Effect of an external field on decoherence

R. F. O'Connell and Jian Zuo
Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001, USA
(Received 26 November 2002; published 26 June 2003)

''Decoherence of quantum superpositions through coupling to engineered reservoirs'' is the topic of a recent paper by Myatt *et al.* [Nature **403**, 269 (2000)] which has attracted much interest because of its relevance to current research in fundamental quantum theory, quantum computation, teleportation, entanglement, and the quantum-classical interface. However, the preponderance of theoretical work on decoherence does not consider the effect of an *external field*. Here, we present an analysis of such an effect in the case of the random δ-correlated force discussed by Myatt *et al.*

DOI: 10.1103/PhysRevA.67.062107 PACS number(s): 03.65.Yz

''Decoherence of quantum superpositions through coupling to engineered reservoirs'' [1] is the topic of a recent paper that has attracted much interest because of its relevance to current research in fundamental quantum theory, quantum computation, teleportation, entanglement, and the quantum-classical interface. As Schleich remarks in an accompanying ''News and Views'' paper, this is a pioneering experiment that engineers decoherence [2]. However, the preponderance of theoretical work on decoherence [3,4] does not consider the effect of an *external field*. Here, we present an analysis of such an effect in the case of the random δ-correlated force discussed in Ref. [1].

Myatt *et al.* [1] used a linear Paul trap to confine single Be ions in a harmonic potential and then prepared various superposition states. Next, they induced decoherence by coupling the single ion to a reservoir which they controlled in various ways. Such a reservoir gives rise to an external force $f(t)$ in the equation of motion of the system, in contrast to the usual intrinsic fluctuation force $F(t)$ that arises from interaction with an ambient thermal dissipative environment [4], which, of course, will always be present, even at $T=0$. Thus, the question arises as to not only what is the dependence of the characteristic decoherence decay time τ_d on the separation d of the superposition components, the temperature T, and the dissipative decay rate γ [all of which come into play when $f(t)=0$, the focus of most theoretical work], but also what is the dependence on the parameters of the engineered reservoirs that give rise to $f(t)$. The existing experiments [1] focused on the dependence of τ_d on d and demonstrated that $\tau_d \sim d^{-2}$. This is a familiar result predicted by the plethora of papers dealing with the $f(t)=0$ situation but it does not give information on the dependence of τ_d on the parameters of the externally superimposed reservoir. More details on the experimental results were given by Turchette *et al.* [5] and these authors also reviewed the theory of the damping of a harmonic oscillator in a dissipative reservoir. Whereas the latter gives rise to a fluctuation force on the oscillator which is related to the dissipation via the fluctuation-dissipation theorem, an externally engineered situation requires an additional analysis, as is made clear in Ref. [6].

Much of the discussion of decoherence [3,4] has been in terms of a particle moving in one dimension that is placed in an initial superposition state (a Schrödinger ''cat'' state) corresponding to two widely separated Gaussian wave packets. The corresponding wave function has the form

$$\psi(x,0)=\frac{1}{(8\pi\sigma^2)^{1/4}(1+e^{-d^2/8\sigma^2})^{1/2}} \times\left(\exp\left\{-\frac{\left(x-\frac{d}{2}\right)^2}{4\sigma^2}\right\}+\exp\left\{-\frac{\left(x+\frac{d}{2}\right)^2}{4\sigma^2}\right\}\right), \tag{1}$$

where d is the separation and σ^2 is the variance of each packet. It is clear that the spatial probability distribution $P(x,0)$ for this superposition of two states consists of the sum of the probability distributions for the individual states plus an interference term. In the absence of dissipation [7,8], one proceeds by calculating $\psi(x,t)$ from which $P(x,t)$ readily follows. However, when dissipation is present it is necessary to use a density-matrix approach [4] which, when combined with the use of quantum probability functions, leads to an expression for $P(x,t)$ of the form

$$P(x,t)=P_1(x,t)+P_2(x,t)+P_I(x,t)\cos[f(t)], \tag{2}$$

where P_1 and P_2 correspond to the time-dependent probability distributions for the separate wave packets and the third term is an interference term. The latter is characterized by a cosine factor (which varies in time according to a known function $f(t)$ [4]) that is multiplied by an amplitude factor $P_I(x,t)$, which is found to decay in time. The disappearance of the interference term, that is, the decoherence, is measured by defining an attenuation coefficient $a(t)$, which is the ratio of the factor multiplying the oscillatory term to twice the geometric mean of the first two terms, i.e.,

$$a(t)=\frac{P_I(x,t)}{2[P_1(x,t)P_2(x,t)]^{1/2}}. \tag{3}$$

We should mention that, in the literature, one finds various measures of decoherence, based on decay of diagonal and off-diagonal density-matrix elements or probability distributions in phase space, momentum space, or coordinate space [9] but we consider the latter to be the most desirable be-

cause it is closest to experiment. Thus, returning to Eq. (3), what we have found [4] is that $a(t)$ depends crucially on the spreading of the wave packets corresponding to the individual states.

