Schmidt numbers of low-rank bipartite mixed states

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Schmidt numbers of bipartite mixed states [B. M. Torhal and P. Horodecki, Phys. Rev. A **61**, R040301 (2001)] characterize the minimum Schmidt ranks of pure states that are needed to construct such mixed states. Schmidt number is the minimum number of entangled degrees of freedom of a bipartite mixed state. We give a lower bound for the Schmidt numbers of low-rank bipartite mixed states and conclude that generic low-rank mixed states have relatively high Schmidt numbers and thus entangled.

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In quantum information theory, the study of bipartite entanglement is of great importance. A bipartite pure state $|\psi\rangle$ can always be described by its Schmidt decomposition, i.e., the representation of $|\psi\rangle$ in an orthogonal product basis with minimum number of terms, $|\psi\rangle=\Sigma_{i=1}^{k}p_i|a_i\rangle\otimes|b_i\rangle$ with positive reals p_i's. The Schmidt rank k is the total number of nonvanishing terms in the representation. It is actually the rank of the reduced density matrix $\rho_B=\mathrm{Tr}_B(|\psi\rangle\langle\psi|)$. The Schmidt ranks of pure states give a clear insight into the number of degrees of freedom that are entangled between two parties [2]. A necessary condition for a pure state to be convertible by local operations and classical communication (LOCC) to another pure state is that the Schmidt rank of the first pure state is larger than or equal to the Schmidt rank of the latter pure state; that is, local operations and classical communication cannot increase the Schmidt rank of a pure state [3]. The characterization of mixed state entanglement is a more difficult task. A great effort has been devoted to detecting the presence of entanglement in a given mixed state (see Refs. [4–7] for the study of separability criteria. Terhal and Horodecki [1] introduced the concept of Schmidt numbers of bipartite mixed states. For a bipartite mixed state ρ, it has Schmidt number k if and only if for any decomposition $\rho=\Sigma_i p_i|v_i\rangle\langle v_i|$ with positive real numbers p_i's and pure states $|v_i\rangle$'s, at least one of the pure states $|v_i\rangle$'s has Schmidt rank at least k, and there exists such a decomposition with all pure states $|v_i\rangle$'s Schmidt rank at most k. Schmidt numbers of bipartite mixed states characterize the minimum Schmidt ranks of pure states that are needed to construct such mixed states. Schmidt number is the minimum number of entangled degrees of freedom of a bipartite mixed state. The mixed states are entangled if and only if their Schmidt numbers are greater than 1. It is proved [1] that Schmidt number is entanglement monotone, i.e., it cannot increase under local operations and classical communication.

Because Schmidt numbers of bipartite mixed states cannot increase under LOCC, it is desirable if people could compute Schmidt numbers of bipartite mixed states effectively and thus it would offer an effective-deciding criterion that two bipartite mixed states cannot be convertible by LOCC. However, in this aspect, Schmidt numbers are only calculated for very few bipartite mixed states, for example, it is calculated for "isotropic states" $\rho=[(1-F)/(N^2-1)](I-|\Psi\rangle\langle\Psi|)+F|\Psi\rangle\langle|\Psi|$ on $H_A^N\otimes H_B^N$, where $|\Psi\rangle=(1/\sqrt{N})\Sigma_{i=1}^{N}|ii\rangle$ in Ref. [1]. Some methods to relate Schmidt numbers of bipartite mixed states to "k-positive" maps and the so-called "Schmidt number witness" have been developed in Refs. [1,8].

On the other hand, a method of decomposing a mixed state by "edge" entangled states and a separable state was developed in Refs. [9,10]. From this method, it is found that the low-rank mixed state entanglement seems to be easier to understand, and it is proved in Ref. [11] that for the mixed states on $H_A^m\otimes H_B^n$ with ranks not larger than $\max\{m,n\}$ the PPT (positive partial transposition) property is equivalent to the separability.

