

Generalized robustness of entanglement

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The robustness of entanglement results of Vidal and Tarrach [Phys. Rev. A **59**, 141 (1999)] considered the problem whereby an entangled state is mixed with a separable state so that the overall state becomes nonentangled. In general, it is known that there are also cases when entangled states are mixed with other entangled states and where the sum is separable. In this paper, we treat a more general case where entangled states can be mixed with any states so that the resulting mixture is unentangled. It is found that entangled pure states for this generalized case have the same robustness as the restricted case of Vidal and Tarrach.

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I. INTRODUCTION

The robustness of entanglement in Ref. [1] examines how much mixing can take place between an entangled state ρ and another state ρ_M , so that the convex combination of the two states is separable. Such work has significance to how robust entangled states are in the presence of interaction mechanisms that could disentangle the state. This work is also of interest in the characterization of the state space in terms of entangled and separable states, and the decomposition is an important result in quantum information theory.

In Ref. [1] the authors restricted ρ_M to be separable. This is a reasonable restriction as long as the states that are being mixed are separable. Another possibility is that a given entangled state would not only interact with separable states, but could also interact with other entangled states. It is known that the mixing of entangled states can also result in the convex combination being separable. Hence the entangled states thought to be robust for the cases where interaction is only with separable states might not be robust when allowed to interact with arbitrary states. This would require the presence of entangled states in the interaction medium that are stable at least within the interaction time. The degree and scale for which entangled states exist are yet to be known and are current areas of research in mesoscopic physics. What is known is that the processes of entanglement and decoherence are continually at work and it becomes more difficult to maintain entanglement with the scale of the entangled state. In general, it is known that microscopic entangled states are found, which are very stable, for example, electron sharing in atomic bonding and two-particle entangled photon states generated by parametric down conversion. Additionally, it is known that certain larger entangled systems can exist. Examples of multiparticle superpositions that are given in Ref. [2] include phonons in solids, superfluids, and superconducting quantum interference devices. It is also known that Bose-Einstein condensates are examples of large scale superpositions and the relation to entanglement is explored in Ref. [3]. Hence, it is known that entangled states exist on the microscopic level and under certain conditions on the mesoscopic level.

Therefore, the restriction that ρ_M be separable is lifted, so that ρ_M can be an arbitrary density matrix. As it is known [4] that the relative volume of the state space is dominated by

entangled states as the dimension of the composite Hilbert space grows, one might initially expect that entangled states would not be as robust when entangled states are allowed to interact with arbitrary states versus only separable states. Numerical simulations in Ref. [4] indicate that the probability of finding a nonentangled state decreases exponentially with the size of the Hilbert space of the composite system. Geometrically, the separable states become sandwiched between two hyperplanes [5]. However, as we will see, the robustness of entanglement of pure entangled states does not change when ρ_M can be an arbitrary state, compared to the case when ρ_M is separable. That is, we find the same expression for the robustness of entanglement for the two cases.

II. GENERALIZED ROBUSTNESS OF ENTANGLEMENT

Consider two systems of particles p_1, p_2 with composite states represented by density matrices ρ, ρ_M that operate on the Hilbert space $\mathcal{C}^n \otimes \mathcal{C}^n$, $N = n^2$. For a given entangled state ρ , Vidal and Tarrach [1] considered the problem of finding the largest $a \in [0, 1]$ for which there exists a $\rho_M \in \mathcal{S}$ and that

$$a\rho + (1-a)\rho_M \in \mathcal{S},$$

where \mathcal{S} denotes the set of separable states. The robustness of entanglement [1] was defined as $R_s(\rho) \triangleq 1/a - 1$, where a is largest. Define the optimal a for a given ρ as $\mathcal{O}_s(\rho)$. Note from the definition that for any $a > \mathcal{O}_s(\rho)$, the matrix $a\rho + (1-a)\rho_M$ is necessarily entangled for all $\rho_M \in \mathcal{S}$. For the case where ρ is a pure state (i.e., rank 1), $\rho = \psi\psi'$ and with ψ having a Schmidt decomposition $\psi = \sum_i \tilde{a}_i |i\rangle \otimes |i\rangle$, $\tilde{a}_i \geq 0$, it was shown in Ref. [1] that

