Surface-impedance approach solves problems with the thermal Casimir force between real metals

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The surface-impedance approach to the description of the thermal Casimir effect in the case of real metals is elaborated starting from the free energy of oscillators. The Lifshitz formula expressed in terms of the dielectric permittivity depending only on frequency is shown to be inapplicable in the frequency region where a real current may arise leading to Joule heating of the metal. The standard concept of a fluctuating electromagnetic field on such frequencies meets difficulties when used as a model for the zero-point oscillations or thermal photons in the thermal equilibrium inside metals. Instead, the surface impedance permits not to consider the electromagnetic oscillations inside the metal but taking the realistic material properties into account by means of the effective boundary condition. An independent derivation of the Lifshitz-type formulas for the Casimir free energy and force between two metal plates is presented within the impedance approach. It is shown that they are free of the contradictions with thermodynamics that are specific to the usual Lifshitz formula for dielectrics in combination with the Drude model. We demonstrate that in the impedance approach the zero-frequency contribution is uniquely fixed by the form of impedance function and does not need any of the *ad hoc* prescriptions intensively discussed in the recent literature. As an example, the computations of the Casimir free energy between two gold plates (or the Casimir force acting between a plate and a sphere) are performed at different separations and temperatures specific for the regions of the anomalous skin effect and infrared optics. The results are in good agreement with those obtained by the use of the tabulated optical data for the complex refraction index and plasma model. It is argued that the surface impedance approach lays a reliable theoretical framework for the future measurements of the thermal Casimir force.

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I. INTRODUCTION

Considerable attention has been focused recently on the Casimir effect [1] which is a rare manifestation of the zero-point oscillations of the electromagnetic field at macroscopic scales. The Casimir force arises as response to the change of the spectrum of zero-point oscillations when material boundaries are present. It acts between the boundaries of these bodies and depends on the parameters of their materials, their geometry (including surface roughness), and on the temperature (for a detailed information see the monographs and reviews [2–6]). Recently, many precision measurements of the Casimir force between metal boundaries have been performed [7–14]. Their results were used for constraining hypothetical forces predicted by unified gauge theories of fundamental interactions [15–17] and in nanotechnological applications [18,19].

With respect to the present state of the art, the theoretical description of the Casimir force calls for a careful account of all material properties and other relevant factors. Surprisingly, it was found that the calculations of the temperature effect on the Casimir force between real metals of finite conductivity run into serious troubles, which have been the subject of much controversy [20–33]. The key contradiction is on whether the term of the Lifshitz formula [34,35] related to the zero Matsubara frequency for the *perpendicular* polarized modes of an electromagnetic field contributes to the physical quantities and, if so, how much would its contribution be. At the moment there are five distinct approaches to the resolution of this problem in the recent literature.

(a) According to the first approach, proposed in Ref. [20] and supported in Refs. [26,27], in the case of real metals the term of the Lifshitz formula with zero Matsubara frequency should be calculated by using the Drude dielectric function. As a result, the perpendicular polarized modes do not contribute to this term.

(b) In the other approach [28] a special modification of the zero-frequency term of the Lifshitz formula, supplemented by the Drude model, was proposed leading to a nonzero contribution of the perpendicular polarized modes. This modification was done by analogy with the prescription of Ref. [36] for an ideal metal but it does not coincide with it.

(c) In the framework of the approaches of Refs. [23–25] the modification of the zero-frequency term of the Lifshitz formula was made identical to that for an ideal metal [36]. As a consequence, the contribution of the perpendicular polarized modes to the zero-frequency term in Refs. [23–25] is nonzero and coincides with that for an ideal metal.

(d) According to Refs. [21,22,29] the contribution of the perpendicular polarized modes with zero Matsubara frequency is nonzero and should be calculated by substituting the free-electron plasma dielectric function into the unmodified Lifshitz formula.

(e) Finally, according to the approach of Refs. [30,31], the description of the thermal Casimir force can be obtained by

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using the Leontovich surface impedance boundary condition (recently this approach was applied also in Ref. [32]). In doing so, the perpendicular polarized modes give a nonzero contribution to the zero-frequency term of the Lifshitz formula prescribed by the form of the impedance.

As to the contribution of the modes with a *parallel* polarization to the zero-frequency term, there is consensus between all these approaches that for real metals it is nonzero and the same as for ideal metals. Note also that some of the viewpoints varied with the time in the framework of the above approaches (a)–(e). For example, in Refs. [23,24] approach (c) was considered as an universal prescription, whereas in Ref. [32] it is restricted by the range of only cryogenic temperatures, and in Ref. [25] by the case of sufficiently large separation distances and by the presence of thin covering metallic films. Approach (b), proposed in Ref. [28] to resolve the contradictions arising for the Drude model in combination with the Lifshitz formula, was considered later in Ref. [31] as unnecessary as the Drude model itself turned out to be irrelevant for the description of the thermal Casimir force between real metals. It should be particularly emphasized that approaches (a) and (c) were proved to be in contradiction with thermodynamics [31,33] since they violate the Nernst heat theorem. A detailed comparison of all the approaches can be found in Refs. [28,31].

The present paper aims to work out quite clearly that the surface impedance approach provides an answer to all the complicated problems with the retarded Casimir force between real metals at both zero and nonzero temperatures. We demonstrate that the main reason why the Drude model in combination with the Lifshitz theory had failed to describe the thermal Casimir force is the inadequacy of the standard concept of a fluctuating electromagnetic field depending only on frequency inside a lossy real metal. Rather than considering fluctuations inside a metal, the surface impedance approach suggests that the effective boundary conditions take into consideration in a noncontradictory way the involved reflection properties from the surface of a real conductor. In this case no additional prescriptions are needed, and the values of the zero-frequency contributions for both parallel and perpendicular modes follow immediately from the explicit form of the impedance function.

On this basis, we present a derivation of the formula for the Casimir free energy and force in a configuration of two parallel plates in the surface impedance approach (in Ref. [30] it was presented without proof by the use of a prescription changing the integration over continuous frequencies for the summation over the discrete Matsubara frequencies). The necessity of this derivation results from the fundamental role played by the concept of the surface impedance in the theory of the Casimir effect between real metals. The relationship between this formula and the Lifshitz formula for the free energy is found (the new formula is obtained by exchanging the reflection coefficients, which appear in the Lifshitz formula, for those derived in the surface impedance approach). Our derivation starts from the free energy of the oscillators and suggests also other means to derive the usual Lifshitz formula for the thermal Casimir force between dielectrics. The obtained formula is applied to compute the Casimir free energy and force at different temperatures and separation distances between the test bodies. To do this, one has to use the impedance functions describing the regions of infrared optics, anomalous or normal skin effect depending on the value of the characteristic frequency giving the main contribution to the Casimir force. It is shown that no contradictions with thermodynamics arise, and no artificial prescriptions for the zero-frequency term are needed (we also demonstrate that the predictions of large temperature corrections to the Casimir force at small separations and cryogenic temperatures, made in Ref. [32], are in error since the impedance of the anomalous skin effect was used in Ref. [32] outside its range of application).

The paper is organized as follows. In Sec. II we demonstrate the inadequacy of the concept of a fluctuating electromagnetic field inside lossy real metals and remind the basic facts from the theory of surface impedance. Section III is devoted to the derivation of the electromagnetic oscillation spectrum between two parallel plates starting from the impedance boundary condition. In Sec. IV the formula for the Casimir free energy is derived in the surface impedance approach. The proposed simple derivation of the usual Lifshitz formula for the thermal Casimir force between dielectrics is also given here. Section V contains the calculations of the Casimir energy and force at zero temperature using the impedance approach. The results are compared with the previously known ones, obtained by the use of the usual Lifshitz formula and the tabulated optical data, and are found to be in agreement. In Sec. VI the computations at nonzero temperature are performed in the impedance approach at different separation distances between the test bodies. They demonstrate good agreement in transition regions between the different analytical expressions for the impedance. In Sec. VII the reader finds conclusions and a discussion on the relationship between the proposed impedance approach to the Casimir effect and the theory of the van der Waals forces valid at small separation distances between the test bodies.

II. THE CONCEPT OF A FLUCTUATING ELECTROMAGNETIC FIELD AND THE SURFACE IMPEDANCE

It is well known that the concept of a fluctuating electromagnetic field works well for the description of zero-point oscillations within media with a frequency-dependent dielectric permittivity where no real electric current arises. We will now look at a conductor in an external electric field, which varies with some frequency ω satisfying the conditions

$$l \ll \delta_n(\omega), \quad l \ll \frac{v_F}{\omega}, \tag{1}$$

where l is the mean free path of a conduction electron, $\delta_n(\omega)=c/\sqrt{2\pi\sigma\omega}$ is the penetration depth of the field inside a metal, σ is the conductivity, and v_F is the Fermi velocity. Equations (1) determine the domain of the normal skin effect [37]. In this frequency region the external field leads to the initiation of a real current of the conduction electrons.

The normal skin effect is characterized by the volume relaxation described by the temperature-dependent relaxation frequency $\gamma(T)$. As a result, the mean free path of the conduction electrons is also temperature dependent, $l=l(T)=v_F/\gamma(T)$, and increases with a decrease in temperature. The interaction of the conduction electrons with the elementary excitations of the crystal lattice (phonons) leads to the occurrence of electric resistance and heating of the metal. The dielectric permittivity of a metal in the domain of the normal skin effect can be modeled by the Drude function

$$\varepsilon(\omega)=1-\frac{\omega_p^2}{\omega[\omega+i\gamma(T)]}, \qquad (2)$$

where ω_p is the plasma frequency of the free electron plasma model (ω_p is temperature independent). Remind that the Drude dielectric function (2) was used in approaches (a)–(c) (see the Introduction) in combination with the Lifshitz formula to describe the thermal Casimir force between real metals for the frequencies both inside and outside the region (1). This has led to difficulties including the violation of Nernst's heat theorem (see the Introduction).