In this paper, we introduce linear subspaces $L_A(\rho)$ and $L_B(\rho)$ of H_A^m and H_B^n for bipartite mixed states ρ on $H_A^m\otimes H_B^n$, which are closely related to the Schmidt numbers of the mixed states ρ. Roughly speaking the smaller the dimensions of these linear subspaces are, the bigger the Schmidt number of ρ is (see Theorem 1). Then, we give a lower bound for the Schmidt numbers of low rank-mixed states ρ from $L_A(\rho)$ and $L_B(\rho)$. On the other hand, it is very easy to compute $L_A(\rho)$ or $L_B(\rho)$ from any representation of ρ as a convex combination $\rho=\Sigma_i p_i|\psi_i\rangle\langle\psi_i|$ with positive reals p_i's and pure states $|\psi_i\rangle$'s. Thus, we give an easy numerical method to give a lower bound for the Schmidt numbers of low-rank bipartite mixed states. Sometimes the lower bound can be used to compute the Schmidt numbers of mixed states exactly. Another implication of our result is that generic rank $r<n$ mixed states on $H_A^m\otimes H_B^n$ (assume $m\leq n$ without loss of generality) have their Schmidt numbers at least $\min\{n/r,m\}$ and thus entangled. We recall the result in Ref. [11], it is proved for mixed states of ranks not larger than $\max\{m,n\}$ on $H_A^m\otimes H_B^n$, the PPT property is equivalent to the separability. It is also well known that there exist rank $\max\{m,n\}+1$ PPT entangled mixed states on $H_A^m\otimes H_B^n$ (see Refs. [5,12]). Combining with our results, here we can see that low-rank mixed state entanglement is easier to understand. On the other hand, it would be interesting to consider if high-rank bipartite mixed states have relatively low Schmidt numbers. In this direction we know from Ref. [13] that there exists an open ball consisting of separable mixed states (thus Schmidt number 1) in the space of rank mn mixed states on $H_A^m\otimes H_B^n$. We

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do not know if this result could be extended to prove that generic rank mn mixed states on $H_A^m \otimes H_B^n$ have relatively low Schmidt numbers.

For any given bipartite mixed state ρ on $H_A^m \otimes H_B^n$, $L_A(\rho)$ $[L_B(\rho)]$ is the set of pure states $|a\rangle$ in H_A^m [pure states $|b'\rangle$ in H_B^n] such that $\langle a\otimes b|\rho|a\otimes b\rangle = 0$ for any pure state $|b\rangle$ in H_B^n $[\langle a'\otimes b'|\rho|a'\otimes b'\rangle = 0$ for any pure state $|a'\rangle$ in $H_A^m]$. Since $\langle a\otimes b|(U_A\otimes U_B)\rho(U_A\otimes U_B)^\dagger|a\otimes b\rangle = \langle (U_A^\dagger a)\otimes(U_B^\dagger b)|\rho|(U_A^\dagger a)\otimes(U_B^\dagger b)\rangle$, $L_A((U_A\otimes U_B)\rho)$ is $U_A^\dagger(L_A(\rho))$, i.e., the dimensions of $L_A(\rho)$ and $L_B(\rho)$ are invariant under local unitary operations. Here † is the adjoint.

For a pure state $\rho = |\psi\rangle\langle\psi|$, from the invariance under local unitary operations, we can compute $L_A(\rho)$ and $L_B(\rho)$ from the Schmidt decomposition of $|\psi\rangle = \Sigma_{i=1}^k p_i|a_i\rangle\otimes|b_i\rangle$, that is, $\langle a\otimes b|\rho|a\otimes b\rangle = \Sigma_{i=1} p_i^2|\langle a|a_i\rangle\langle b|b_i\rangle|^2 = 0$ implies that $L_A(\rho)$ $[L_B(\rho)]$ is the orthogonal complementary in H_A^m of the space span by pure states $|a_i\rangle$'s $[|b_i\rangle$'s]. Thus, the Schmidt rank k of the pure state ρ is just the codimensions of the linear subspaces $L_A(\rho)$ and $L_B(\rho)$, that is, the linear subspaces we introduced above can be thought as the degrees of freedom, which are not entangled in the pure state. This point is manifested in the following result, which asserts that the Schmidt number (i.e., the minimum number of entangled degrees of freedom) of a bipartite mixed state ρ is relatively high if the dimension of $L_A(\rho)$ or $L_B(\rho)$ is small.