$$R_s(\rho) = \left(\sum_i \tilde{a}_i \right)^2 - 1. \quad (1)$$

For the case of generalized robustness of entanglement, we define $\mathcal{O}_g(\rho)$ as the largest value of a for which there exists $\rho_M \in \text{DM}(N)$ with

$$a\rho + (1-a)\rho_M \in \mathcal{S}, \quad (2)$$

where $\text{DM}(N)$ is the set of N by N density matrices. Note that since $\mathcal{S} \subset \text{DM}(N)$, then

$$\mathcal{O}_g(\rho) \geq \mathcal{O}_s(\rho). \quad (3)$$

The generalized robustness of entanglement is defined as $R_g(\rho) \triangleq 1/\mathcal{O}_g(\rho) - 1$.

Several of the results in Ref. [1] can be extended to the generalized robustness of entanglement case. The proofs of these extensions are straightforward and are shown in the Appendix. We state these results here.

Theorem 1. $R_g(\rho)$ is convex, i.e., $R_g(t\rho_1 + (1-t)\rho_2) \leq tR_g(\rho_1) + (1-t)R_g(\rho_2)$.

Theorem 2. $R_g(\rho) = R_g(U_L \rho U_L)$, where U_L is a local unitary transformation of the form $U_L = U_1 \otimes U_2$.

Corollary. Let $\rho = \psi\psi'$, ρ act on $\mathbb{C}^n \otimes \mathbb{C}^n$ and $\psi = \sum_{i=1}^n \tilde{a}_i \gamma_i^A \otimes \gamma_i^B$ is a Schmidt decomposition of ψ , where γ_i^A is an orthonormal basis for subsystem A and γ_i^B is an orthonormal basis for subsystem B . Then if $\tilde{\rho} = \tilde{\psi}\tilde{\psi}'$, $\tilde{\psi} = \sum_{i=1}^n \tilde{a}_i |i\rangle \otimes |i\rangle$, where $\{|1\rangle, |2\rangle, \dots, |n\rangle\}$ is the natural basis of \mathbb{C}^n , then the robustness is the same, i.e., $R_g(\tilde{\rho}) = R_g(\rho)$.

Proof. Define U_1 by the map $\psi_i^A \rightarrow |i\rangle$ and similarly define U_2 for subsystem B . Since change of orthogonal basis mappings are unitary, one can apply Theorem 2 and the result follows. ■

Lemma 1. Let ρ be a pure state acting on $\mathbb{C}^n \otimes \mathbb{C}^n$. The eigenvalues of ρ^{pt} consist of up to $n(n-1)/2$ negative eigenvalues, where ρ^{pt} denotes the partial transpose [6] of ρ . If $\rho = \psi\psi'$, $\psi = \sum_{i=1}^n \tilde{a}_i |i\rangle \otimes |i\rangle$, then these eigenvalues are given by $\{-\tilde{a}_r \tilde{a}_s\}$, $r < s$. The corresponding eigenvectors are $\tilde{e}_{f(r,s)} = (1/\sqrt{2})(|r\rangle|s\rangle - |s\rangle|r\rangle)$, $r < s$, where $f(i,j) = (j-i) + n(i-1) - i(i-1)/2$, $i < j$.

Proof. The proof is given in Ref. [1], Eqs. (B17) and (B18).

A main result is that for pure states, the generalized robustness of entanglement is the same as the Vidal and Tarrach robustness of entanglement.

Theorem 3. Let ρ be a pure state. Then $\mathcal{O}_g(\rho) = \mathcal{O}_s(\rho) = 1/[1 + R_s(\rho)]$.