For $f(t)=0$, this spreading arises from the possible intrinsic spreading associated with the uncertainty principle, thermal spreading and spreading due to dissipative γ effects. Explicitly, for a free particle described by a single wave packet, the width after a time t is $w(t)$, given by [4]

$$w^2(t)=\sigma^2-\frac{[x(t_1),x(t_1+t)]^2}{4\sigma^2}+s_0(t), \tag{4}$$

where σ is the initial width and $s_0(t)$ is the mean-square displacement (discussed in more detail below). For the attenuation coefficient in the case of a *free* particle, we have the formula [4]

$$a(t)=\exp\left\{-\frac{s_0(t)d^2}{8\sigma^2 w^2(t)}\right\}. \tag{5}$$

In addition, the characteristic time for decay to occur, τ_d say, is defined as usual [4,7,8] as the time at which $a(t)=\exp(-1)$.

We now turn to the case where $f(t)\neq 0$ and we generalize from the case of a free particle to that of an *oscillator* potential, corresponding to the experiment described in Ref. [1]. For $f(t)\neq 0$, there is an additional spreading of the wave packets, which we will now calculate. Afterwards, we will turn to the role it plays in the calculation of $a(t)$.

Let $x(t)$ be the dynamical variable corresponding to the coordinate of the wave function of the superposition state of the oscillator of Myatt *et al.* [1]. As shown in Ref. [10], in the presence of an external force $f(t)$ in addition to the fluctuation force $F(t)$, the steady-state motion can be described by means of a generalized quantum Langevin equation

$$m\ddot{x}+\int_{-\infty}^{t}dt'\mu(t-t')\dot{x}(t')+Kx=F(t)+f(t), \tag{6}$$

where $\mu(t)$ is the memory function, K is the oscillator force constant ($K=m\omega_0^2$), where ω_0 is the oscillator frequency, and $F(t)$ is a fluctuating operator force with mean $\langle F(t)\rangle=0$. The steady-state solution of Eq. (6) can be written as

$$x(t)=\int_{-\infty}^{t}dt'G(t-t')[F(t')+f(t')]\equiv x_s(t)+x_d(t), \tag{7}$$

where $x_s(t)$ is the stationary solution and x_d is due to the driven motion. Also, $G(t)$, the Green function, is given by

$$G(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\alpha(\omega+i0^+)e^{-i\omega t}, \tag{8}$$

with $\alpha(z)$ the familiar response function

$$\alpha(z)=\frac{1}{-mz^2-iz\tilde{\mu}(z)+K}. \tag{9}$$

In addition,

$$\tilde{\mu}(z)=\int_0^{\infty}dt\mu(t)e^{izt}\equiv m\gamma(z) \tag{10}$$

is the Fourier transform of the memory function and it characterizes the reservoir [4,10]. The fact that $\alpha(z)$ does not depend on $f(t)$ follows simply by taking the Fourier transform of Eq. (7) which enables the solution to be written in Fourier transform language as

$$\tilde{x}(\omega)=\alpha(\omega)[\tilde{F}(\omega)+\tilde{f}(\omega)], \tag{11}$$

where superposed tildes indicate Fourier transforms.

Because of the linearity of the oscillator, it is clear that the motion of the driven oscillator will be a superposition of a driven mean motion and a motion about the mean that is identical with the motion about the equilibrium state [6]. The starting point of our calculation is the correlation

$$\begin{aligned}\frac{1}{2}\langle x(t)x(t')+x(t')x(t)\rangle &\equiv C(t-t')\equiv C_0+C_d\\ &=\frac{\hbar}{\pi}\int_0^{\infty}d\omega\,\mathrm{Im}\{\alpha(\omega+i0^+)\}\coth\frac{\hbar\omega}{2kT}\cos\omega(t-t')+C_d,\end{aligned} \tag{12}$$

where C_0 and C_d are the contribution due to $F(t)$ and $f(t)$, respectively. It follows that the mean-square displacement (which characterizes the spreading of the wave packet) is

$$\begin{aligned}s(t)\equiv\langle[x(t)-x(0)]^2\rangle&=2\{C(0)-C(t)\}\\ &=\frac{2\hbar}{\pi}\int_0^{\infty}d\omega\,\mathrm{Im}\{\alpha(\omega+i0^+)\}\coth\frac{\hbar\omega}{2kT}(1-\cos\omega t)+s_d,\end{aligned} \tag{13}$$

where $C(t)$ is given by Eq. (12) and s_d is the contribution due to the "driven motion."

