g'|^2C(\varphi) & -ig'\sin\varphi \\ -ig^*\sin\varphi & -ig'^*\sin\varphi & \cos\varphi \end{pmatrix}, \tag{5}$$

where $\varphi=\Omega t$ is an adimensional interaction time, $C(\varphi)=\cos\varphi-1$, and $\Omega^2=|\kappa'|^2+|\kappa|^2$. We have introduced the notations $g=\kappa/\Omega$ and $g'=\kappa'/\Omega$, where $\kappa=\Omega_{40}\Omega_{42}^*/\Delta$ and $\kappa'=\Omega_{30}\Omega_{31}^*/\Delta$. This evolution operator allows implementing all the required coherent operations between any two logical states. For instance, to activate the transition $|1\rangle\rightarrow|2\rangle$, we assume $\varphi=\pi$ in Eq. (5), so that

$$U_1=\begin{pmatrix} \cos\alpha & -e^{i\beta_1}\sin\alpha & 0 \\ -e^{-i\beta_1}\sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{6}$$

where we have defined

$$\cos\alpha=\frac{|\kappa'|^2-|\kappa|^2}{|\kappa'|^2+|\kappa|^2} \text{ and } e^{i\beta_1}=\kappa\kappa'^*/|\kappa\kappa'^*|. \tag{7}$$

Other transitions are addressed by manipulating the κ couplings in the transition $|0\rangle\rightarrow|2\rangle$, or κ' in the transition $|0\rangle\rightarrow|1\rangle$. For example, the transition $|0\rangle\rightarrow|2\rangle$ is addressed by assuming $\kappa'=0$ so that

$$U_2=\begin{pmatrix} \cos\varphi_2 & 0 & -ie^{-i\beta_2}\sin\varphi_2 \\ 0 & 1 & 0 \\ -ie^{i\beta_2}\sin\varphi_2 & 0 & \cos\varphi_2 \end{pmatrix}, \tag{8}$$

where $\kappa=e^{i\beta_2}|\kappa|$ and $\varphi_3=|\kappa|t$. Finally, for the transition $|0\rangle\rightarrow|1\rangle$, we assume $\kappa=0$:

$$U_3=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_3 & -ie^{i\beta_3}\sin\varphi_3 \\ 0 & -ie^{-i\beta_3}\sin\varphi_3 & \cos\varphi_3 \end{pmatrix}, \tag{9}$$

where $\kappa'=e^{i\beta_3}|\kappa'|$ and $\varphi_3=|\kappa'|t$. In order to generate any SU(3) operator [26], we need to have a decomposition of height-independent parameters. In the previous cases, we have three operations involving six independent parameters. By connecting interactions of classical fields with $|0\rangle\rightarrow|1\rangle$ and the $|0\rangle\rightarrow|2\rangle$ transitions in the far off-resonance limit, we obtain the following dispersive evolution:

$$U_D=\begin{pmatrix} e^{i\varrho} & 0 & 0 \\ 0 & e^{i\varepsilon} & 0 \\ 0 & 0 & e^{-i(\varrho+\varepsilon)} \end{pmatrix}, \tag{10}$$

which provides two additional parameters.

In particular, we can now focus on the calculation of the quantum Fourier transform for one qutrit, which is a unitary operation defined by

$$F|j\rangle=\frac{1}{\sqrt{3}}\sum_{l=0}^{2} e^{2i\pi lj/3}|l\rangle. \tag{11}$$

Explicitly, the transformed states $|\overline{j}\rangle=F|j\rangle$ read as

$$|\overline{0}\rangle=\frac{1}{\sqrt{3}}(|0\rangle+|1\rangle+|2\rangle), \tag{12}$$

$$|\overline{1}\rangle=\frac{1}{\sqrt{3}}(|0\rangle+e^{2i\pi/3}|1\rangle+e^{-2i\pi/3}|2\rangle),$$

$$|\overline{2}\rangle=\frac{1}{\sqrt{3}}(|0\rangle+e^{-2i\pi/3}|1\rangle+e^{2i\pi/3}|2\rangle).$$