The physical reason for these difficulties becomes quite clear when one observes that the usual alternating electric field with frequencies characteristic for the normal skin effect inevitably leads to heating of a metal as it penetrates through the skin layer. By contrast, the thermal photons in thermal equilibrium with a metal plate or, much less, the virtual photons (giving rise to the Casimir effect) cannot, under any circumstances, lead to the initiation of a real current and heating of the metal (of course, this is strictly prohibited by thermodynamics). Hence the standard concept of a fluctuating electromagnetic field penetrating inside a metal described by the Drude dielectric function fails to model virtual and thermal photons in the frequency region (1). As a consequence, the Lifshitz formula cannot be applied in combination with the Drude dielectric function (2) to describe the thermal Casimir force even in the domain of the normal skin effect.

These arguments are supported also by considering the other frequency regions. At higher frequencies or larger l (lower temperatures) for most of the metals the anomalous skin effect holds, which is characterized by the inequalities

$$\delta_a(\omega)\ll l, \quad \delta_a(\omega)\ll\frac{v_F}{\omega}, \qquad (3)$$

where the skin depth is given by [37]

$$\delta_a(\omega)=\left(\frac{4\pi c^2\hbar^3}{\omega e^2 S_F}\right)^{1/3}, \qquad (4)$$

and S_F is the total area of the Fermi surface [in fact, within the inequalities (3) "much less" can be replaced by "less" thereby preserving Eq. (4) with a good precision]. In the frequency region of Eqs. (3) the volume relaxation is not significant, but the connection between the electric field and current becomes nonlocal. Because of this, a metal cannot be described by any dielectric function depending only on the frequency. As with the normal skin effect, this leads to the inapplicability of the standard concept of a fluctuating field spreading inside a medium described by $\varepsilon(\omega)$ and to the impossibility to calculate the Casimir force on this theoretical basis. If one would like to preserve the role of a fluctuating field within the domain of the anomalous skin effect, some nonlocal generalization of this concept is required. [Note that for some metals, especially for alloys, instead of Eq. (3), inequalities $v_F/\omega\ll l\ll\delta$ hold, specifying the so-called relaxation domain [38,39]; here the space dispersion is also essential.]

On further rise of the frequency, the following inequalities hold:

$$\frac{v_F}{\omega}\ll\delta_r\ll l, \qquad (5)$$

where $\delta_r=c/\omega_p$, which determines the domain of infrared optics (note that condition $\hbar\omega\ll\varepsilon_F$ is also supported where $\varepsilon_F=\hbar\omega_p$ is the Fermi energy). In this domain the volume relaxation does not play any role and the space dispersion is absent. Under conditions (5), metals can be described in the framework of the free-electron plasma model with a frequency-dependent dielectric permittivity

$$\varepsilon(\omega)=1-\frac{\omega_p^2}{\omega^2}. \qquad (6)$$

According to the plasma model, the conductivity is pure imaginary and hence there is not any real current or heating due to an electric field penetrating the metal. Because of this, the standard concept of a fluctuating electromagnetic field, as a model for the virtual and thermal photons, works well in the domain of infrared optics. Note that the plasma model dielectric permittivity in combination with the Lifshitz formula was used to calculate the thermal Casimir force [see approach (d) from the Introduction]. This approach did not meet any difficulties or contradictions with the basic principles of thermodynamics and has led to physically reasonable results. The domain of infrared optics is followed by the domain of ultraviolet frequencies where metals become transparent.

As is evident from the foregoing, the standard concept of a fluctuating electromagnetic field, penetrating a metal described by the dielectric permittivity depending only on frequency, cannot be used as an adequate model for the virtual and thermal photons of some frequencies. There is a frequency region (the domain of the normal skin effect) where this model is in conflict with the basic properties of the virtual and thermal photons at equilibrium which, among other things, cannot lead to heating of a metal. In addition, in the region of the anomalous skin effect and relaxation domain a metal cannot be described by the dielectric permittivity depending only on frequency. Because of this, another theoretical basis is preferred to find the thermal Casimir force between real metals different from that used in the case of dielectrics. Here we show that this basis is given by the concept of the surface impedance introduced by Leontovich [35,40].

The fundamental difference of the surface impedance approach from the usual approaches is that it does not permit one to consider the electromagnetic fluctuations inside a metal. Instead, the appropriate boundary conditions are imposed taking into account the properties of real metal

$$\boldsymbol{E}_t = Z(\omega)[\boldsymbol{B}_t \times \boldsymbol{n}], \tag{7}$$

where $Z(\omega)$ is the surface impedance of the conductor, $\boldsymbol{E}_t$ and $\boldsymbol{B}_t$ are the tangential components of electric and magnetic fields, and $\boldsymbol{n}$ is the unit normal vector to the surface (pointing inside the metal). The boundary condition (7) can be used to determine the electromagnetic field outside a metal. Note, that impedance $Z(\omega)$ and condition (7) suggest a more universal description than that by means of ε. They still hold under inequalities (3), where a description in terms of the dielectric permittivity $\varepsilon(\omega)$ is impossible. For an ideal metal we have $Z \equiv 0$ and for real nonmagnetic metals $|Z| \ll 1$ holds [40].

The calculation of the surface impedance over the whole frequency axis is based on the kinetic theory [35] and is rather cumbersome. However, in the domains of the normal and the anomalous skin effect, and also for infrared optics, simple asymptotic expressions follow, which are of great help to compute the Casimir force between real metals. Thus, in the domain of the normal skin effect, given by Eq. (1), the surface impedance is [35]

$$Z_n(\omega) = (1-i)\sqrt{\frac{\omega}{8\pi\sigma}}. \tag{8}$$

In the domain of the anomalous skin effect, determined by Eq. (3), the impedance depends on the shape of the Fermi surface [41]. For a polycrystalline metal, composed of many single crystal grains of different orientations, one obtains an approximately spherical Fermi surface and the impedance is [41]

$$Z_a(\omega) = \frac{2(1-i\sqrt{3})}{3\sqrt{3}} \frac{\omega\delta_a(\omega)}{c}, \tag{9}$$

where $\delta_a(\omega)$ was defined in Eq. (4).

In the domain of infrared optics [see Eq. (5)] the impedance is given by [35]

$$Z_r(\omega) = -i\frac{\omega}{\sqrt{\omega_p^2 - \omega^2}}. \tag{10}$$

Now one can impose the impedance boundary condition (7) on the surface of metal plates, find the oscillation spectrum in the space between the plates, and calculate the Casimir free-energy density and force without consideration of a fluctuating electromagnetic field inside the metal. This was performed at zero temperature in Ref. [42] (see also Ref. [3]). Another approach, being similar in spirit, was used at $T=0$ in Ref. [43] where the reflection coefficients in the Lifshitz formula were expressed in terms of $Z(\omega)$.

Now, the question arises what expression for impedance (8), (9), or (10) should be used to calculate the Casimir effect. To answer this question, it is good to bear in mind that the main contribution to the Casimir free energy and force is given by the frequency region centered around the so-called characteristic frequency $\omega_c = c/(2a)$, where a is the space separation between the two bodies, parallel plates for instance. The value of ω_c may belong to the frequency region given by Eqs. (1), (3), or (5), with the result that functions (8), (9), or (10), respectively, should be used, defining the impedance in the domains of the normal or anomalous skin effect and infrared optics.

By way of example, for most of the metals (Au, for instance) at room temperature the application region (1) of the normal skin effect with impedance (8) extends up to the frequencies of order 10^{12} rad/s. The application region (3) of the anomalous skin effect at $T=300$ K is very narrow and extends up to around $(6-7)\times 10^{13}$ rad/s. It should be stressed, however, that with the decrease of temperature the application region of the normal skin effect practically disappears and the anomalous skin effect extends to all frequencies lesser than 10^{12} rad/s. The reason is that l increases and $\delta_n(\omega)$ decreases with the decrease of temperature. As a result, the first inequality in Eq. (1) breaks down, whereas the first inequality in Eq. (3) is satisfied also at smaller frequencies. Finally, the impedance of the infrared optics (10) is applicable up to the frequencies of order $0.1\omega_p$ (for Au, for instance, $\omega_p = 1.37\times 10^{16}$ rad/s). It should be particularly emphasized that the transition frequency between the anomalous skin effect and infrared optics does not depend on temperature, because all the parameters in the second inequality of Eq. (3) are temperature independent (at temperatures much smaller than the Fermi temperature, which is of order 10^5 K). In Secs. V and VI we will discuss with more details which impedance function should be used for the calculation of the Casimir force at different separation distances between the test bodies (also the transition regions between different impedance functions will be considered).

III. ELECTROMAGNETIC OSCILLATION SPECTRUM BETWEEN TWO PLATES IN THE SURFACE IMPEDANCE APPROACH

Here the derivation of the photon eigenfrequencies in the framework of the surface impedance approach is presented. They are needed to derive the Lifshitz-type formula for the free energy in the case of real metals.