Theorem 1. Let ρ be a rank r mixed state on $H_A^m\otimes H_B^n$ with Schmidt number k. Then, $k \geqslant [m-\dim(L_A(\rho))]/r$ and $k \geqslant [n-\dim(L_B(\rho))]/r$.

For any bipartite mixed state ρ on $H_A^m\otimes H_B^n$, $\langle a\otimes b|\rho|a\otimes b\rangle = 0$ is equivalent to $|a\rangle\otimes|b\rangle$ is in the kernel of ρ, thus it is equivalent to $|a\rangle\otimes|b\rangle$ is orthogonal to the range of ρ. This observation implies that $L_A(\rho) = \cap_{i=1}^r L_A(|v_i\rangle)$ for pure states $|v_1\rangle,\dots,|v_r\rangle$ in $H_A^m\otimes H_B^n$ if they span the range of ρ. We can now recall a lemma in Ref. [5], which asserts that for a mixed state of the form $\rho = \Sigma p_i|\phi_i\rangle\langle\phi_i|$, where p_i's are positive reals and $|\phi_i\rangle$'s are pure states, the range of ρ is the linear span of pure states $|\phi_i\rangle$'s. Combining these two points together, we can see that we can know some properties of the Schmidt ranks of $|\phi_i\rangle$'s for *any representation* of ρ as $\Sigma p_i|\phi_i\rangle\langle\phi_i|$ from $L_A(\rho)$ or $L_B(\rho)$.

Proof of Theorem 1. We just prove the conclusion $k \geqslant [m-\dim L_A(\rho)]/r$. Another conclusion can be proved similarly. Take any representation of the mixed state ρ as $\rho = \Sigma_{i=1}^t p_i|v_i\rangle\langle v_i|$, where p_i's are positive and the maximal Schmidt rank of $|v_i\rangle$'s is k. As observed above, we only need to take r linear independent vectors in $\{|v_1\rangle,\dots,|v_t\rangle\}$, say $|v_1\rangle,\dots,|v_r\rangle$, to compute $L_A(\rho) = \cap_{i=1}^r L_A(|v_i\rangle)$. From our observation of pure states $\dim L_A(|v_i\rangle) = m - k(v_i)$, where $k(v_i)$ is the Schmidt rank of $|v_i\rangle$. Thus, we know that $\dim L_A(\rho) \geqslant m - \Sigma_{i=1}^r k(v_i) \geqslant m - rk$, since $k(v_i) \leqslant k$. The conclusion is proved.

For the convenience is using Theorem 1, we consider the coordinate form of the above linear subspaces $L_A(\rho)$ and $L_B(\rho)$. Let $\{|1\rangle,\dots,|m\rangle\}$ and $\{|1\rangle,\dots,|n\rangle\}$ be the standard orthogonal bases of H_A^m and H_B^n, then $\{|11\rangle,\dots,|1n\rangle,\dots,|m1\rangle,\dots,|mn\rangle\}$ be the standard orthogonal basis of $H_A^m\otimes H_B^n$. We represent the matrix of ρ in the basis $\{|11\rangle,\dots|1n\rangle,\dots,|m1\rangle,\dots,|mn\rangle\}$, and consider ρ as a blocked matrix $\rho = (\rho_{ij})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant m}$ with each block ρ_{ij} a $n\times n$ matrix corresponding to the $|i1\rangle,\dots,|in\rangle$ rows and the $|j1\rangle,\dots,|jn\rangle$ columns. For any fixed pure state $|a\rangle = r_1|1\rangle+\dots+r_m|m\rangle$ in H_A^m, the matrix of the Hermitian linear form $\langle a\otimes b|\rho|a\otimes b\rangle$ of $|b\rangle$ in H_B^n, with respect to the basis $|1\rangle,\dots,|n\rangle$, is $\Sigma_{i,j} r_i r_j^\dagger \rho_{ij}$. Let $\rho = \Sigma_{l=1}^t p_l|v_l\rangle\langle v_l|$ be any given representation of ρ as a convex combination of projections with $p_1,\dots,p_t>0$. Suppose $v_l = \Sigma_{i,j=1}^{m,n} a_{ijl}|ij\rangle$, $A = (a_{ijl})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n, 1\leqslant l\leqslant t}$ is the $mn\times t$ matrix. Then, it is clear that the matrix representation of ρ with the basis $\{|11\rangle,\dots,|1n\rangle,\dots,|m1\rangle,\dots,|mn\rangle\}$ is $APA^\dagger$, where P is the diagonal matrix with diagonal entries $p_1,\dots,p_t$. We may consider the $mn\times t$ matrix A as a $m\times 1$ blocked matrix with each block A_w, where $w = 1,\dots,m$, a $n\times t$ matrix corresponding to $\{|w1\rangle,\dots,|wn\rangle\}$. Then, $\rho_{ij} = A_iPA_j^\dagger$ and thus $\Sigma_{ij} r_i r_j^\dagger \rho_{ij} = \Sigma_{ij} r_i r_j^\dagger A_i P A_j^\dagger = (\Sigma_i r_i A_i)P(\Sigma_j r_j A_j)^\dagger$. We note that P is a strictly positive definite matrix, thus $L_A(\rho)$ is just the set of pure states $|a\rangle = r_1|1\rangle+\cdots+r_m|m\rangle$ in H_A^m such that the matrix $\Sigma_i r_i A_i$ is the zero matrix (of size $n\times t$). Similarly, we can have the coordinate form of $L_B(\rho)$.