Proof. Consider the eigenvectors \tilde{e}_i from Lemma 1. If $\mathcal{O}_g(\rho) = t$, then there exists a $\rho_M \in \text{DM}(N)$, with

$$\tilde{e}_i' [t\rho^{pt} + (1-t)\rho_M^{pt}] \tilde{e}_i \geq 0, \quad i = 1, \dots, n(n-1)/2, \quad (4)$$

where x' denotes the conjugate transpose of x . It will be shown that $t \leq \mathcal{O}_s(\rho)$, otherwise at least one term in Eq. (4) will be negative for every ρ_M . To this end, let $\tilde{e}_{f(i,j)}' [t\rho^{pt} + (1-t)\rho_M^{pt}] \tilde{e}_{f(i,j)} \geq 0$ for $j > i$. Then

$$t \leq [1 + \tilde{a}_i \tilde{a}_j' / (\tilde{e}_{f(i,j)}' \rho_M^{pt} \tilde{e}_{f(i,j)})]^{-1}, \quad j > i. \quad (5)$$

Let $h_{i,j} = \tilde{e}_{f(i,j)}' \rho_M^{pt} \tilde{e}_{f(i,j)} / \tilde{a}_i \tilde{a}_j$. Since Eq. (5) is true for all $j > i$,

$$t \leq \min_{j>i} \left(1 + \frac{1}{h_{i,j}} \right)^{-1}. \quad (6)$$

For a given ρ_M , Eq. (6) must be satisfied if $t\rho + (1-t)\rho_M$ is separable. The largest t for which there exists a $\rho_M \in \text{DM}(N)$ and where $t\rho + (1-t)\rho_M$ is separable is upper bounded by the maximum of the right-hand side (rhs) of Eq. (6). That is,

$$\mathcal{O}_g(\rho) \leq \max_{\rho_M} \min_{j>i} \left(1 + \frac{1}{h_{i,j}} \right)^{-1}$$

or

$$\mathcal{O}_g(\rho) \leq \left(1 + \frac{1}{\max_{\rho_M} \min_{j>i} h_{i,j}} \right)^{-1}. \quad (7)$$

We will now consider the max-min problem

$$\mathcal{T} \triangleq \max_{\rho_M} \min_{j>i} h_{i,j}. \quad (8)$$

Let $\rho_M = \sum_i \lambda_i e_i e_i'$ be a spectral decomposition of ρ_M , $\|e_i\| = 1$, $e_i' e_j = 0$, $i \neq j$, and where $\|\cdot\|$ denotes the L2 norm. Denote the function $c(i,j) = m$, where m is the indice where the vector $|i\rangle \otimes |j\rangle$ is equal to one, $|i\rangle, |j\rangle \in \mathbb{C}^n$. For example, if $n=2$, $|1\rangle \otimes |2\rangle = (0100)'$, hence $c(1,2) = 2$. Clearly, then $c(i,j) = n(i-1) + j$.

Now rewrite the eigenvectors of ρ_M as $e_i = \sum_{r,s} e_{i,c(r,s)} |r\rangle \otimes |s\rangle$. Now

$$e_i e_i' = \left(\sum_{r,s} e_{i,c(r,s)} |r\rangle \otimes |s\rangle \right) \left(\sum_{t,u} e_{i,c(t,u)} \langle t| \otimes \langle u| \right),$$

$$\rho_M = \sum_i \lambda_i \sum_{r,s} \sum_{t,u} e_{i,c(r,s)} e_{i,c(t,u)}^* |r\rangle \langle t| \otimes |s\rangle \langle u|,$$

where $*$ denotes conjugate. The second partial transpose [7] of ρ_M is

$$\rho_M^{pt} = \sum_i \lambda_i \sum_{r,s} \sum_{t,u} e_{i,c(r,s)} e_{i,c(t,u)}^* |r\rangle \langle t| \otimes |u\rangle \langle s|. \quad (9)$$