This transformation is obtained by using the general operator we have calculated before, for the carrier transition, and combining with adiabatic transitions in both polarization channels. After some calculations, it is found that the Fourier transform is decomposed into the form

$$F=iU_DU_2U_3U_1, \tag{13}$$

where each one of these operations is obtained from the above process: In U_D, if $\varrho=\pi/3$, $\varepsilon=\pi/6$, we get

$$U_D=\begin{pmatrix} e^{i\pi/3} & 0 & 0 \\ 0 & e^{i\pi/6} & 0 \\ 0 & 0 & e^{-i\pi/2} \end{pmatrix}. \tag{14}$$

In U_1 with $\alpha=\pi/4,\beta_1=-2\pi/3$,

$$U_1=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -e^{-2i\pi/3} & 0 \\ -e^{2i\pi/3} & -1 & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}. \tag{15}$$

If $\varphi_2=\pi/4,\beta_2=-\pi/3$ in U_2,

$$U_2=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & ie^{2i\pi/3} \\ 0 & \sqrt{2} & 0 \\ ie^{-2i\pi/3} & 0 & 1 \end{pmatrix}. \tag{16}$$

Finally, assuming $\varphi_3=-\pi/3,\beta_3=7\pi/6$ in U_3,

$$U_3=\frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & -\sqrt{2} & ie^{-i\pi/6} \\ 0 & ie^{i\pi/6} & -\sqrt{2} \end{pmatrix}. \tag{17}$$

In the same way, any arbitrary one-qutrit gate is decomposed as in the case of the Fourier transform using the evolution operator given by Eq. (5).

III. CONDITIONAL TWO-QUTRIT GATE

The Fourier transform that we have found plays the same role that the Hadamard gate plays for qubits. For example, the entangled states of two qubits are generated by the application of a Hadamard gate followed by a conditional phase-shift gate. In what follows we will describe the accomplishment of a conditional phase shift between two qutrits, which allows the achievement of the $\mathrm{XOR}^{(3)}$ gate we have introduced in Sec. I. A basic requirement for performing a conditional gate between two qutrits is providing a mechanism to distinguish independent quantum paths, in order to satisfy the conditional change in the target qutrit depending on the state of the control. In our case, we have defined the $\mathrm{XOR}^{(3)}$ in such a way that the target changes only when the control qutrit is in state $|1\rangle$ or $|2\rangle$, provided that the state of the target will be $|j\oplus 0\rangle=|j\rangle$ when the control qubit is in state $|0\rangle$. Thus, we only need to implement a protocol considering independent quantum channels through states $|1\rangle$ and $|2\rangle$ of the control qutrit. In a set of ions in a linear trap, in the electronic configuration given in Fig. 2, such quantum channels are established with the compromise of the collective center-of-mass (CM) motion of ions inside the trap.

In the ion-trap quantum computer with qubits [3], the quantum channel between ions is established through the center-of-mass motion, which is addressed by adjusting a Raman transition to a given red sideband. As we shall see in what follows, we can proceed along the same line of reasoning, considering the ion model described in Fig. 2. Let us assume that we adjust the field amplitudes such that $\Omega_{04}=\Omega_{24}=0$, and Ω_{31}, $\Omega_{03}\neq 0$; or $\Omega_{04}=\Omega_{24}\neq 0$, and Ω_{31}, $\Omega_{03}=0$. In both cases, after eliminating the upper excited level and adjusting to the first red sideband transition, we obtain the Hamiltonian describing the ion center of mass coupled to the electronic transition $|0\rangle\rightarrow|q\rangle$:

$$H_{n,q}=\frac{\Omega_q\eta}{2}[|q\rangle_n\langle 0|ae^{-i\delta t-i\phi}+a^{\dagger}|0\rangle_n\langle q|e^{i\delta t+i\phi}]. \tag{18}$$