We consider the configuration of two parallel uncharged metal plates, separated by a distance a, at temperature T in thermal equilibrium. Let their nearest boundary planes be described by equations $z = \pm a/2$. We impose the boundary condition (7) on planes $z = \pm a/2$ and determine the eigenfrequencies of the electromagnetic field in the free space between the plates. The solutions of the Maxwell equations in vacuum can be found in the forms

$$\boldsymbol{E}_\alpha(t,\boldsymbol{r}) = \boldsymbol{e}_p(k_\perp, z)\exp(i\boldsymbol{k}_\perp \cdot \boldsymbol{r}_\perp - i\omega t),$$

$$\boldsymbol{B}_\alpha(t,\boldsymbol{r}) = \boldsymbol{g}_p(k_\perp, z)\exp(i\boldsymbol{k}_\perp \cdot \boldsymbol{r}_\perp - i\omega t). \tag{11}$$

Here $\boldsymbol{r}=(x,y,z)=(\boldsymbol{r}_\perp,z)$, $\alpha=\{p,\boldsymbol{k}_\perp,\omega\}$, $\boldsymbol{k}_\perp=(k_1,k_2)$ is the wave vector in plane (x,y), ω is a frequency, and index $p=\|,\perp$ labels the two independent polarization states ($\|$ stands for the electric field parallel to the plane formed by $\boldsymbol{k}_\perp$ and the z axis, and $\perp$ stands for the electric field perpendicular to this plane). From the Maxwell equations the oscillatory equations for the functions $\boldsymbol{e}_p$, $\boldsymbol{g}_p$ follow,

$$\boldsymbol{e}_p''(k_\perp,z)-q^2(k_\perp,\omega)\boldsymbol{e}_p(k_\perp,z)=0,$$
$$\boldsymbol{g}_p''(k_\perp,z)-q^2(k_\perp,\omega)\boldsymbol{g}_p(k_\perp,z)=0, \quad (12)$$

where $q^2(k_\perp,\omega)\equiv q^2\equiv k_\perp^2-\omega^2/c^2$; the prime denotes the derivative with respect to z, and also the first-order equations

$$e_{p,3}'(k_\perp,z)+ik_1e_{p,1}(k_\perp,z)+ik_2e_{p,2}(k_\perp,z)=0,$$
$$g_{p,3}'(k_\perp,z)+ik_1g_{p,1}(k_\perp,z)+ik_2g_{p,2}(k_\perp,z)=0 \quad (13)$$

(lower indices 1,2,3 after a comma stand for the projections of vectors $\boldsymbol{e}_p$, $\boldsymbol{g}_p$ onto axes x,y,z, respectively).

Substituting Eqs. (11) into the boundary condition (7), using the Maxwell equations to express the magnetic field and taking the direction of the normal into account [$\boldsymbol{n}=(0,0,1)$ at the plane $z=a/2$, and $\boldsymbol{n}=(0,0,-1)$ at $z=-a/2$], we find at boundaries $z=\pm a/2$, respectively,

$$e_{p,1}\left(k_\perp,\pm\frac{a}{2}\right)=\pm\frac{iZc}{\omega}\left[ik_1e_{p,3}\left(k_\perp,\pm\frac{a}{2}\right)-e_{p,1}'\left(k_\perp,\pm\frac{a}{2}\right)\right],$$
$$e_{p,2}\left(k_\perp,\pm\frac{a}{2}\right)=\pm\frac{iZc}{\omega}\left[ik_2e_{p,3}\left(k_\perp,\pm\frac{a}{2}\right)-e_{p,2}'\left(k_\perp,\pm\frac{a}{2}\right)\right]. \quad (14)$$

The same boundary conditions for $\boldsymbol{g}_p$ can be obtained also.

Now, let us consider separately the cases of parallel and perpendicular polarizations beginning with the parallel one. Without loss of generality, we temporarily assume that $k_2=0$. In this case $e_{\|,2}(k_\perp,z)\equiv 0$, and the solution of Eq. (12) has the form

$$e_{\|,1}(k_\perp,z)=B\sinh qz, \quad e_{\|,3}(k_\perp,z)=B\cosh qz, \quad (15)$$

where A and B do not depend on z. From Eq. (13) it follows $Aq+ik_1B=0$. For the sake of convenience, we choose

$$A=-\frac{ik_1}{q}e^{-aq/2}, \quad B=e^{-aq/2}. \quad (16)$$

Substituting Eqs. (15) and (16) into the impedance boundary condition (14), one obtains the dispersion equation for the spectrum of the electromagnetic oscillations between plates:

$$\Delta_\|^{(1)}(\omega,k_\perp)\equiv e^{-aq/2}\left(\sinh\frac{aq}{2}-\frac{iZ\omega}{cq}\cosh\frac{aq}{2}\right)=0. \quad (17)$$

Equations (12) and (13) also have the solutions

$$e_{\|,1}(k_\perp,z)=e^{-aq/2}\cosh qz, \quad e_{\|,2}(k_\perp,z)=0,$$
$$e_{\|,3}(k_\perp,z)=-\frac{ik_1}{q}e^{-aq/2}\sinh qz. \quad (18)$$

After substitution of Eq. (18) into Eq. (14) a further dispersion equation for the modes with parallel polarization is obtained:

$$\Delta_\|^{(2)}(\omega,k_\perp)\equiv e^{-aq/2}\left(\cosh\frac{aq}{2}-\frac{iZ\omega}{cq}\sinh\frac{aq}{2}\right)=0. \quad (19)$$

It is obvious that in Eqs. (17) and (19) $\boldsymbol{k}_\perp=(k_1,k_2)$ can be now considered as arbitrary.

Exactly the same procedure is applicable to the case of the perpendicular polarization. Once again, assuming temporarily $k_2=0$, we obtain the solutions of Eqs. (12) and (13) in the form

$$e_{\perp,1}(k_\perp,z)=e_{\perp,3}(k_\perp,z)=0 \quad (20)$$

and

$$e_{\perp,2}(k_\perp,z)=e^{-aq/2}\sinh qz$$

or

$$e_{\perp,2}(k_\perp,z)=e^{-aq/2}\cosh qz. \quad (21)$$

Substituting these solutions into Eq. (14), we arrive at two dispersion equations for the determination of the electromagnetic eigenfrequencies with perpendicular polarization

$$\Delta_\perp^{(1)}(\omega,k_\perp)\equiv e^{-aq/2}\left(\sinh\frac{aq}{2}+\frac{iZcq}{\omega}\cosh\frac{aq}{2}\right)=0,$$
$$\Delta_\perp^{(2)}(\omega,k_\perp)\equiv e^{-aq/2}\left(\cosh\frac{aq}{2}+\frac{iZcq}{\omega}\sinh\frac{aq}{2}\right)=0. \quad (22)$$

Let us denote the solutions of the transcendental equations (17) and (19) by $\omega_{k_\perp,n}^\|$, and the solutions of the transcendental equations (22) by $\omega_{k_\perp,n}^\perp$. Multiplying Eqs. (17) and (19), we can finally find $\omega_{k_\perp,n}^\|$ from the equation

$$\Delta_\|(\omega,k_\perp)\equiv\Delta_\|^{(1)}(\omega,k_\perp)\Delta_\|^{(2)}(\omega,k_\perp)$$
$$=\frac{1}{2}e^{-aq}(1-\eta^2)$$
$$\times\left(\sinh aq-\frac{2i\eta}{1-\eta^2}\cosh aq\right)=0, \quad (23)$$

where $\eta=\eta(\omega)=Z\omega/(cq)$.

In perfect analogy to this, by multiplication of Eqs. (22), one can find the equation for the determination of $\omega_{k_\perp,n}^\perp$:

$$\Delta_\perp(\omega,k_\perp)\equiv\Delta_\perp^{(1)}(\omega,k_\perp)\Delta_\perp^{(2)}(\omega,k_\perp)$$
$$=\frac{1}{2}e^{-aq}(1-\kappa^2)$$
$$\times\left(\sinh aq+\frac{2i\kappa}{1-\kappa^2}\cosh aq\right)=0, \quad (24)$$

where $\kappa=\kappa(\omega)=Zcq/\omega$.

Note that we have obtained the conditions for the determination of the electromagnetic oscillation spectrum by the use of equations for $\boldsymbol{e}_p$. Exactly the same spectrum is obtained if the equations for $\boldsymbol{g}_p$ are used.

IV. CASIMIR FREE ENERGY IN THE SURFACE IMPEDANCE APPROACH

Now we are in a position to present a rigorous derivation of Lifshitz-type formulas for the Casimir free energy and force for the configuration of two plates at temperature T in thermal equilibrium in the surface impedance approach. As shown below, these formulas are well adapted for the calculation of the Casimir effect between real metals and are not subject to the disadvantages of the approaches (a)–(d) discussed in the Introduction.

First we consider the case of real eigenfrequencies $\omega_{k_\perp,n}^{\parallel}$, $\omega_{k_\perp,n}^{\perp}$ (this is fulfilled for the pure imaginary impedance). The total free energy of the electromagnetic oscillations is given by the sum of the free energies of separate oscillators over all possible values of their quantum numbers,

$$\mathcal{F}=\sum_\alpha\left[\frac{\hbar\omega_\alpha}{2}+k_BT\ln(1-e^{-\hbar\omega_\alpha/k_BT})\right], \quad (25)$$

where k_B is the Boltzmann constant. Identically, Eq. (25) can be rewritten as

$$\mathcal{F}=k_BT\sum_\alpha\ln\left(2\sinh\frac{\hbar\omega_\alpha}{2k_BT}\right). \quad (26)$$

It is clear that at $T\to 0$, the value of $\mathcal{F}$ from Eqs. (25) and (26) coincides with the sum of the zero-point energies, which is the traditional starting point in theoretical investigations of the Casimir effect at zero temperature.

Applying this to the electromagnetic oscillations between metal plates, where $\alpha=\{p,\boldsymbol{k}_\perp,n\}$, we obtain

$$\mathcal{F}=k_BT\int_0^\infty\frac{k_\perp dk_\perp}{2\pi}\sum_n\left[\ln\left(2\sinh\frac{\hbar\omega_{k_\perp,n}^{\parallel}}{2k_BT}\right)+\ln\left(2\sinh\frac{\hbar\omega_{k_\perp,n}^{\perp}}{2k_BT}\right)\right]. \quad (27)$$

According to the calculations of Sec. III, the eigenfrequencies of the electromagnetic field between plates with parallel and perpendicular polarizations are determined by Eqs. (23) and (24), respectively.