From Lemma 1, we have $\tilde{e}_{f(j,k)}' = 1/\sqrt{2}(\langle j| \otimes \langle k| - \langle j| \otimes \langle k|)$, so that

$$\begin{aligned} \tilde{e}_{f(j,k)}' \rho_M^{pt} \tilde{e}_{f(j,k)} &= \frac{1}{2} \sum_i \lambda_i \sum_{r,s} \sum_{t,u} e_{i,c(r,s)} e_{i,c(t,u)}^* (\langle j|r\rangle \langle t|j\rangle \otimes \langle k|u\rangle \langle s|k\rangle - \langle j|r\rangle \langle t|k\rangle \otimes \langle k|u\rangle \langle s|j\rangle - \langle k|r\rangle \langle t|j\rangle \otimes \langle j|u\rangle \langle s|k\rangle \\ &\quad + \langle k|r\rangle \langle t|k\rangle \otimes \langle j|u\rangle \langle s|j\rangle) \\ &= \frac{1}{2} \sum_i \lambda_i [e_{i,c(j,k)} e_{i,c(j,k)}^* - e_{i,c(j,j)} e_{i,c(k,k)}^* - e_{i,c(k,k)} e_{i,c(j,j)}^* + e_{i,c(k,j)} e_{i,c(k,j)}^*] \\ &= \frac{1}{2} \sum_i \lambda_i [|e_{i,c(j,k)}|^2 + |e_{i,c(k,j)}|^2 - 2 \text{Re}(e_{i,c(j,j)} e_{i,c(k,k)}^*)], \end{aligned}$$

where $\text{Re}(x)$ denotes the real part of x . Define $g_{f(j,k)}^{(i)} \triangleq |e_{i,c(j,k)}|^2 + |e_{i,c(k,j)}|^2 - 2 \text{Re}(e_{i,c(j,j)} e_{i,c(k,k)}^*)$, and the above becomes $\tilde{\rho}'_{f(j,k)} \rho_M^{\rho'} \tilde{\rho}_{f(j,k)} = \frac{1}{2} \sum_i \lambda_i g_{f(j,k)}^{(i)}$. Consider the matrix

$$A = \begin{pmatrix} \frac{\lambda_1 g_1^{(1)}}{2\tilde{a}_1 \tilde{a}_2} & \frac{\lambda_2 g_1^{(2)}}{2\tilde{a}_1 \tilde{a}_2} & \dots & \frac{\lambda_N g_1^{(1)}}{2\tilde{a}_1 \tilde{a}_2} \\ \frac{\lambda_1 g_2^{(1)}}{2\tilde{a}_1 \tilde{a}_3} & \frac{\lambda_2 g_2^{(2)}}{2\tilde{a}_1 \tilde{a}_3} & \dots & \frac{\lambda_N g_2^{(N)}}{2\tilde{a}_1 \tilde{a}_3} \\ \vdots & \ddots & & \vdots \\ \frac{\lambda_1 g_{f(n-1,n)}^{(1)}}{2\tilde{a}_{n-1} \tilde{a}_n} & \frac{\lambda_2 g_{f(n-1,n)}^{(2)}}{2\tilde{a}_{n-1} \tilde{a}_n} & \dots & \frac{\lambda_N g_{f(n-1,n)}^{(N)}}{2\tilde{a}_{n-1} \tilde{a}_n} \end{pmatrix}. \quad (10)$$