Here a and $a^{\dagger}$ are the annihilation and creation operators of the CM phonons, respectively, Ω_q is the effective Rabi frequency after adiabatic elimination of upper excited levels, ϕ is the laser phase, $\delta=\omega_2-\omega_0-\nu_2+\nu_1+\nu_x=\omega_1-\omega_0-\nu_2+\nu_1+\nu_x$, and $\eta=\sqrt{\hbar k_\theta^2/(2M\nu_x)}$ is the Lamb-Dicke parameter ($k_\theta=l\cos\theta$, with $\vec{k}$ the laser wave vector and θ the angle between the X axis and the direction of laser propagation). The subscript $q=1,2$ refers to the transition excited by the laser in the nth ion. The center-of-mass motion coupled to electronic transitions is coherently manipulated by the selection of the effective interaction time and laser polarizations. This Hamiltonian allows implementing the coherent interaction between qutrits and the collective center-of-mass motion. In particular, in order to implement a conditional phase shift, the following operations are needed:

$$U_m^{l,q}(\phi)|0\rangle_m|0\rangle=|0\rangle_m|0\rangle,$$

$$U_m^{l,q}(\phi)|0\rangle_m|1\rangle=\cos\left(\frac{l\pi}{2}\right)|0\rangle_m|1\rangle-ie^{-i\phi}\sin\left(\frac{l\pi}{2}\right)|q\rangle_m|0\rangle,$$

$$U_m^{l,q}(\phi)|q\rangle_m|0\rangle=\cos\left(\frac{l\pi}{2}\right)|q\rangle_m|0\rangle-ie^{i\phi}\sin\left(\frac{l\pi}{2}\right)|0\rangle_m|1\rangle,$$

where we have defined $\Omega_q\eta t/2=l\pi/2$. As we shall see, these coherent operations allow selecting the quantum channel for transferring the information to the center-of-mass. After this it is necessary to introduce a phase change in the qutrit state, which depends on the energy of the center-of-mass state. This phase change is accomplished through the dispersive regime of the first red sideband in Eq. (18), that is,

$$D_m^q(\varphi)=e^{i\varphi aa^{\dagger}}|q\rangle_m\langle q|+e^{-i\varphi a^{\dagger}a}|0\rangle_m\langle 0|, \tag{19}$$

where $\varphi=(\Omega_q\eta)^2/4\delta$, which allows for an intensity-dependent phase shift of the electronic levels.

From Eq. (12) it is not difficult to figure out that the conditional phase shift needed to implement the $\mathrm{XOR}^{(3)}|i\rangle_m|j\rangle_n=|i\rangle_m|j\ominus i\rangle_n$ is given by

$$\begin{array}{lcl}
|0\rangle_m|0\rangle_n & & |0\rangle_m|0\rangle_n\\
|0\rangle_m|1\rangle_n & & |0\rangle_m|1\rangle_n\\
|0\rangle_m|2\rangle_n & & |0\rangle_m|2\rangle_n\\
|1\rangle_m|0\rangle_n & & |1\rangle_m|0\rangle_n\\
|1\rangle_m|1\rangle_n & \xrightarrow{P_{mn}^{(2)}P_{mn}^{(1)}} & e^{4i\pi/3}|1\rangle_m|1\rangle_n\\
|1\rangle_m|2\rangle_n & & e^{2i\pi/3}|1\rangle_m|2\rangle_n\\
|2\rangle_m|0\rangle_n & & |2\rangle_m|0\rangle_n\\
|2\rangle_m|1\rangle_n & & e^{2i\pi/3}|2\rangle_m|1\rangle_n\\
|2\rangle_m|2\rangle_n & & e^{4i\pi/3}|2\rangle_m|2\rangle_n,
\end{array} \tag{20}$$

where $P_{mn}^{(1)}$ and $P_{mn}^{(2)}$ are defined as follows:

$$\begin{aligned}
P_{mn}^{(1)}(\phi_1,\phi_2)&=R_{00'}(\pi)U_m^{1,1}(3\pi/2)\mathcal{D}_n^2(\xi_2)D_n^2(\phi_2)\mathcal{D}_n^1(\xi_1)\\
&\quad\times D_n^1(\phi_1)U_m^{1,1}(\pi/2)R_{00'}(\pi),\\
P_{mn}^{(2)}(\phi_2,\phi_1)&=R_{00'}(\pi)U_m^{1,2}(3\pi/2)\mathcal{D}_n^2(\xi_1)D_n^2(\phi_1)\mathcal{D}_n^1(\xi_2)\\
&\quad\times D_n^1(\phi_2)U_m^{1,2}(\pi/2)R_{00'}(\pi),
\end{aligned} \tag{21}$$