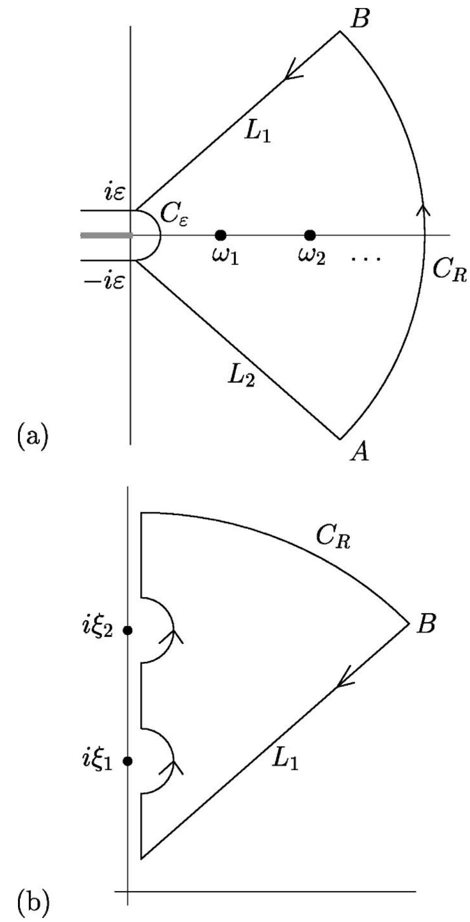

FIG. 1. Integration paths C_1 (a) and C_2 (b) in the plane of complex frequency. The Matsubara frequencies are ξ_l and the photon eigenfrequencies are ω_n.

The expression in the right-hand side of Eq. (27) is evidently infinite. Before performing a renormalization, let us equivalently represent the sum over the eigenfrequencies $\omega_{k_\perp,n}^{\parallel,\perp}$ by the use of the argument theorem, as is usually done in the derivation of the Lifshitz formula at zero temperature by the method of surface modes [6,44,45]. Then Eq. (27) can be rewritten as

$$\mathcal{F}=k_BT\int_0^\infty\frac{k_\perp dk_\perp}{2\pi}\frac{1}{2\pi i}\oint_{C_1}\ln\left(2\sinh\frac{\hbar\omega}{2k_BT}\right)d[\ln\Delta_\parallel(\omega,k_\perp)+\ln\Delta_\perp(\omega,k_\perp)]. \quad (28)$$

Here, the closed contour C_1 is bypassed counterclockwise. It consists of two arcs, one having an infinitely small radius ε and the other having an infinitely large radius R, and two straight lines L_1, L_2 inclined at angles ± 45 degrees to the real axis [see Fig. 1(a)]. The quantities $\Delta_{\parallel,\perp}(\omega,k_\perp)$, having their roots at the photon eigenfrequencies, are defined in Eqs. (23) and (24). Note that, unlike the usual derivation of the Lifshitz formula at nonzero temperature [2], the function under the integral in Eq. (28) has branch points rather than poles at the imaginary frequencies $\omega_l=i\xi_l$, where

$$\xi_l=\frac{2\pi k_BTl}{\hbar},\quad l=0,\pm1,\pm2,\ldots \tag{29}$$

are the Matsubara frequencies. The contour C_1 in Fig. 1(a) is chosen so as to avoid all these branch points and to enclose all the photon eigenfrequencies.

The integral in Eq. (28) can be calculated as follows:

$$\begin{aligned}I_{\parallel,\perp}&\equiv\frac{1}{2\pi i}\oint_{C_1}\ln\left(2\sinh\frac{\hbar\omega}{2k_BT}\right)d\ln\Delta_{\parallel,\perp}(\omega,k_\perp)\\&=\frac{1}{2\pi i}\left[\int_{L_2}\ln\left(2\sinh\frac{\hbar\omega}{2k_BT}\right)d\ln\Delta_{\parallel,\perp}(\omega,k_\perp)\right.\\&\quad+\int_{C_R}\ln\left(2\sinh\frac{\hbar\omega}{2k_BT}\right)d\ln\Delta_{\parallel,\perp}(\omega,k_\perp)\\&\quad+\int_{L_1}\ln\left(2\sinh\frac{\hbar\omega}{2k_BT}\right)d\ln\Delta_{\parallel,\perp}(\omega,k_\perp)\\&\quad\left.+\int_{C_\varepsilon}\ln\left(2\sinh\frac{\hbar\omega}{2k_BT}\right)d\ln\Delta_{\parallel,\perp}(\omega,k_\perp)\right].\end{aligned} \tag{30}$$

The integral along the arc of infinitely large radius C_R vanishes, which follows from Eqs. (23) and (24) under the natural conditions

$$\lim_{\omega\to\infty}Z(\omega)=\text{const},\quad \lim_{\omega\to\infty}\frac{dZ(\omega)}{d\omega}=0. \tag{31}$$

Integrating by parts in the right-hand side of Eq. (30), one obtains

$$\begin{aligned}I_{\parallel,\perp}&=\frac{1}{2\pi i}\left[\ln\left(2\sinh\frac{\hbar\omega}{2k_BT}\right)\ln\Delta_{\parallel,\perp}(\omega,k_\perp)\Big|_{-i\varepsilon}^{A}\right.\\&\quad-\frac{\hbar}{k_BT}\int_{L_2}\coth\frac{\hbar\omega}{2k_BT}\ln\Delta_{\parallel,\perp}(\omega,k_\perp)d\omega\\&\quad+\ln\left(2\sinh\frac{\hbar\omega}{2k_BT}\right)\ln\Delta_{\parallel,\perp}(\omega,k_\perp)\Big|_{B}^{i\varepsilon}\\&\quad-\frac{\hbar}{k_BT}\int_{L_1}\coth\frac{\hbar\omega}{2k_BT}\ln\Delta_{\parallel,\perp}(\omega,k_\perp)d\omega\\&\quad+\ln\left(2\sinh\frac{\hbar\omega}{2k_BT}\right)\ln\Delta_{\parallel,\perp}(\omega,k_\perp)\Big|_{i\varepsilon}^{-i\varepsilon}\\&\quad-\frac{\hbar}{k_BT}\int_{C_\varepsilon}\coth\frac{\hbar\omega}{2k_BT}\ln\Delta_{\parallel,\perp}(\omega,k_\perp)d\omega,\end{aligned} \tag{32}$$

where contours $L_{1,2}$ and points A,B are shown in Fig. 1(a). It is evident that all terms, besides the integrals, cancel each other or are equal to zero (at points A,B). The integral along L_1 can be calculated by the application of the Cauchy theorem to the closed contour C_2 [see Fig. 1(b)], inside which the function under consideration is analytic,

$$\begin{aligned}&-\int_{L_1}\coth\frac{\hbar\omega}{2k_BT}\ln\Delta_{\parallel,\perp}(\omega,k_\perp)d\omega\\&\quad=-\int_{i\infty}^{i\varepsilon}\coth\frac{\hbar\omega}{2k_BT}\ln\Delta_{\parallel,\perp}(\omega,k_\perp)d\omega.\end{aligned} \tag{33}$$

Here we assume that the integral along C_R vanishes. Path $(i\infty;i\varepsilon)$ contains semicircles of radius ε about the singular points $i\xi_l$ (poles) of the function $\coth(\hbar\omega/2k_BT)$. The analogous formula for the integral along line L_2 is

$$\begin{aligned}&-\int_{L_2}\coth\frac{\hbar\omega}{2k_BT}\ln\Delta_{\parallel,\perp}(\omega,k_\perp)d\omega\\&\quad=-\int_{-i\varepsilon}^{-i\infty}\coth\frac{\hbar\omega}{2k_BT}\ln\Delta_{\parallel,\perp}(\omega,k_\perp)d\omega.\end{aligned} \tag{34}$$

Substituting Eqs. (33), (34) into Eq. (32), one arrives at

$$I_{\parallel,\perp}=-\frac{\hbar}{2\pi ik_BT}\int_{i\infty}^{-i\infty}\coth\frac{\hbar\omega}{2k_BT}\ln\Delta_{\parallel,\perp}(\omega,k_\perp)d\omega. \tag{35}$$

The integration in Eq. (35), involving poles at the points $i\xi_l$, leads to

$$\begin{aligned}I_{\parallel,\perp}&=\frac{i\hbar}{2\pi k_BT}\int_{\infty}^{-\infty}\cot\frac{\hbar\xi}{2k_BT}\ln\Delta_{\parallel,\perp}(\xi,k_\perp)d\xi\\&\quad-\pi\sum_{l=-\infty}^{\infty}\text{res}\left[\coth\frac{\hbar\omega}{2k_BT}\ln\Delta_{\parallel,\perp}(\omega,k_\perp);i\xi_l\right],\end{aligned} \tag{36}$$

where functions $\Delta_{\parallel,\perp}(\xi,k_\perp)$ are obtained from $\Delta_{\parallel,\perp}(\omega,k_\perp)$ by the substitution $\omega=i\xi$. In the case of real eigenfrequencies, which is under consideration now, $\Delta_{\parallel,\perp}$ are even functions of ω (and ξ). As a consequence, the seemingly pure imaginary integral in the right-hand side of Eq. (36) vanishes. After the calculation of the residues, and using the evenness of functions $\Delta_{\parallel,\perp}(\omega,k_\perp)$, the result is

$$I_{\parallel,\perp}=\sum_{l=0}^{\infty}{}'\ \ln\Delta_{\parallel,\perp}(\xi_l,k_\perp), \tag{37}$$

where the prime on the summation sign means that the term for $l=0$ has to be multiplied by 1/2.