Then Eq. (8) is identically

$$\mathcal{T} = \max_{\rho_M} \min_i \sum_j A_{ij}. \quad (11)$$

Before the final step in the proof, consider

$$\begin{aligned} \sum_{j=1}^{n(n-1)/2} g_j^{(i)} &= \sum_{j < k} g_{f(j,k)}^{(i)} = \sum_{j < k} |e_{i,c(j,k)}|^2 + |e_{i,c(k,j)}|^2 \\ &\quad - e_{i,c(j,j)} e_{i,c(k,k)}^* - e_{i,c(k,k)} e_{i,c(j,j)}^* \\ &= \sum_{j \neq k} |e_{i,c(j,k)}|^2 - e_{i,c(j,j)} e_{i,c(k,k)}^* \\ &= \left(\sum_{j \neq k} |e_{i,c(j,k)}|^2 + \sum_j |e_{i,c(j,j)}|^2 \right) \\ &\quad - \left(\sum_j |e_{i,c(j,j)}|^2 + \sum_{j \neq k} e_{i,c(j,j)} e_{i,c(k,k)}^* \right) \\ &= 1 - \left(\sum_j |e_{i,c(j,j)}|^2 + \sum_{j \neq k} e_{i,c(j,j)} e_{i,c(k,k)}^* \right) \\ &= 1 - \left| \sum_j e_{i,c(j,j)} \right|^2 \leq 1. \end{aligned} \quad (12)$$

Now, Eq. (11) can be written as

$$\mathcal{T} = \max_{\rho_M} \min_{j < k} \sum_i \frac{\lambda_i g_{f(j,k)}^{(i)}}{2\tilde{a}_j \tilde{a}_k}. \quad (13)$$

Let $\alpha_{j,k} = \sum_i \lambda_i g_{f(j,k)}^{(i)}$. Then Eq. (13) becomes $\mathcal{T} = \max_{\rho_M} \min_{j < k} (\alpha_{j,k} / 2\tilde{a}_j \tilde{a}_k)$. Summing over $\alpha_{j,k}$ with $j < k$ we have

$$\begin{aligned} \sum_{j < k} \alpha_{j,k} &= \sum_{j < k} \sum_i \lambda_i g_{f(j,k)}^{(i)} \\ &= \sum_i \lambda_i \sum_{j < k} g_{f(j,k)}^{(i)} \leq \sum_i \lambda_i \quad [\text{from Eq. (12)}] \\ &= 1. \end{aligned}$$

Hence \mathcal{T} in Eq. (13) can be upper bounded by

$$\mathcal{T} \leq \max_{\rho_M} \min_i \delta_i \beta_i, \quad \text{where } \beta_{f(j,k)} = \frac{1}{2\tilde{a}_j \tilde{a}_k}, \quad j < k$$

$$\delta_{f(j,k)} = \alpha_{j,k}, \quad j < k, \quad \sum_i \delta_i \leq 1$$

or

$$\mathcal{T} \leq \max_{\delta_i \leq 1} \min_i \delta_i \beta_i, \quad \text{where } \beta_{f(j,k)} = \frac{1}{2\tilde{a}_j \tilde{a}_k}, \quad j < k$$

and rewritten as

$$\mathcal{T} \leq \max y, \quad y \leq \delta_i \beta_i, \quad \sum_i \delta_i \leq 1. \quad (14)$$

Consider the candidate solution to Eq. (14) of $\delta_i = \delta_i^{(1)} = K / \beta_i$, where K is a constant. Since $\sum_i \delta_i^{(2)} \leq 1$, it follows that $K \leq 1 / (\sum_{j < k} 2\tilde{a}_j \tilde{a}_k)$. Note that the denominator is equal to $(\sum_i \tilde{a}_i)^2 - 1$, which is equal to $R_s(\rho)$ from Eq. (1). Hence

$$K \leq 1 / R_s(\rho). \quad (15)$$

For this choice of $\delta_i^{(1)}$, the rhs of Eq. (14) is maximized when K is on the boundary in Eq. (15), i.e., $K = 1 / R_s(\rho)$ and for which $\sum_i \delta_i^{(1)} = 1$. Then $\delta_i^{(1)} = 1 / R_s(\rho) \beta_i$, and