with $\xi_i=2\pi-\phi_i$. The operation $R_{00'}(\pi)$ is a rotation which only impinges on the ion when it is in the $|0\rangle$ level, sending it to the $|0'\rangle$ level, avoiding producing any phase shift in this state. The dispersive operations affecting transitions $|0\rangle\rightarrow|1\rangle$ and $|0\rangle\rightarrow|2\rangle$ are given by

$$D_n^1(\phi_1)=\begin{pmatrix}1 & 0 & 0\\ 0 & e^{i\phi_1 aa^{\dagger}} & 0\\ 0 & 0 & e^{-i\phi_1 a^{\dagger}a}\end{pmatrix},$$

$$\mathcal{D}_n^1(\varphi_1)=\begin{pmatrix}1 & 0 & 0\\ 0 & e^{i\varphi_1} & 0\\ 0 & 0 & e^{-i\varphi_1}\end{pmatrix},$$

$$D_n^2(\phi_2)=\begin{pmatrix}e^{i\phi_2 aa^{\dagger}} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{-i\phi_2 a^{\dagger}a}\end{pmatrix},$$

$$\mathcal{D}_n^2(\varphi_2)=\begin{pmatrix} e^{i\varphi_2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\varphi_2} \end{pmatrix}.$$

Thus, the particular phase shift in Eq. (20) is achieved for $P_{mn}^{(1)}(4\pi/3,2\pi/3)$ and for $P_{mn}^{(2)}(2\pi/3,4\pi/3)$. Finally, the effective conditional change of the state of the target qutrit gives rise to the $\mathrm{XOR}^{(3)}$ gate. In brief,

$$\mathrm{XOR}_{mn}^{(3)}=F_n^{-1}P_{mn}^{(2)}P_{mn}^{(1)}F_n. \tag{22}$$

In this way, the $\mathrm{XOR}_{mn}^{(3)}$ produces the following evolution of state of two qutrits:

$$\begin{array}{ccccccc}
|0\rangle|0\rangle & & |0\rangle|\overline{0}\rangle & & |0\rangle|\overline{0}\rangle & & |0\rangle|0\rangle \\
|0\rangle|1\rangle & & |0\rangle|\overline{1}\rangle & & |0\rangle|\overline{1}\rangle & & |0\rangle|1\rangle \\
|0\rangle|2\rangle & & |0\rangle|\overline{2}\rangle & & |0\rangle|\overline{2}\rangle & & |0\rangle|2\rangle \\
|1\rangle|0\rangle & & |1\rangle|\overline{0}\rangle & & |1\rangle|\overline{2}\rangle & & |1\rangle|2\rangle \\
|1\rangle|1\rangle & \overset{F_n}{\rightarrow} & |1\rangle|\overline{1}\rangle & \xrightarrow{P_{mn}^{(2)}P_{mn}^{(1)}} & |1\rangle|\overline{0}\rangle & \overset{F_n^{-1}}{\rightarrow} & |1\rangle|0\rangle \\
|1\rangle|2\rangle & & |1\rangle|\overline{2}\rangle & & |1\rangle|\overline{1}\rangle & & |1\rangle|1\rangle \\
|2\rangle|0\rangle & & |2\rangle|\overline{0}\rangle & & |2\rangle|\overline{1}\rangle & & |2\rangle|1\rangle \\
|2\rangle|1\rangle & & |2\rangle|\overline{1}\rangle & & |2\rangle|\overline{2}\rangle & & |2\rangle|2\rangle \\
|2\rangle|2\rangle & & |2\rangle|\overline{2}\rangle & & |2\rangle|\overline{0}\rangle & & |2\rangle|0\rangle.
\end{array} \tag{23}$$

Universal quantum computation requires, in addition, a measurement scheme in the computational basis. In our case, von Neumann measurements distinguishing among three directions $|0\rangle$, $|1\rangle$, $|2\rangle$ are accomplished by connecting resonant interactions from $|1\rangle$, $|2\rangle$ to states $|3\rangle$, $|4\rangle$, respectively. Fast decay of excited optical levels through separated polarization channels allows us to discriminate between occupation of levels $|1\rangle$, $|2\rangle$, when fluorescence is observed; or level $|0\rangle$ when nothing is observed.