We can now return to the pure state case. Let $\rho = |v\rangle\langle v|$, $|v\rangle = \Sigma_{i=1,j=1}^{m,n} a_{ij}|ij\rangle$, be a pure state on $H_A^m\otimes H_B^n$. Consider $A = (a_{ij})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$ as a $m\times n$ matrix, then $L_A(\rho)$ is just the pure states $r_1|1\rangle+\cdots+r_m|m\rangle$ (in H_A^m) such that $(r_1,\dots,r_m)A = 0$ and $L_B(\rho)$ is just the pure states $r_1|1\rangle+\cdots+r_n|n\rangle$ (in H_B^n) such that $A(r_1,\dots,r_n)^\tau = 0$ (Here, τ is the transposition.) These are of codimension *rank* (A) (equal to the Schmidt rank of $|v\rangle$ as well known).

This can be easily generalized to mixed state case. Let $\rho = \Sigma_{k=1}^t p_i|v_k\rangle\langle v_k|$, where p_i's are positive real numbers, be a mixed state on $H_A^m\otimes H_B^n$, $|v_k\rangle = \Sigma_{i=1,j=n}^{m,n} a_{ij}^k|ij\rangle$ be the expansion in the standard basis. We arrange the t matrices of size $m\times n$, $A^k = (a_{ij}^k)_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$, $k=1,\dots,t$, in two different ways. $T_1 = (A^1,\dots,A^t)$ is a matrix of size $m\times tn$ and $T_2 = (A^1,\dots,A^t)^\tau$ is a matrix of size $tm\times n$ (note in each A^k no transposition imposed). Then, $L_A(\rho)$ is just the pure states $r_1|1\rangle+\cdots+r_m|m\rangle$ (in H_A^m) such that $(r_1,\dots,r_m)T_1 = 0$ and $L_B(\rho)$ is just the pure states $r_1|1\rangle+\cdots+r_n|n\rangle$ (in H_B^n) such that $T_2(r_1,\dots,r_n)^\tau = 0$. Theorem 1 can be reread, as the Schmidt number of ρ is at least rank$(T_1)/r$ and rank$(T_2)/r$. From this point of view, Theorem 1 can be thought as a natural extension of the previously well-known result that the Schmidt rank of $|v\rangle$ is just the rank of the matrix A. It is immediate we have the following result.

Proposition 1. Let $\rho = \Sigma_{k=1}^t p_k|v_k\rangle\langle v_k|$ be a rank $r<m$ mixed state on $H_A^m\otimes H_B^m$, where $p_1,\dots,p_t$ are positive reals and $|v_k\rangle = \Sigma_{ij} a_{ij}^k|ij\rangle$. Consider $A^k = (a_{ij}^k)_{1\leqslant i,j\leqslant m}$, $k = 1,2,\dots,t$, as matrices of size $m\times m$. Suppose the linear span of all tm rows of $m\times m$ matrices $A^1,\dots,A^t$ is of dimension m. *Then the Schmidt number of ρ is at least m/r, thus entangled.*

Proof. We consider the $tm\times m$ matrix $T_2 = (A^1,\dots,A^t)^\tau$. It is of rank m since tm rows of the matrices $A^1,\dots,A^t$ span a dimension m space. From the coordinate form of Theorem 1, the Schmidt number of ρ is at least $\text{rank}(T_2)/r$, thus the conclusion is proved.