$$\max \geq \max = 1 / R_s(\rho),$$

$$y \leq \delta_i \beta_i, \quad y \leq \delta_i^{(1)} \beta_i,$$

$$\sum_i \delta_i \leq 1, \quad \sum_i \delta_i^{(1)} = 1.$$

We will now show that no other larger solutions to Eq. (14) exist. Suppose that such a solution y_2 exists and let $\delta_i = \delta_i^{(2)}$ be the associated parameters in Eq. (14). Since $y_2 > 1 / R_s(\rho)$, there must exist a $\delta_e > 0$ whereby $y_2 = 1 / R_s(\rho) + \delta_e$. From Eq. (14) it follows that $y_2 \leq \delta_i^{(2)} \beta_i \quad \forall i$, $1 / R_s(\rho) + \delta_e \leq \delta_i^{(2)} \beta_i \quad \forall i$, $\delta_i^{(1)} \beta_i + \delta_e \leq \delta_i^{(2)} \beta_i \quad \forall i$, $0 < \delta_e \leq (\delta_i^{(2)} - \delta_i^{(1)}) \beta_i \quad \forall i$, from which it follows that $\delta_i^{(2)} > \delta_i^{(1)} \quad \forall i$. Summing both sides $\sum_i \delta_i^{(2)} > \sum_i \delta_i^{(1)}$, which implies $\sum_i \delta_i^{(2)} > 1$. This contradicts the assumption in Eq. (14) that $\sum_i \delta_i^{(2)} > 1$. Hence $\mathcal{T} \leq 1 / R_s(\rho)$, and so we have upper bounded the rhs of Eq. (7) to arrive at $\mathcal{O}_g(\rho) \leq 1 / (1 + R_s(\rho))$. Since from Eq. (3) we know that $\mathcal{O}_g(\rho) \geq \mathcal{O}_s(\rho)$, the result follows. ■

An interesting question is whether or not an optimal ρ_M in Eq. (2) (optimal in the sense of maximizing \mathcal{O}_g) is necessarily separable. We already know that such separable states ρ_M can be constructed via the construction given in Ref. [1]. It is shown by counterexample that one can also construct ρ_M that are entangled. To construct the counterexample, consider an entangled pure state ρ with $a = \mathcal{O}_g(\rho)$. Assume that ρ is in the simplified form $\rho = \psi \psi'$, with $\psi = \sum_i \tilde{a}_i |i\rangle \otimes |i\rangle$, where $\tilde{a}_i \geq 0$ are the Schmidt coefficients of ψ and $|i\rangle$ is in the natu-

ral basis. Let $G = -a\rho/(1-a)$ and note that all the elements in G satisfy $G_{i,j} \leq 0$, where $G_{i,j}$ represents the i th column and j th row of G . Let $G^{(2)}$ be the matrix G except with the diagonal elements removed. Now $G^{(2)}$ is generally not positive definite, so we will replace the diagonal elements with values from the Gersgoren disks ([8], p. 344); that is, $G_{i,i}^{(2)} = -\sum_k G_{i,k}$. This will guarantee that $G^{(2)}$ is positive semidefinite. Now, $a\rho + (1-a)G^{(2)}$ is clearly a diagonal matrix. It can also be seen that the diagonal elements are positive and can be verified, the sum of the diagonal elements is 1 and $G^{(2)}$ is typically entangled. Hence $a\rho + (1-a)G^{(2)}$ is separable and there exist entangled matrices ρ_M that optimize the generalized robustness of entanglement. Additionally, note that if there are two optimal solutions for ρ_M in Eq. (2), call them ρ_{M_1} and ρ_{M_2} , then it is easily proven that any convex combination, i.e., $t\rho_{M_1} + (1-t)\rho_{M_2}$, $0 \leq t \leq 1$, also is an optimal solution for ρ_M in Eq. (2). Since we have seen that both entangled and nonentangled solutions exist, this further implies that there can be an infinite number of solutions for ρ_M .

III. CONCLUSIONS

Vidal and Tarrach considered robustness by considering how much one can mix a state with an arbitrary separable state such that the combination is separable. This result has been extended by allowing any state to mix with the state such that the combination is separable. It was found that the state is just as robust as before. This is somewhat a surprising result since the volume of state space becomes dominated by the entangled states in large dimension, as seen in Ref. [4]. The author initially expected that the robustness as measured by $R_g(\rho)$ would decrease as compared to $R_s(\rho)$. However, it was found that there is no degradation. This result shows that the entangled states that are generally robust (in terms of this definition) when only nonentangled states are present would be expected to robust in similar situations where there are other entangled states present that can mix with the state to cause disentanglement. Extensions to this work would be to determine the generalized robustness of entanglement for mixed states.