As a final remark, it should be stated that a conditional modular addition $|j\oplus i\rangle$, as the conditional operation between qutrits, is defined instead of a modular difference operation, just by adjusting the conditional phase shift in Eq. (20) to

$$|1\rangle_m|1\rangle_n\rightarrow e^{2i\pi/3}|1\rangle_m|1\rangle_n,$$

$$|1\rangle_m|2\rangle_n\rightarrow e^{4i\pi/3}|1\rangle_m|2\rangle_n,$$

$$|2\rangle_m|1\rangle_n\rightarrow e^{4i\pi/3}|2\rangle_m|1\rangle_n,$$

$$|2\rangle_m|2\rangle_n\rightarrow e^{2i\pi/3}|2\rangle_m|2\rangle_n.$$

IV. THE FOURIER TRANSFORM FOR *n* QUTRITS

A natural extension of the previous sections is to consider the analysis of the general protocol for the quantum Fourier transform for a system of *n* qutrits. We can benefit, of course, from the mathematical guidelines developed to define the Fourier transform for a system of *n* qubits [1]. In basis 3, the general expression for the Fourier transform is given by

$$|\overline{j}\rangle=\frac{1}{3^{n/2}}\sum_{k=0}^{3^n-1} e^{2\pi ij(k/3^n)}|k\rangle. \tag{24}$$

In this context, one important element is the decomposition of an integer number $0\leqslant j\leqslant 3^n-1$ in basis 3, which is conveniently given as

$$j=j_1 3^{n-1}+j_2 3^{n-2}+\cdots+j_n 3^0. \tag{25}$$

The fraction $k/3^n$ can be written in the 3-basis as

$$\frac{k}{3^n}=\frac{k_1 3^{n-1}}{3^n}+\frac{k_2 3^{n-2}}{3^n}+\cdots+\frac{k_n 3^0}{3^n}=\sum_{l=1}^{n} k_l 3^{-l}. \tag{26}$$

Following an analysis similar to that carried out for a system of *n* qubit, so that the Fourier transform now is written in the form

$$|\overline{j}\rangle=\frac{1}{3^{n/2}}\otimes_{l=1}^{n}\left[\sum_{k_l=0}^{2} e^{2\pi ijk_l 3^{-l}}|k_l\rangle\right]. \tag{27}$$

If the condition $l<n$ is satisfied, we get

$$\begin{aligned}\frac{j}{3^l}&=\mathrm{int}+\frac{j_{n+1-l}}{3}+\frac{j_{n+2-l}}{3^2}+\cdots+\frac{j_n}{3^l}\\ &=\mathrm{int}+0j_{n+1-l}j_{n+2-l}\cdots j_n,\end{aligned} \tag{28}$$

where ‘‘int’’ denotes an integer number. Then

$$|\overline{j}\rangle=\frac{1}{3^{n/2}}\otimes_{l=1}^{n}\left[\sum_{k_l=0}^{2} e^{2\pi ik_l[0j_{n+1-l}j_{n+2-l}\cdots j_n]}|k_l\rangle\right]. \tag{29}$$

This product is written in an equivalent form as

$$\begin{aligned}|\overline{j}\rangle=&\frac{1}{3^{n/2}}\left(\sum_{k_1=0}^{2} e^{2\pi ik_1 0j_n}|k_1\rangle\right)\left(\sum_{k_2=0}^{2} e^{2\pi ik_2 0j_{n-l}j_n}|k_2\rangle\right)\\ &\cdots\left(\sum_{k_n=0}^{2} e^{2\pi ik_n 0j_1j_2\cdots j_n}|k_n\rangle\right).\end{aligned} \tag{30}$$

If we expand the summation for each factor, for example, in the last term we have

$$\begin{aligned}&\sum_{k_n=0}^{3-1} e^{2\pi ik_n(0j_1j_2\cdots j_n)}|k_n\rangle\\ &=\frac{1}{\sqrt{3}}(|0\rangle+e^{2\pi i(0j_1j_2\cdots j_n)}|1\rangle+e^{4\pi i(0j_1j_2\cdots j_n)}|2\rangle)\\ &=\frac{1}{\sqrt{3}}(|0\rangle+e^{2\pi i(j_1/3+j_2/3^2\cdots j_n/3^n)}|1\rangle\\ &\quad+e^{4\pi i(j_1/3+j_2/3^2\cdots j_n/3^n)}|2\rangle).\end{aligned} \tag{31}$$