In the space of all rank r mixed states $\rho=\Sigma_{k=1}^{r}p_k|v_k\rangle\langle v_k|$ on $H_A^m\otimes H_B^n$ (where $p_1,\ldots,p_r$ are eigenvalues, $|v_1\rangle,\ldots,|v_r\rangle$ are eigenvectors, and $|v_k\rangle=\Sigma_{i=1,j=1}^{m}a_{ij}^k|ij\rangle$), the rank of the $m\times m$ matrix $A^k=(a_{ij}^k)_{1\le i,j\le m}$ is m for generic $|v_k\rangle$. In fact, the condition $\text{rank}(A^k)<m$, i.e., $\det(A^k)=0$ (where det is the determinant of a square matrix), is a polynomial equation of variables a_{ij}^k's. Thus, the set of points $(a_{ij}^k)_{1\le i,j\le m}$ satisfying $\det(A^k)=0$ is of measure zero in the space of all possible (a_{ij}^k)'s if this set is not the whole space. Hence, we can see that the condition of Proposition 1 is satisfied by generic rank r mixed states.

Theorem 1 also implies that if a mixed state is mixed by not too many pure states and one of these pure states has the highest possible Schmidt rank, then the mixed state has a relatively high Schmidt number. This can be sometimes used as an easy way to detect the entanglement in low rank mixed states.

Proposition 2. Let $\rho=\Sigma_{k=1}^{t}p_k|v_k\rangle\langle v_k|$ be a rank r mixed state on $H_A^m\otimes H_B^n$ *with* $r<m\le n$, where $p_1,\ldots,p_t$ are positive real numbers and the Schmidt rank of $|v_1\rangle$ is *m*. Then the Schmidt number of ρ is at least m/r and thus entangled.

Proof. Let $|v_k\rangle=\Sigma_{i=1,j=1}^{m,n}a_{ij}^k|ij\rangle$ and $A^k=(a_{ij}^k)_{1\le i\le m,1\le j\le n}$ is $m\times n$ matrix for $k=1,\ldots,t$. We consider the $m\times tn$ matrix $T_1=(A^1,\ldots,A^t)$. It is clear that A^1 is of rank *m* since the Schmidt rank of $|v_1\rangle$ is *m*. Thus, the rank of the matrix T_1 is *m*. From the coordinate form of Theorem 1 the Schmidt number of ρ is at least $\text{rank}(T_1)/r$, thus the conclusion is proved.

Sometimes the lower bound in Theorem 1 can be used to give *exact* results about Schmidt numbers for some bipartite mixed states.

Example 1. Let $\rho=\Sigma_{k=1}^{u}p_k|\phi_k\rangle\langle\phi_k|$ be a mixed state on $H_A^{uv}\otimes H_B^{uv}$, where $p_1,\ldots,p_u$ are positive reals and $|\phi_k\rangle=\Sigma_{i=1,j=1}^{v}a_{ij}^k|[(k-1)v+i][(k-1)v+j]\rangle$ for $k=1,\ldots,u$. From this representation of ρ, we know that the Schmidt number of ρ is at most v, since the Schmidt ranks of $|\phi_k\rangle$'s are at most v. Consider $A^k=(a_{ij}^k)_{1\le i\le v,1\le j\le v}$, $k=1,\ldots,u$, as matrices of size $v\times v$, if $\Sigma_{k=1}^{u}\text{rank}(A^k)\ge uv-u+1$ then the Schmidt number of ρ is exactly v. In fact the matrix T_1 in the coordinate form of Theorem 1 is a blocked diagonal matrix with u blocks $A^1,\ldots,A^u$, thus $\text{rank}(T_1)=\Sigma_{k=1}^{u}\text{rank}(A^k)\ge uv-u+1$. From the coordinate form of Theorem 1, the Schmidt number of ρ is at least $v-1+1/u$ and thus at least v. We get the conclusion. We can observe the following case.