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Note added in proof. After this work was submitted, the author learned of two recent relevant works [9,10]. The main result on the generalized robustness of entanglement for pure states found in this paper was also found independently by A. Harrow and M. Nielsen and derived in a rather nice compact manner [9]. The authors also extend the notion of state robustness for the application of quantum logic gate robustness. In [10], F. Verstraete and H. Verschelde examined the problem of the maximal achievable fidelity optimized over all local operations and classical communication (LOCC) operations. They found an interesting equivalence between this latter problem and the generalized robustness of entanglement for the case of two qubits. They also developed a solution for the case of two qubits in a mixed state, which reduces to the solution in this paper when the two qubits are pure.

APPENDIX

Proofs of Theorem 1 and 2. These proofs follow from Ref. [1] with minor modification.

Theorem 1. $R_g(\rho)$ is convex, i.e., $R(p\rho_1 + (1-p)\rho_2) \leq pR(\rho_1) + (1-p)R(\rho_2)$.

Proof. Define $t \triangleq pR(\rho_1) + (1-p)R(\rho_2)$ and $\rho \triangleq p\rho_1 + (1-p)\rho_2$. From the definition of robustness, there exists $\rho_{M,1} \subset \text{DM}(N)$ and $\rho_{M,2} \subset \text{DM}(N)$, where $\text{DM}(N)$ represents the set of N by N density matrices, such that $\rho_1 = [1 + R_g(\rho_1)]\rho_{s,1} - R_g(\rho_1)\rho_{M,1}$, $\rho_2 = [1 + R_g(\rho_2)]\rho_{s,2} - R_g(\rho_2)\rho_{M,2}$, and for which $\rho_{s,1}, \rho_{s,2} \in \mathcal{S}$. Now, note that the matrix ρ_s as defined by $\rho_s \triangleq [1/(1+t)]\{p[1 + R_g(\rho_1)]\rho_{s,1} + (1-p)[1 + R_g(\rho_2)]\rho_{s,2}\}$ is separable. It can be shown that

$$\rho = (1+t)\rho_s - t\rho_{M,3},$$

where $\rho_{M,3} \triangleq 1/t[pR_g(\rho_1)\rho_{s,1} + (1-p)R_g(\rho_2)\rho_{s,2}]$. Hence $R_g(\rho) \leq t$. ■

Theorem 2. $R_g(\rho) = R_g(U_L\rho U_L^\dagger)$, where U_L is a local unitary transformation of the form $U_L = U_1 \otimes U_2$.

Proof. By definition, there exists a decomposition $\rho = [1 + R_g(\rho)]\rho_s - R_g(\rho)\rho_M$ with $\rho_s \in \mathcal{S}$. Then $U_L\rho U_L^\dagger = [1 + R_g(\rho)]U_L\rho_s U_L^\dagger - R_g(\rho)U_L\rho_M U_L^\dagger$. Since $U_L\rho U_L^\dagger \in \mathcal{S}$, then $R_g(U_L\rho U_L^\dagger) \leq R_g(\rho)$. On the other hand, if one considers $\rho_2 \triangleq U_L\rho U_L^\dagger$, there exists by definition a decomposition $\rho_2 = [1 + R_g(\rho_2)]\rho_{s,2} - R_g(\rho_2)\rho_{M,2}$. Multiplying both sides by U_L^\dagger and U_L gives $\rho = [1 + R_g(\rho_2)]U_L^\dagger\rho_{s,2}U_L - R_g(\rho_2)U_L^\dagger\rho_{M,2}U_L$, which implies $R_g(\rho) \leq R_g(\rho_2)$. ■

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