Here, we have used the fact that since $\langle F(t)\rangle=0$ and since there is no correlation between $F(t)$ and $f(t)$, it is clear that

$$s(t)=s_0(t)+s_d(t), \tag{14}$$

where s_0 denotes the contributions due to $F(t)$. Since $s_0(t)$ has been calculated in detail, in Ref. [4], which considers entanglement between the system and the environment at the initial time $t=0$, we will henceforth concentrate on s_d. Consider that the external force is applied at $t=0$. It follows from Eq. (7) that

$$x_d(t)=\int_0^t dt'G(t-t')f(t'). \tag{15}$$

Since $x_d(0)=0$, it follows that

$$s_d(t)=\langle x_d^2(t)\rangle=\int_0^t dt'\int_0^t dt'' G(t-t')G(t-t'')g(t'-t''), \tag{16}$$

where

$$g(t'-t'')=\langle f(t')f(t'')\rangle. \tag{17}$$

Further progress clearly depends on the nature of $f(t)$ but, keeping in mind the existing experiments [1,5], let us consider a random δ-correlated force so that

$$g(t'-t'')=g\,\delta(t'-t''), \tag{18}$$

where g is time independent. Hence, substituting Eq. (18) in Eq. (16), we obtain

$$s_d=g\int_0^t dt' G^2(t'). \tag{19}$$

In the case of the oscillator potential of Myatt *et al.* [1], we find that in the case of Ohmic coupling [$\gamma(\omega)=\gamma=$const]

$$G(t)=e^{-(\gamma t/2)}\frac{\sin\omega_1 t}{m\omega_1}, \tag{20}$$

where

$$\omega_1^2=\omega_0^2-(\gamma/2)^2. \tag{21}$$

Thus, substituting Eq. (20) in Eq. (19), it follows that

$$s_d=\frac{g}{4m^2\gamma\omega_0^2\omega_1^2}\{(1-e^{-\gamma t})2\omega_1^2-e^{-\gamma t}(\gamma^2\sin^2\omega_1 t + \gamma\omega_1\sin 2\omega_1 t)\}. \tag{22}$$

In the absence of dissipation ($\gamma\to 0$) (which approximates the experiment of Myatt *et al.*), Eqs. (21) and (22) give

$$s_d\to\frac{g}{2m^2\omega_0^2}t\left\{1-\frac{\sin 2\omega_0 t}{2\omega_0 t}\right\}. \tag{23}$$

Again, for $\gamma\to 0$ and $T\to 0$ (absence of dissipation and for negligibly low temperatures), it readily follows that

$$a(t)=\exp\left\{-\frac{s_d(t)d^2}{8\sigma^2[\sigma^2+s_d(t)]}\right\}, \tag{24}$$

where σ is the initial width of the individual wave packets. Thus, the dependence on d^2 in the numerator always emerges, regardless of the value of s_d. We also note the absence of a term analogous to the second term in Eq. (4), corresponding to the fact that, when $f(t)=0$, the width of the oscillator wave function is constant in time whereas that of the free particle continually increases.

It is clear from Eq. (24) that the relative magnitudes of s_d and the initial variance σ^2 play a crucial role. In particular,

$$a(t)\approx\exp\left\{-\frac{d^2}{8\sigma^2}\right\}\ \text{ if }\ s_d\gg\sigma^2, \tag{25}$$

and

$$a(t)\approx\exp\left\{-\frac{s_d}{\sigma^2}\frac{d^2}{8\sigma^2}\right\}\ \text{ if }\ s_d\ll\sigma^2. \tag{26}$$