In the case $u=3,v=4$, consider the following two mixed states: $\rho_1=\frac{1}{4}\Sigma_{i=1}^{4}|\phi_i\rangle\langle\phi_i|$ and $\rho_2=\frac{1}{3}\Sigma_{i=1}^{3}|\psi_i\rangle\langle\psi_i|$ on $H_A^{12}\otimes H_B^{12}$, where

$$|\phi_1\rangle=\frac{1}{\sqrt{3}}(|11\rangle+|22\rangle+|33\rangle),$$

$$|\phi_2\rangle=\frac{1}{\sqrt{2}}(|44\rangle+|55\rangle),$$

$$|\phi_3\rangle=\frac{1}{\sqrt{3}}(|77\rangle+|88\rangle+|99\rangle),$$

$$|\phi_4\rangle=\frac{1}{\sqrt{2}}(|(11)(11)\rangle+|(12)(12)\rangle),$$

$$|\psi_1\rangle=\tfrac{1}{2}(|11\rangle+|22\rangle+|33\rangle+|44\rangle),$$

$$|\psi_2\rangle=\frac{1}{\sqrt{3}}(|55\rangle+|66\rangle+|77\rangle),$$

$$|\psi_3\rangle=\frac{1}{\sqrt{3}}(|(10)(10)\rangle+|(11)(11)\rangle+|(12)(12)\rangle).$$

Thus, we know that the Schmidt number of ρ_1 is 3 and the Schmidt number of ρ_2 is 4.

Example 2. Let $\rho=\Sigma_{i=1}^{r}p_i|\phi_i\rangle\langle\phi_i|$, where $p_1,\ldots,p_r$ are positive real numbers, be a rank r mixed state on $H_A^m\otimes H_B^n$ with $rm\le n$. Suppose the matrix T_2 (size $rm\times n$) of the above representation of ρ is of full rank rm. From the coordinate form of Theorem 1, the Schmidt number of ρ is at least $\text{rank}(T_2)/r=m$. Since *m* is the highest possible value of Schmidt numbers, the Schmidt number of ρ is exactly *m*.

Let $\rho=\Sigma_{i=1}^{3}p_i|\phi_i\rangle\langle\phi_i|$, where p_1,p_2,p_3 are positive real numbers and

$$|\phi_1\rangle=\tfrac{1}{3}(|11\rangle+|12\rangle+|13\rangle+|15\rangle+|17\rangle+|22\rangle+|28\rangle+|33\rangle+|39\rangle),$$

$$|\phi_2\rangle=\frac{1}{\sqrt{7}}(|14\rangle+|15\rangle+|16\rangle+|25\rangle+|27\rangle+|36\rangle+|39\rangle),$$

$$|\phi_3\rangle=\tfrac{1}{2}(|17\rangle+|19\rangle+|28\rangle+|39\rangle),$$

be a rank 3 mixed state on $H_A^3\otimes H_B^9$. We can check that its T_2 is a rank 9 matrix of size 9×9. Thus, the Schmidt number of ρ is 3.

For a given mixed state $\rho=\Sigma_i^t p_i|v_i\rangle\langle v_i|$ with positive reals p_i's, we know from the definition that the Schmidt number of ρ is at most $\max\{k(v_1),\ldots,k(v_t)\}$, where $k(v_i)$ is the Schmidt rank of the pure state $|v_i\rangle$ for $i=1,\ldots,t$. This gives an upper bound for the Schmidt numbers of bipartite mixed states. On the other hand, there is a lower bound for the Schmidt numbers of bipartite mixed states from Theorem 1. Hence, in some cases, we can compare Schmidt numbers from the above fact and know the first one cannot be convertible to the latter mixed state by local operations and classical communication.

Example 3. Let $\rho=\Sigma_{i=1}^{m-2}p_i|\phi_i\rangle\langle\phi_i|$, where p_i's are positive real numbers and $|\phi_i\rangle=1/\sqrt{3}[|ii\rangle+|(i+1)(i+1)\rangle$

$+|(i+2)(i+2)\rangle]$, for $i=1,\ldots,m-2$, be a mixed state on $H_A^m\otimes H_B^m$. It is clear that the Schmidt number of ρ is at most 3.