Substituting the values (37) of the integrals (30) into Eq. (28), we find the equivalent but more simple expression for the Casimir free energy,

$$\mathcal{F}=\frac{k_BT}{2\pi}\int_0^\infty k_\perp dk_\perp\sum_{l=0}^{\infty}{}'\ [\ln\Delta_\parallel(\xi_l,k_\perp)+\ln\Delta_\perp(\xi_l,k_\perp)]. \tag{38}$$

Expression (38) is still infinite. To remove the divergences, we subtract from the right-hand side of Eq. (38) the free energy in the case of infinitely separated interacting bodies $(a\to\infty)$. Then the physical, renormalized, free energy vanishes for infinitely remote plates. From Eqs. (23) and (24), after the substitution $\omega \to i\xi_l$ in the limit $a\to\infty$, it follows

$$\Delta_{\parallel}^{\infty}(\xi_l,k_\perp)=\frac{1}{4}(1+\eta_l^2)\left(1+\frac{2\eta_l}{1+\eta_l^2}\right),$$

$$\Delta_{\perp}^{\infty}(\xi_l,k_\perp)=\frac{1}{4}(1+\kappa_l^2)\left(1+\frac{2\kappa_l}{1+\kappa_l^2}\right). \tag{39}$$

The renormalization prescription is equivalent to the change of $\Delta_{\parallel,\perp}(\xi_l,k_\perp)$ in Eq. (38) for

$$\Delta_{\parallel,\perp}^{R}(\xi_l,k_\perp)\equiv\frac{\Delta_{\parallel,\perp}(\xi_l,k_\perp)}{\Delta_{\parallel,\perp}^{\infty}(\xi_l,k_\perp)}=1-r_{\parallel,\perp}^2(\xi_l,k_\perp)e^{-2aq_l}, \tag{40}$$

where quantities $r_{\parallel,\perp}(\xi_l,k_\perp)$ have the meaning of the reflection coefficients and are given by

$$r_{\parallel}^2(\xi_l,k_\perp)=\left(\frac{1-\eta_l}{1+\eta_l}\right)^2=\left(\frac{cq_l-Z_l\xi_l}{cq_l+Z_l\xi_l}\right)^2,$$

$$r_{\perp}^2(\xi_l,k_\perp)=\left(\frac{1-\kappa_l}{1+\kappa_l}\right)^2=\left(\frac{\xi_l-Z_lcq_l}{\xi_l+Z_lcq_l}\right)^2. \tag{41}$$

Here $Z_l\equiv Z(i\xi_l)$ and $q_l^2=k_\perp^2+\xi_l^2/c^2$. The reflection coefficients (41) are in accordance with Ref. [40], where the reflection of a plane electromagnetic wave incident from vacuum onto the plane surface of the metal was described in terms of the surface impedance.

In such a manner the final renormalized expression for the Casimir free energy in the surface impedance approach is given by

$$\mathcal{F}_R=\frac{k_BT}{2\pi}\int_0^\infty k_\perp\,dk_\perp\sum_{l=0}^{\infty}{}'\,\{\ln[1-r_{\parallel}^2(\xi_l,k_\perp)e^{-2aq_l}]+\ln[1-r_{\perp}^2(\xi_l,k_\perp)e^{-2aq_l}]\}. \tag{42}$$

where the reflection coefficients are given by Eq. (41).

The Casimir force, acting between plates, is obtained from Eq. (42),

$$F=-\frac{\partial\mathcal{F}_R}{\partial a}=-\frac{k_BT}{\pi}\int_0^\infty k_\perp dk_\perp\sum_{l=0}^{\infty}{}'\;q_l\{[r_{\parallel}^{-2}(\xi_l,k_\perp)\times e^{2aq_l}-1]^{-1}+[r_{\perp}^{-2}(\xi_l,k_\perp)e^{2aq_l}-1]^{-1}\}. \tag{43}$$

The above derivation was performed under the assumption that the photon eigenfrequencies are real. This is, however, not the case for arbitrary complex impedance. If the photon eigenfrequencies are complex, the free energy is not given by Eq. (26) (which is already clear from the complexity of the right-hand side of this equation). For arbitrary complex impedance, the correct expression for the free energy should be determined from the solution of an auxiliary electrodynamic problem [46]. It turns out that the Casimir free energy and force are the functionals of the impedance even when the impedance has a nonzero real part taking absorption into account. The solution of the auxiliary electrodynamic problem leads to the conclusion [46] that the correct free energy is obtained from Eqs. (38)–(42) by analytic continuation to arbitrary complex impedance, i.e., to arbitrary oscillation spectra. The qualitative reason for the validity of this statement is that the free energy depends only on the behavior of $Z(\omega)$ at the imaginary frequency axis, where $Z(\omega)$ is always real [see, e.g., Eqs. (8)–(10)]. Note that in the case of complex eigenfrequencies, exactly Eqs. (42) and (43) should be used written in terms of summations from zero to infinity. Although for real eigenfrequencies the summations over l from $-\infty$ to ∞ can be equivalently used, it is not so for complex ω_α as the dispersion functions $\Delta_{\parallel,\perp}$ cease to be even any more [46].

It is necessary to stress that the above derivation of the free energy in the impedance approach can be simply modified in order to present the new derivation of the usual Lifshitz formula describing the thermal Casimir force between dielectrics. In fact, nothing should be changed in the presentation of this section, except for the explicit expressions of the dispersion functions $\Delta_{\parallel,\perp}$ in Eq. (38) and thus of the reflection coefficients $r_{\parallel,\perp}$ in Eqs. (42) and (43). The dispersion functions should be determined not according to Sec. III but from the consideration of a fluctuating electromagnetic field both inside and outside the dielectric plates with the usual boundary conditions at the interfaces [2,6,44,45]. As a result, the Lifshitz reflection coefficients take forms

$$r_{\parallel,L}^2(\xi_l,k_\perp)=\left(\frac{\varepsilon_lq_l-k_l}{\varepsilon_lq_l+k_l}\right)^2,$$

$$r_{\perp,L}^2(\xi_l,k_\perp)=\left(\frac{q_l-k_l}{q_l+k_l}\right)^2, \tag{44}$$

where $\varepsilon_l\equiv\varepsilon(i\xi_l)$, $\varepsilon(\omega)$ is the dielectric permittivity of the plate material, and $k_l^2\equiv k_\perp^2+\varepsilon_l\xi_l^2/c^2$. Then the Lifshitz expressions for the Casimir free energy and force between dielectrics are given by Eqs. (42) and (43), where the substitution $r_\parallel,r_\perp\to r_{\parallel,L},r_{\perp,L}$ is made. In such a manner, we have performed also a new derivation of the usual Lifshitz formula between dielectric plates starting from the free energy of an oscillator. Conversely, the free energy and force in the framework of the impedance approach are obtained from the Lifshitz formula if the Fresnel-type reflection coefficients $r_{\parallel,L},r_{\perp,L}$ are changed for those obtained by the use of the impedance boundary condition.

It should be stressed, however, that the reflection coefficients (44) differ essentially from the impedance coefficients (41). To take an example, it is not possible to obtain coefficients Eq. (41) from (44) even if both descriptions in terms of $\varepsilon(\omega)$ and $Z(\omega)$ are applicable and the impedance is expressed in terms of the dielectric permittivity by means of relation $Z(\omega)=1/\sqrt{\varepsilon(\omega)}$ (which holds, e.g., in the region of infrared optics). This underlines the fundamental role of the impedance boundary condition as an alternative to the consideration of a fluctuating field inside a medium described by $\varepsilon(\omega)$ in the case of real metals.

We conclude this section by remarking that the obtained expression (42) for the free energy gives the possibility also to find the thermal Casimir force in configuration of a sphere (spherical lens) above a plate made of real metals in the surface impedance approach

$$F(a)=2\pi R\mathcal{F}_R(a), \tag{45}$$

where R is the sphere radius. The approximate expression (45) is obtained by the application of the proximity force theorem [6] and has an accuracy around a fraction of 1% for configurations used in precision experiments on the measurement of the Casimir force [7–12,14,18]. Thus, the impedance approach provides the theoretical basis for the measurements of the thermal Casimir force between real metals to be performed in the near future.