This state is generated starting with the application of a Fourier transform on the first qutrit,

$$F_1^{(3)}|j_1\rangle|j_2\rangle,|j_3\rangle\cdots|j_n\rangle$$

$$=\frac{1}{\sqrt{3}}(|0\rangle+e^{2\pi i j_1/3}|1\rangle+e^{4\pi i j_1/3}|2\rangle)|j_2\rangle,|j_3\rangle\cdots|j_n\rangle,$$

and after applying conditional phase transformations on this qutrit state, conditioned to the initial state of remaining $|j_2\rangle,|j_3\rangle,\ldots,|j_n\rangle$ qutrit states,

$$R_{n1}\cdots R_{31}R_{21}F_1^{(3)}|j_1\rangle|j_2\rangle|j_3\rangle\cdots|j_n\rangle$$

$$=\frac{1}{\sqrt{3}}(|0\rangle+e^{2\pi i(j_1/3+j_2/3^2\cdots j_n/3^n)}|1\rangle$$

$$+e^{4\pi i(j_1/3+j_2/3^2\cdots j_n/3^n)}|2\rangle)|j_2\rangle|j_3\rangle\cdots|j_n\rangle.$$

The conditional phase is given by

$$R_{21}=P_{21}^{(2)}\left(\frac{4\pi}{3^2},\frac{8\pi}{3^2}\right)P_{21}^{(1)}\left(\frac{2\pi}{3^2},\frac{4\pi}{3^2}\right), \quad (32)$$

$$R_{31}=P_{31}^{(2)}\left(\frac{4\pi}{3^3},\frac{8\pi}{3^3}\right)P_{31}^{(1)}\left(\frac{2\pi}{3^3},\frac{4\pi}{3^3}\right),$$

$$\ldots.$$

$$R_{k1}=P_{k1}^{(2)}P_{k1}^{(1)},$$

$$P_{k1}^{(j_k)}=\left(\frac{2j_k\pi}{3^k},\frac{4j_k\pi}{3^k}\right). \quad (33)$$

In the same way, the state

$$\sum_{k_n=0}^{2} e^{2\pi i k_n(0j_2j_3\cdots j_n)}|k_n\rangle$$

$$=\frac{1}{\sqrt{3}}(|0\rangle+e^{2\pi i(0j_2j_3\cdots j_n)}|1\rangle+e^{4\pi i(0j_2j_3\cdots j_n)}|2\rangle)$$

$$=\frac{1}{\sqrt{3}}(|0\rangle+e^{2\pi i(j_2/3+j_3/3^2+\cdots+j_n/3^{n-1})}|1\rangle$$

$$+e^{4\pi i(j_2/3+j_3/3^2+\cdots+j_n/3^{n-1})}|2\rangle) \quad (34)$$

is generated by applying a Fourier transform on the qutrit $|j_2\rangle$ and after applying conditional phase operations on this qutrit state, conditioned to the state of the remaining $|j_3\rangle,\ldots,|j_n\rangle$ qutrit states.

V. SUMMARY

We have described a physical implementation of a universal qutrit quantum computer based on trapped ions. The logical states of qutrits are codified into the electronic levels of trapped ions. We have shown how to implement an arbitrary one-qutrit gate. Besides, we have built a two-qutrit gate, which is decomposed into a quantum Fourier transform and a phase-shift gate. The main result of this work is that the same kind of physical setup used for a qubit quantum computer is suitable for a qutrit quantum computer, which gives an exponential increase of the available Hilbert space for the same amount of physical resources. In principle, this scheme also allows for entanglement distributions between qutrits allocated in widely distant nodes of a quantum network.

ACKNOWLEDGMENTS

We acknowledge A. Delgado for stimulating comments. This work was partially supported by grants from CONACYT (Mexico) and CONICYT (Chile) ''Información Cuántica e Interacciones Colectivas,'' Milenio Grant No. ICM P99-135F and FONDECYT Grant No. 1010010. R.G. thanks to MECESUP Grant No. USA0108.

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