Let $\rho'=\frac{1}{2}(|\psi_1\rangle\langle\psi_1|+|\psi_2\rangle\langle\psi_2|)$ be a mixed state on $H_A^n\otimes H_B^n$ with $n\geqslant 8$, where $|\psi_1\rangle$ and $|\psi_2\rangle$ are pure states in $H_A^n\otimes H_B^n$. Suppose the Schmidt rank of $|\psi_1\rangle$ or $|\psi_2\rangle$ is n. From Proposition 2, the Schmidt number of ρ' is at least $n/2\geqslant 4$. Thus, ρ cannot be convertible to ρ' by local operations and classical communication.

Because of the above discussion about the coordinate form of Theorem 1, we can easily calculate the lower bound of Schmidt numbers of low-rank bipartite mixed states. This gives us the following conclusion.

Theorem 2. Generic rank $r<n$ mixed states on $H_A^m\otimes H_B^n$ (assume $m\leqslant n$ without loss of generality) have their Schmidt numbers at least $\min\{n/r,m\}$ and thus entangled.

Proof. We consider the spectral decomposition $\rho=\Sigma_{k=1}^r p_i|v_k\rangle\langle v_k|$ of rank r mixed states on $H_A^m\otimes H_B^n$, where p_k and $|v_k\rangle=\Sigma_{i=1,j=1}^{m,n}a_{ij}^k|ij\rangle$, $k=1,\ldots,r$, are eigenvalue and corresponding eigenvector, respectively. Then, the space of the rank r mixed states can be identified with the space $X=X_1\times X_2$, where $X_1=\{(p_1,\ldots p_r)\colon p_1>0,\ldots,p_r>0,p_1+\cdots+p_r=1\}$ and $X_2=\{(a_{ij}^1,\ldots,a_{ij}^r)\in C^{rmn}\colon\Sigma_{ij}|a_{ij}^1|^2=1,\ldots,\Sigma_{ij}|a_{ij}^r|^2=1,\Sigma_{ij}a_{ij}^{k_1}a_{ij}^{k_2}=0,k_1\neq k_2\}$. (Strictly speaking, there is a group action of permutation group of r symbols on the space X and the space of rank r mixed states should be identified with the quotient space of this group action. However, it is clear in the discussion below this point can be neglected.) As above discussion $A^k=(a_{ij}^k)_{1\leqslant i\leqslant m,1\leqslant j\leqslant n}$, $k=1,\ldots,r$, are considered as size $m\times n$ matrix.

It is clear that the matrix $T_2=(A^1,\ldots,A^r)^\tau$ (of size $rm\times n$) can reach the highest possible rank $\min\{rm,n\}$ on the space X, since we can take $\min\{rm,n\}$ rows of T_2 (i.e., rows of $A^1,\ldots,A^r$) to be the orthogonal vectors in C^n with suitable lengths. The rank r mixed states corresponding to these points in X have their Schmidt numbers at least $\min\{n/r,m\}$ from the coordinate form of Theorem 1. In the space X_2, the condition $\text{rank}(T_2)<\min\{rm,n\}$ is just $C_{\max\{rm,n\}}^{\min\{rm,n\}}$ (binomial coefficient) polynomial equations (i.e., the determinants of all size $\min\{rm,n\}$ square submatrices of T_2 are zero) of variables a_{ij}^k's. This set S defined by these polynomial equations is of measure zero in X_2, since it is an algebraic set not equal to the whole space X_2. Thus, the Schmidt numbers of generic mixed states corresponding to points outside the measure zero set $X_1\times S$ are at least $\min\{n/r,m\}$ from the coordinate form of Theorem 1. The conclusion is proved.

In conclusion, we proved a lower bound for the Schmidt numbers of bipartite mixed states. This lower bound can be applied easily to low-rank bipartite mixed states. From this lower bound, it is known that generic low-rank bipartite mixed states have relatively high Schmidt numbers and thus entangled. We can compute Schmidt numbers exactly for some mixed states by this lower bound as shown in Examples. This lower bound can also be used effectively to determine that some mixed states cannot be convertible to other mixed states by local operations and classical communication.

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