V. CALCULATION OF THE CASIMIR ENERGY IN THE SURFACE IMPEDANCE APPROACH

First, we apply the obtained general formulas at zero temperature. In this case, Eq. (42) for the free energy transforms to the double integral representing the Casimir energy between plates [or, according to Eq. (45), the Casimir force acting between a sphere and a plate]

$$E(a)=\frac{\hbar}{4\pi^2}\int_0^\infty k_\perp dk_\perp\int_0^\infty d\xi\{\ln[1-r_\parallel^2(\xi,k_\perp)e^{-2aq}] + \ln[1-r_\perp^2(\xi,k_\perp)e^{-2aq}]\}, \tag{46}$$

where the reflection coefficients in terms of the impedance are given by Eq. (41) with the substitution

$$q_l\to q=\sqrt{k_\perp^2+\frac{\xi^2}{c^2}},\quad Z_l\to Z(i\xi),\quad \xi_l\to\xi. \tag{47}$$

Let us calculate quantity (46) obtained in the impedance approach and compare the results with the available data found by the traditional computations using the Lifshitz formula. For the purpose of numerical computations, it is convenient to rearrange Eq. (46) to the form [30]

$$E(a)=\frac{\hbar c}{32\pi^2a^3}\int_0^\infty d\zeta\int_\zeta^\infty ydy\left\{2\ln(1-e^{-y}) + \ln\left[1+\frac{X^{\parallel}(\zeta,y)}{e^y-1}\right]+\ln\left[1+\frac{X^{\perp}(\zeta,y)}{e^y-1}\right]\right\}, \tag{48}$$

where the dimensionless variables ζ,y are defined as

$$\zeta=\frac{\xi}{\omega_c}=\frac{2a\xi}{c},\quad y=2qa, \tag{49}$$

and quantities $X^{\parallel,\perp}(\zeta,y)$ are given by

$$X^{\parallel}(\zeta,y)=\frac{4\zeta yZ}{(y+\zeta Z)^2},$$

$$X^{\perp}(\zeta,y)=\frac{4\zeta yZ}{(\zeta+yZ)^2},\quad Z\equiv Z\left(i\frac{c\zeta}{2a}\right). \tag{50}$$

The first contribution in the right-hand side of Eq. (48) describes the case of an ideal metal,

$$E^{(0)}(a)=\frac{\hbar c}{16\pi^2a^3}\int_0^\infty d\zeta\int_\zeta^\infty ydy\ln(1-e^{-y})=-\frac{\pi^2\hbar c}{720a^3}, \tag{51}$$

the others are the corrections due to the finite conductivity.

As was stressed in Sec. II, with the decrease of temperature the range of application of the normal skin effect (1) reduces to zero, and at $T=0$ only the anomalous skin effect and infrared optics occur with the frequency regions given by Eqs. (3) and (5), respectively. The transition frequency Ω between the two effects can be obtained from equations

$$\delta_a(\Omega)=\frac{v_F}{\Omega}=\delta_r=\frac{c}{\omega_p}, \tag{52}$$

where, according to Eq. (4), $\delta_a(\Omega)=C_a/\Omega^{1/3}$. All computations given below are performed for Au with $\omega_p=1.37\times10^{16}$ rad/s [47] and $v_F=1.4\times10^6$ m/s (see, e.g., Ref. [48]). Then from Eq. (52) we obtain the values of both $C_a=8.8\times10^{-4}$ m rad$^{1/3}$/s$^{1/3}$ and $\Omega=6.36\times10^{13}$ rad/s. If to consider Ω as the characteristic frequency giving the main contribution to the Casimir effect ($\Omega=\omega_c=c/2a_{tr}$), the transition separation distance between the two effects turns out to be equal to $a_{tr}=2.36$ μm. Then it follows that at distances $\lambda_p<a\ll a_{tr}=2.36$ μm the impedance of the infrared optics determines the value of the Casimir energy and force, whereas at $a\gg a_{tr}=2.36$ μm the impedance of the anomalous skin effect is applicable ($\lambda_p=137$ nm is the plasma wavelength for Au). Direct calculations by Eqs. (48) and (50) show that the main contribution to the Casimir energy is given by the narrow frequency interval around the characteristic frequency ω_c. Thus, the interval $(0.1\omega_c,10\omega_c)$ contributes 94% of the total energy in the wide separation region. What is even more important, the remainder does not depend on the form of the impedance function outside interval $(0.1\omega_c,10\omega_c)$, to within the error of about 0.5%. From this it follows that at each separation distance between the plates one should, first, determine the characteristic frequency ω_c and, second, fix the proper impedance function. Thereafter the chosen impedance function can be used at all frequencies when performing the integration in Eq. (48). At zero temperature this prescription is optional. At $T\neq0$, however, it takes on great significance (see Sec. VI).

In Fig. 2 the correction factor to the Casimir energy $E(a)/E^{(0)}(a)$ is plotted, which is computed by Eqs. (48), (50), and (51) as a function of the separation distance. The solid line is obtained with the impedance of the infrared optics (10), and the dotted line is obtained with the impedance of the anomalous skin effect (9). Both lines are plotted

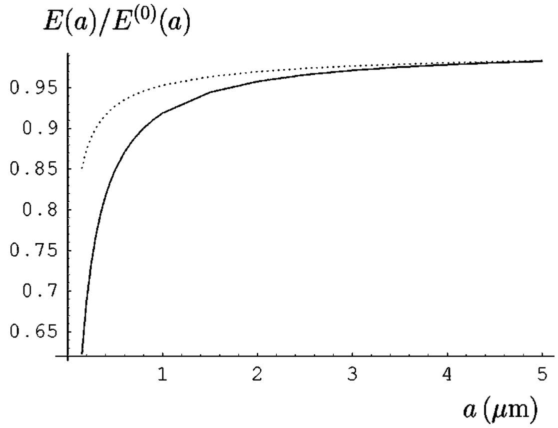

FIG. 2. Correction factor to the Casimir energy between two Au plates at zero temperature computed by the use of the impedance of infrared optics (solid line) and of anomalous skin effect (dotted line) versus surface separation.

at all separations $a > \lambda_p$ to make sure that each impedance function is applicable within its own frequency region and to follow their applicability at the transition separations around a_{tr}. It must be emphasized that the solid line coincides with the correction factor to the Casimir energy computed on the basis of the usual Lifshitz formula in combination with the dielectric function of the plasma model (this was demonstrated in detail in Ref. [30]). Thus, both the impedance approach and the Lifshitz formula combined with the plasma model lead to one and the same result for the Casimir energy at separations $a > \lambda_p$.

As is seen from Fig. 2, at $\lambda_p < a \ll 2.36$ μm, the pointed line computed with the impedance of the anomalous skin effect (which is inapplicable in this region) significantly underestimates the correction factor due to the finite conductivity. For example, at $a = 0.15$ μm the values of the correction factors, given by the solid and dotted lines, are 0.623 and 0.851, respectively, i.e., the error introduced by the use of the impedance of the anomalous skin effect is almost 37%. At a separation $a = 0.5$ μm this error is more than 9%, and decreases with increasing separation. Notice that the computations on the basis of the usual Lifshitz formula and optical tabulated data for the complex refraction index [which are used to obtain $\varepsilon(i\xi)$ through the dispersion relation] also practically coincide with those given by the surface impedance in the region of the infrared optics (solid line in Fig. 2). Thus, at the separations of 0.2 μm, 0.5 μm, and 3 μm, the correction factor obtained by the tabulated data and Lifshitz formula is equal to 0.69, 0.85, and 0.97, respectively [45,47], whereas in the impedance approach it takes the values 0.689, 0.849, and 0.972.

At larger separations ($a \gg a_{tr} = 2.36$ μm) the impedance function of the anomalous skin effect should be used to compute the Casimir energy (dotted line in Fig. 2). As is seen from that figure, at these separations the impedance of the infrared optics overestimates the role of the finite conductivity corrections to the Casimir energy. This overestimation is, however, to within a fraction of 1%. In the transition region $a = 2$–2.5 μm the results given by both impedance functions are in agreement, bringing the discrepancies of about 1% only. This leads us to the conclusion that at zero temperature both impedance functions work well in their respective application regions. In the transition region each of them can be applied, and the results are in agreement within an error of 1%. It is seen also that "much less" or "much larger" in the above inequalities, in fact, means two or three times less (larger).

As regards the region of infrared optics, the Lifshitz formula in combination with the plasma model or optical tabulated data for the complex refraction index leads to the same results as the impedance approach. It gives rather good results even in the region of the anomalous skin effect, where, strictly speaking, the description in terms of ε is not applicable (see Sec. II). The feasibility of the Lifshitz formula is explained by the fact that at zero temperature the normal skin effect is practically absent and the problems connected with the heating of a metal due to the real electric current are not relevant. As a result, both the impedance approach and the usual Lifshitz formula are applicable. At nonzero temperature, however, the surface impedance approach acquires a new meaning and solves the problems formulated in the Introduction (see the following section).

VI. CALCULATION OF THE CASIMIR FREE ENERGY IN THE SURFACE IMPEDANCE APPROACH

Here we calculate the Casimir free energy for the configuration of two parallel plates made of Au at temperature T at thermal equilibrium. The starting point is Eq. (42), where the reflection coefficients are expressed in terms of the surface impedance by Eq. (41). Introducing the dimensionless variables by analogy with Eq. (49), we transform Eq. (42) to a form convenient for numerical computations:

$$\begin{aligned}\mathcal{F}_R &= \mathcal{F}_R(a,T)\\ &= \frac{k_B T}{8\pi a^2}\sum_{l=0}^{\infty}{}' \int_{\zeta_l}^{\infty} y\,dy \Bigg\{ 2\ln(1-e^{-y})\\ &\quad + \ln\left[1+\frac{X^{\parallel}(\zeta_l,y)}{e^y-1}\right] + \ln\left[1+\frac{X^{\perp}(\zeta_l,y)}{e^y-1}\right]\Bigg\}, \qquad (53)\end{aligned}$$

where $X^{\parallel}, X^{\perp}$ are given by Eq. (50) with the change $\zeta \to \zeta_l$, $Z \to Z_l = Z(ic\zeta_l/2a)$. Notice that the first contribution in the right-hand side of Eq. (53) presents the Casimir free energy for the ideal metal [6]

$$\begin{aligned}\mathcal{F}_R^{I}(a,T) &= \frac{k_B T}{4\pi a^2}\sum_{l=0}^{\infty}{}' \int_{\zeta_l}^{\infty} y\,dy \ln(1-e^{-y})\\ &= E^{(0)}(a)\Bigg\{1+\frac{45}{\pi^3}\sum_{l=1}^{\infty}\left[\left(\frac{T}{T_{eff}}\right)^3 \frac{1}{l^3}\coth\left(\pi l\,\frac{T_{eff}}{T}\right)\right.\\ &\quad \left. + \pi\left(\frac{T}{T_{eff}}\right)^2 \frac{1}{l^2}\sinh^{-2}\left(\pi l\,\frac{T_{eff}}{T}\right)\right] - \left(\frac{T}{T_{eff}}\right)^4\Bigg\}, \qquad (54)\end{aligned}$$

where $E^{(0)}(a)$ is defined in Eq. (51) and the effective temperature $k_B T_{eff} = \hbar\omega_c = \hbar c/(2a)$.