Thus, in the former case, the result for $a(t)$ is independent of s_d, i.e., independent of the external force $f(t)$. In the latter case, using Eq. (23), we see that

$$a(t)=\exp\left\{-\frac{t}{\tau_0}\left(1-\frac{\sin 2\omega_0 t}{2\omega_0 t}\right)\right\}, \tag{27}$$

where

$$\tau_0=\frac{16\sigma^4 m^2\omega_0^2}{d^2 g}. \tag{28}$$

For small times ($2\omega_0 t\ll 1$) after the initial time $t=0$, we see that

$$a(t)=\exp\left\{-\frac{t}{\tau_0}\frac{(2\omega_0 t)^2}{6}\right\}=\exp\left\{-\frac{g d^2}{24m^2\sigma^4}t^3\right\}\ \text{ if }\ \omega_0 t\ll 1, \tag{29}$$

in which case the decay rate of decoherence is independent of ω_0, corresponding to free particle behavior. However, when t further increases there is a change in the time behavior until at the end of the first cycle at $2\omega_0 t=2\pi$, we see from Eq. (27) that

$$a(t)=\exp\left\{-\frac{t}{\tau_0}\right\}. \tag{30}$$

In fact, as we go into the next and subsequent cycles, the $\sin(2\omega_0 t)/2\omega_0 t$ term becomes more and more negligible so that Eq. (30) becomes more and more accurate as we go beyond the first cycle.

It should also be noted that Eq. (18) also corresponds to a white-noise spectrum. However, it is very different in nature than the white-noise spectrum associated with the fluctuation force $F(t)$. A random c-number field feeds energy into the quantum particle (and, in fact, for a particle with negligibly weak coupling to a heat bath and for either a zero or oscillator potential, it may be shown that the energy of the particle increases linearly in time). On the other hand, in the case of a fluctuation force, we are necessarily dealing with a heat bath; in other words, we have a dynamical system in which the particle also loses energy due to the emission of bath excitations. Thus, for example, in the case where the white-noise spectrum is associated with an equilibrium temperature [10]

$$\langle F(t')F(t'')\rangle=2m\gamma T\delta(t'-t''), \tag{31}$$

for the case of constant γ and in the classical limit. Moreover, the rate of work being done by the fluctuation force, P_F say, is given by Ref. [11]

$$P_F = kT\gamma. \tag{32}$$

Thus, the rate of work being done by the fluctuation force is proportional to the dissipation. This is a manifestation of the general principle that, at equilibrium, the energy lost by a particle due to dissipation is compensated by the energy received from the fluctuation force. Thus, there is a crucial difference between the effects of $f(t)$ and $F(t)$ so that, in particular, an external field that has a white-noise spectrum cannot be approximated by a weakly-coupled thermal reservoir and, as a result, one must use the analysis given above.

Finally, it is clear that in order to explore the larger parameter space (such as dependence on T, γ, and various choices of $f(t)$ as well as on the potential), both further experiments and theoretical work will be needed. Some recent work has made inroads into this multidimensional parameter space. First, for the problem considered above, we find that a nonrandom external force does not cause decoherence. Second, in the absence of an external field, the quantum superposition of states (Schrödinger cat state) has been examined for the case of an oscillator potential at high temperature [12] and also for the case of a free particle subject to the effects of the zero-point oscillations of the electromagnetic field [13].

We are pleased to thank Professor G. W. Ford for many enlightening discussions.

[1] C.J. Myatt, B.E. King, Q.A. Turchette, C.A. Sackett, D. Kielpinski, W.M. Itano, C. Monroe, and D.J. Wineland, Nature (London) **403**, 269 (2000).
[2] W.P. Schleich, Nature (London) **403**, 256 (2000).
[3] D. Giulini, E. Joos, C. Kiefer, J. Kupisch, I.-O. Stamatescu, and H. D. Zeh, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer, New York, 1996).
[4] G.W. Ford, J.T. Lewis, and R.F. O'Connell, Phys. Rev. A **64**, 032101 (2001).
[5] Q.A. Turchette, C.J. Myatt, B.E. King, C.A. Sackett, D. Kielpinski, W.M. Itano, C. Monroe, and D.J. Wineland, Phys. Rev. A **62**, 053807 (2000).
[6] G.W. Ford, and R.F. O'Connell, Opt. Commun. **179**, 451 (2000). See especially Sec. 4.
[7] G.W. Ford and R.F. O'Connell, Phys. Lett. A **286**, 87 (2001).
[8] G.W. Ford and R.F. O'Connell, Am. J. Phys. **70**, 319 (2002).
[9] M. Murakami, G.W. Ford, and R.F. O'Connell, Laser Phys. **13**, 180 (2003).
[10] G.W. Ford, J.T. Lewis, and R.F. O'Connell, Phys. Rev. A **37**, 4419 (1988).
[11] X.L. Li, G.W. Ford, and R.F. O'Connell, Phys. Rev. E **48**, 1547 (1993).
[12] G. W. Ford and R. F. O'Connell, Proceedings of the Wigner Centennial Conference, Pecs, Hungary, 2002 [Acta Phys. Hung. (to be published)].
[13] G. W. Ford and R. F. O'Connell, J. Opt. B, special issue on quantum computing (to be published).