First of all, let us demonstrate that in the impedance approach there is no problem with the contribution of the zero Matsubara frequency which was the subject of much recent controversy (see the Introduction). We start from the lowest characteristic frequencies where the impedance of the normal skin effect, given by Eq. (8), is applicable. Substituting it into Eq. (41) and putting $\xi_l = 0$, one obtains

$$r_{\parallel}^2(0,k_{\perp}) = r_{\perp}^2(0,k_{\perp}) = 1, \tag{55}$$

i.e., the same result as for an ideal metal; namely, this becomes clear since quantities $X^{\parallel,\perp}$, defined in Eq. (50) and given for the impedance of the normal skin effect as

$$X^{\parallel}(0,y) = X^{\perp}(0,y) = 0, \tag{56}$$

when inserted into Eq. (53) obviously lead to the same zero-frequency contribution as it holds for an ideal metal. It should be stressed that all functions $r_{\parallel,\perp}^2(\xi,k_{\perp})$ and $X^{\parallel,\perp}(\zeta,y)$ are continuous functions of two variables including the point (0,0). Thus, the case of an ideal metal is achieved as a limiting case of a real metal with increase of the conductivity when the real metal is described in the framework of the impedance approach. Note that this is not the case when the real metal is described by the Drude dielectric function (2) and the Lifshitz formula for dielectrics is used to calculate the Casimir free energy [approach (a) from the Introduction]. In fact, if doing so it follows from Eq. (44)

$$r_{\parallel,L}^2(0,k_{\perp}) = 1, \quad r_{\perp,L}^2(0,k_{\perp}) = 0, \tag{57}$$

and there is a break of continuity between the properties of real metals and of ideal metal [28].

At higher characteristic frequencies the anomalous skin effect holds with an impedance function as from Eqs. (4) and (9). If we extend this function to all frequencies (to zero Matsubara frequency in that case), we ensure that both Eqs. (55) and (56) are valid once again. As a result, in both regions of the normal and the anomalous skin effect the thermal corrections to the Casimir free energy and force for real metals are very close to those for an ideal metal. As one would expect, at large separations (characteristic for the anomalous and especially for the normal skin effect) all metals behave like an ideal one [this is, however, not the case in the framework of the approach (a)].

If the characteristic frequencies increase further, the infrared optics with an impedance function of Eq. (10) takes place. The extension of it to zero Matsubara frequency leads to

$$r_{\parallel}^2(0,k_{\perp}) = 1, \quad r_{\perp}^2(0,k_{\perp}) = \left(\frac{\omega_p - ck_{\perp}}{\omega_p + ck_{\perp}}\right)^2. \tag{58}$$

In this case there occurs a dependence of the perpendicular reflection coefficient at zero frequency on the properties of the real metal through the value of the plasma frequency. This is reasonable, because the real properties of a metal are most pronounced at small separations characteristic for the infrared optics. In the limit $\omega_p \to \infty$, the result for an ideal metal is reproduced from Eq. (58).

Before performing the computations, it must be emphasized that the surface impedance approach is in perfect agreement with thermodynamics. In the impedance approach, the entropy, defined as

$$S(a,T) = -\frac{\partial \mathcal{F}_R(a,T)}{\partial T}, \tag{59}$$

is positive and equal to zero at zero temperature in accordance with the Nernst heat theorem [note that this is not the case in the approaches (a) and (c)]. The validity of the Nernst heat theorem in the impedance approach can be demonstrated in the regions of both the infrared optics and the anomalous skin effect (as noted above, the region of the normal skin effect dies out with decreasing temperature). According to the results of Ref. [30], in the region of infrared optics the Lifshitz formula combined with the plasma model leads to exactly the same perturbation results for the Casimir free energy and force as the impedance approach. At $T \ll T_{eff}$ the free energy is given by [31]

$$\mathcal{F}_R(a,T) = E(a) - \frac{\hbar c \zeta(3)}{16\pi a^3}\left[\left(1 + 2\frac{\delta_r}{a}\right)\left(\frac{T}{T_{eff}}\right)^3 - \frac{\pi^3}{45\zeta(3)}\left(1 + 4\frac{\delta_r}{a}\right)\left(\frac{T}{T_{eff}}\right)^4\right], \tag{60}$$

where $E(a)$ is the Casimir energy at $T=0$ defined in Eq. (46). After the substitution in Eq. (59), this leads to the simple expression for the Casimir entropy

$$S(a,T) = \frac{3k_B\zeta(3)}{8\pi a^2}\left(\frac{T}{T_{eff}}\right)^2\left\{1 - \frac{4\pi^3}{135\zeta(3)}\frac{T}{T_{eff}} + 2\frac{\delta_r}{a}\left[1 - \frac{8\pi^3}{135\zeta(3)}\frac{T}{T_{eff}}\right]\right\}, \tag{61}$$

which is positive and equal to zero at zero temperature.

The impedance approach in the region of the anomalous skin effect was used recently in Ref. [32]. The asymptotic of the entropy at very low temperatures, obtained in Ref. [32], demonstrates that it is positive and has zero value at zero temperature in accordance with the requirements of thermodynamics.

By the way of an example, here we perform the numerical computations of the relative thermal correction to the Casimir free energy defined as $[\mathcal{F}_R(a,T) - E(a)]/E(a)$. This quantity has also the meaning of the relative thermal correction to the Casimir force in the configuration of a sphere (spherical lens) above a plate used in precision experiments on the Casimir effect. If the characteristic frequency ω_c belongs to the region of the normal skin effect, the results practically coincide with those obtained for an ideal metal [30], and the free energy is given by Eq. (54). If the characteristic frequency belongs to the regions of the anomalous skin effect or infrared optics, the computational results for the relative thermal correction are obtained by Eqs. (48), (53)

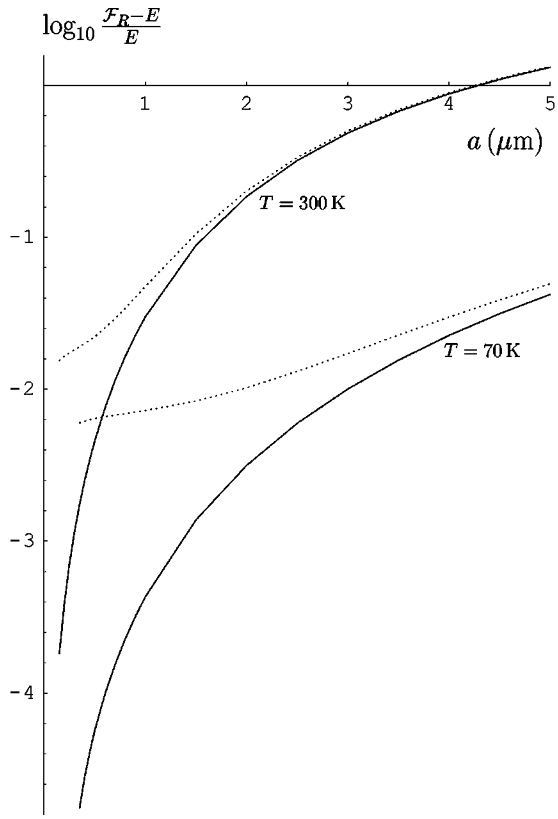

FIG. 3. Relative thermal correction to the Casimir free energy between two Au plates computed by the use of the impedance of infrared optics (solid lines) and of anomalous skin effect (dotted lines) versus surface separation at two different temperatures.

and are presented in Fig. 3. The solid lines are computed with the impedance of the infrared optics (10), and the dotted lines are computed with the impedance of the anomalous skin effect (9). All computations are performed for Au at two temperatures $T = 300$ K and $T = 70$ K with numerical parameters as listed in Sec. V. Both pairs of lines are plotted at all separations $a > \lambda_p$ for a better visualization of the application range of each impedance function.

Note that at separations between the plates $\lambda_p < a < a_{tr}$, where $a_{tr} = 2.36$ μm does not depend on the temperature, the impedance of the infrared optics is applicable, and at separations $a > a_{tr}$ the impedance of the anomalous skin effect should be used. It is seen from Fig. 3 that at small separations the use of the impedance function of the anomalous skin effect significantly overestimates the value of the thermal correction. Thus, at $a = 0.15$ μm the values of the relative thermal corrections given by the dotted and solid lines are 1.55×10^{-2} and 1.82×10^{-4}, respectively, at $T = 300$ K, and 4.85×10^{-3} and 2.76×10^{-6}, respectively, at $T = 70$ K. What this means is the thermal correction, predicted by the impedance of the anomalous skin effect in the region of the infrared optics, where this impedance is not applicable, is 85 times greater at $T = 300$ K and 1757 times greater at $T = 70$ K than the correct values.

At separations $a > a_{tr} = 2.36$ μm the dotted lines present the correct dependence of the thermal correction on the separation distance. The difference between the free energies computed by the use of two impedance functions is, however, to within a fraction of 1%. In the transition region the results, given by the impedance function of the infrared optics and anomalous skin effect, are in agreement with a sufficient accuracy. For example, at $a = 2.5$ μm the ratio of the relative thermal corrections obtained by the use of different impedance functions is 1.05 at $T = 300$ K and 2.19 at $T = 70$ K. As a result, the discrepancies in the values of the free energy are about 1.2% ($T = 300$ K) and 0.7% ($T = 70$ K).

Our results for the thermal correction to the Casimir free energy are in disagreement with the conclusion of Ref. [32] about the existence of large thermal corrections at low temperature made in the framework of the impedance approach. As correctly argued in Ref. [32], the description of metals with the impedance in the region of the anomalous skin effect is more appropriate than with the dielectric permittivity. However, the conclusion about the existence of large thermal corrections at separations 100–500 nm at $T \leqslant 70$ K made in that paper is in error. To obtain this conclusion, the impedance function of the anomalous skin effect was applied in Ref. [32] at separations much less than $a_{tr} = 2.36$ μm, i.e., in the separation range of the infrared optics. This was explained by the fact that at temperatures $T \leqslant 70$ K the inequality $l \gg \delta_r$ holds, which, from the standpoint of Ref. [32], guarantees the applicability of the impedance of the anomalous skin effect. In actual, this inequality is not sufficient. In fact, one additional inequality, $\delta_a(\omega) \ll v_F/\omega$, must be fulfilled in order that the anomalous skin effect holds [see Eq. (3) and Ref. [37]]. Because of this, the frequency Ω [see definition in Eq. (52)], considered in Ref. [32] as the characteristic frequency of the anomalous skin effect, is actually the transition frequency to the region of infrared optics. As a result, all computations performed in Ref. [32] correspond to the dotted line at $T = 70$ K of our Fig. 3 at separations $a < a_{tr} = 2.36$ μm. According to our computations, in this separation range the dotted line at $T = 70$ K overestimates the value of the thermal correction by a factor of 2000, whereas the correct results are given by the solid lines obtained by the use of the impedance function of infrared optics. Note that the characteristic frequencies, corresponding to the separations 100–500 nm, fall into the interval $\omega_c = (0.3 - 1.5) \times 10^{15}$ rad/s$\gg \Omega$, i.e., belong to the region of the infrared optics (see Sec. II).

At the end of this section, we would like to stress that in the sums, such as Eqs. (42), (43), and (53), the form of the impedance at the characteristic frequencies must be substituted and extended to all other frequencies. At zero temperature, as was shown in Sec. V, the frequency region $[0, 1\omega_c, 10\omega_c]$, where the characteristic frequency is $\omega_c = c/(2a)$, gives most of the contributions to the result. Calculations show that at nonzero temperature the Matsubara frequencies from ξ_0 to $\xi_N \approx 10\omega_c$ give the dominant contribution. For example, at $a = 0.15$ μm ($\omega_c = 10^{15}$ rad/s), $T = 300$ K the first 41 Matsubara frequencies determine the total result. Here $\xi_1 = 2.5 \times 10^{14}$ rad/s and $\xi_{40} = 10^{16}$ rad/s. All nonzero Matsubara frequencies belong to the region of infrared optics. With a decrease of the temperature some of the Matsubara frequencies may fall within the frequency region of the anomalous skin effect (at $T = 70$ K, for instance, the first Matsubara frequency $\xi_1=5.75\times10^{13}$ rad/s$<\Omega=6.36\times10^{13}$ rad/s for Au). At small separations, however, the differences in the contributions of several first Matsubara frequencies computed by the use of different impedance functions are negligible. At $T=0$ any extension of the impedance function outside the above interval leads to approximately one and the same value of the Casimir energy (in the integral one point $\xi=0$ is of no significance). At $T\neq0$, however, the contribution of the zero Matsubara frequency $\xi_0=0$ becomes dominant at large separations (high temperatures), and at room temperature, for instance, it determines the total value of the free energy at $a\geqslant5$ μm.

The basic challenge is whether the actual reflection properties of plate materials at very low, quasistatic, frequencies are responsible for the Casimir force in the high-temperature limit. The point is that materials are at hand (e.g., indium tin oxide), which are good conductors at quasistatic frequencies but transparent to visible and near infrared light. To consider a pair of plates made of indium tin oxide (ITO) at a separation $a=5$ μm, and the other pair of plates at the same separation made of Au, one runs into difficulties. If the actual low-frequency reflection properties should be substituted into the zero-frequency term, the impedance of the normal skin effect from Eq. (8) must be used. As a result, the thermal Casimir force at $a=5$ μm will be equal for both pairs of plates (and practically the same as for an ideal metal). This is in contradiction with the physical intuition as around the characteristic frequency $\omega_c=3\times10^{13}$ rad/s (computed at the separation 5 μm) ITO is a poor reflector. A better physical result would be obtained if one extends the characteristic impedance at ω_c (of the anomalous skin effect for Au and of the infrared optics for ITO) to zero Matsubara frequency. If this is done, the zero-frequency term for Au plates will be the same as for an ideal metal in accordance with Eq. (55). For ITO plates the zero-frequency term will contain the value of ω_p^{ITO} according to Eq. (58). Taking into account the large value of the penetration depth for ITO, the magnitude of the Casimir force between the ITO plates will be less than between the plates made of Au, as the intuition suggests. In fact, the question of whether the values of the Casimir force for the above two pairs of plates at separation 5 μm are equal or different, can be answered experimentally using the measurement scheme suggested recently in Ref. [49]. We expect that the experimental result will be in accordance with the suggestion of the physical intuition (note that this example with two pairs of plates was used with another aim in Ref. [31]).

VII. CONCLUSIONS AND DISCUSSION

In the foregoing we have presented the surface impedance approach to the theory of the Casimir effect at both zero and nonzero temperature. In the impedance approach the effective boundary condition is imposed taking the real properties of the metal into account. Previously this approach was considered as nothing more than a useful approximation to the more complete Lifshitz theory using the concept of a fluctuating electromagnetic field both outside and inside the boundary of the bodies. Our conclusion is that the standard concept of a fluctuating field inside a metal, described by the dielectric permittivity depending only on frequency, in the region where a real current may arise, cannot serve as an adequate model for the zero-point oscillations and thermal photons. It follows from the fact that the vacuum oscillations and the thermal photons in equilibrium under no circumstances can lead to a heating of the metal. If this fact is overlooked, contradictions with the thermodynamics arise when one substitutes into the Lifshitz formula for the Casimir free energy and force the Drude dielectric function, which takes into account the volume relaxation and, consequently, the Joule heating. This situation reflects the nontrivial character of quantum fluctuations in the nonhomogeneous case involving both vacuum and real metals, containing conduction electrons, in different spatial regions. In fact, in such cases the quantized electromagnetic field at nonzero temperature may not be represented in terms of (quasi)particles [31]. As a result, the concept of a fluctuating field becomes not so transparent as in nonlossy dielectric media.

In the light of this conclusion, the surface impedance approach takes on fundamental importance as (at present) the only self-consistent description of the Casimir effect between real metals. It does not need any prescription for the zero-frequency contribution to the Casimir energy and force. In all cases the correct expressions for the values of both reflection coefficients with two different polarizations at zero frequency are deduced from the general theoretical framework using the explicit form of the impedance function (see Sec. VI). Thus, a long discussion in the recent literature concerning the most adequate modification of the zero-frequency term of the Lifshitz formula [21,23–28,31–33] can be finalized.

The surface impedance approach solves the puzzle with the violation of the Nernst heat theorem and with negative values of entropy which appears when one substitutes the Drude dielectric function into the usual Lifshitz formula. In the impedance approach the entropy is in all cases nonnegative and takes zero value at zero temperature. Thus, the general formulas given by Eqs. (41)–(43), (46), (48), and (53) lay the theoretical basis for the calculation of the thermal Casimir effect with respect to the needs of future precision experiments. The computations performed in Secs. V and VI are in good agreement with the previous results obtained by the use of the optical tabulated data and the plasma model.

The obtained results allow us to remove the doubts that something is wrong with the Lifshitz formula [32]. In fact, the above formulas in the framework of the impedance approach coincide with the Lifshitz formula. The only difference is that for real metals in the frequency regions, where the electromagnetic oscillations initiate a real current or where the space dispersion is essential, one must express the reflection coefficients not in terms of the dielectric permittivity but in terms of the surface impedance. The usual Lifshitz formula, formulated in terms of the dielectric permittivity, preserves, however, major importance not only in applications to dielectrics, but also in the theory of the nonretarded van der Waals forces between metallic surfaces. As was mentioned in Sec. II, the surface impedance approach is applicable with the proviso that $\omega_c < 0.1\omega_p$, i.e., the separation distances between the test bodies must satisfy the condition $a > 5\lambda_p/(2\pi) \approx \lambda_p$. In essence, the frequency region $\omega > 0.1\omega_p$ is a subject of the optics of real metals near the plasma frequency [50]. At separations $a < \lambda_p$ between the test bodies, the temperature effects are negligible. This is a region of the ultraviolet transparency where metals can be described on the same basis as dielectrics. The most adequate approach to the theory of the van der Waals forces at so small separations is given by the hydrodynamical description of an inhomogeneous electron gas [51]. This is a more general approach if compared with the Lifshitz theory, because it does not start with a model description of a metal in terms of the bulk dielectric permittivity. In the local limit, however, when the spatial dispersion is absent, the hydrodynamical approach leads to the usual Lifshitz formula at zero temperature [51]. As a consequence, the usual Lifshitz formula is well adapted for the calculation of the van der Waals forces between real metals at separations $a < \lambda_p$ if $\varepsilon(i\xi)$ is obtained by the use of optical tabulated data for the complex refraction index (the extension of the available tabulated data into the region of small frequencies makes almost no effect on the computational results).

In conclusion, it may be said that the Lifshitz formula in combination with the impedance approach gives a solid foundation for the investigation of thermal effects onto the Casimir force. This approach does not lead to contradictions and can be used as the theoretical basis for the needs of future experiments.

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