Algebraic approach to the Tavis-Cummings problem

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An algebraic method is introduced for an analytical solution of the eigenvalue problem of the Tavis-Cummings Hamiltonian, based on polynomially deformed su(2), i.e., $su_n(2)$ algebras. In this method the eigenvalue problem is solved in terms of a specific perturbation theory, developed here up to third order. Generalization to the $\mathcal{N}$-atom case of the Rabi frequency and dressed states is also provided. A remarkable enhancement of spontaneous emission of $\mathcal{N}$ atoms in a resonator is found to result from collective effects.

I. INTRODUCTION

We consider here the collective behavior of the system of $\mathcal{N}$ two-level atoms coupled to a single mode of the electromagnetic field in a resonator. The useful form for the atom-field interaction was proposed in the rotating wave approximation (RWA) by Tavis and Cummings [1]. In their model, $\mathcal{N}$ identical two-level atoms interact via dipole coupling with a single-mode quantized radiation field at resonance, so that the Hamiltonian is given by

$$H = H_0 + V, \quad H_0 = \omega a^\dagger a + \omega_0 \left(S_3 + \frac{\mathcal{N}}{2} \right),$$

$$V = g(a^\dagger S_- + a S_+). \tag{1}$$

Here ω is a frequency of the electromagnetic field and ω_0 is the level splitting of the two-level atoms. The operators $S_3, S_\pm$ are collective spin variables of $\mathcal{N}$ two-level atoms. These operators are defined as

$$S_3 = \sum_{j=1}^{\mathcal{N}} \sigma_3^j, \quad S_\pm = \sum_{j=1}^{\mathcal{N}} \sigma_\pm^j, \tag{2}$$

where σ's are Pauli matrices. They satisfy the su(2) algebra. $a, a^\dagger$ are the annihilation and creation operators of the field. Due to historical reasons, the Tavis-Cummings (TC) model is often called the Dicke model [2]. We concentrate here on the case of exact resonance, i.e., $\omega = \omega_0$. In this case, the system exhibits a most interesting collective behavior. For simplicity, time will be measured in units of the coupling constant g, i.e., we assume in the following that $g = 1$.

The one-atom version of the TC model, i.e., the Jaynes-Cummings model (JC) [3] has recently attracted much attention because of, e.g., the spectacular experiments reported in

Ref. [4]. In this work the properties of a dilute flux of atoms excited to the Rydberg states and interacting with a single-quantum mode of the electromagnetic field were analyzed in a high-Q cavity. The results reported in Ref. [4] demonstrate that the physical assumptions underlying the JC model can indeed be experimentally realized. This triggered a lot of theoretical interest towards the one-atom micromaser (micro-laser) (see Refs. [5–7] and references therein). The analysis of the JC model is greatly simplified by the fact that the eigenvalue problem of the Jaynes-Cummings Hamiltonian can be solved exactly. An algebraic approach to the solution of the generalized Jaynes-Cummings types of models can be found in Refs. [8,9]. The approach to the Tavis-Cummings-type models based on the algebraic bethe ansatz is formulated in Ref. [10].

An analytical solution of the $\mathcal{N}$-atom case, i.e., the TC model, is much more complicated than that of the JC model and is thus still far from completion. In this paper, we analyze this model and develop an analytical technique that is applicable to a variety of problems of TC type.

To begin with, we notice that Hamiltonian (1) belongs to a class of operators that can be expressed in the form:

$$H = f(A_0) + g(A_+ + A_-). \tag{3}$$

Here $f(x)$ is an analytic function of x with real coefficients, while the operators $A_\pm$ satisfy the commutation relations

$$[A_0, A_\pm] = \pm A_\pm. \tag{4}$$

We also assume the operator H to be self-adjoint. This means that, for any irreducible representation (irrep), the operators $A_0, A_\pm$ must satisfy the conditions

$$(A_0)^\dagger = A_0, \quad (A_-)^\dagger = A_+. \tag{5}$$

It is worth mentioning that Hamiltonians of type (1) are also used in description of the Raman and the Brillouin scattering, and frequency conversion, as well as parametric amplification that involves trilinear boson operators [11]:

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$$H = \omega a^\dagger a + \sum_{i=1}^{2} \omega_i b_i^\dagger b_i + g(b_1^\dagger b_2 a + b_1 b_2^\dagger a^\dagger). \quad (6)$$

The equivalence of Hamiltonians (1) and (6) can be readily seen by applying the Schwinger transformation

$$S_+ = b_1^\dagger b_2, \quad S_- = b_1 b_2^\dagger, \quad S_3 = \frac{1}{2}(b_1^\dagger b_1 - b_2^\dagger b_2). \quad (7)$$

The Hamiltonians of the type of Eq. (3) are usually analyzed by approximating them with an exactly solvable one. The solvable Hamiltonian is usually quadratic in the boson operators or linear in the generators of a classical Lie algebra. In this approach, the so-called parametric approximation or its variations [12–14] are often used. The basic assumption of this approximation is that one of the quantum modes or subsystems is prepared in a highly excited (often coherent) state, or in a state close to the vacuum. This approach, however, puts certain restrictions on the type of possible initial conditions as well as on the timespan over which the quantum dynamics can be followed. Another type of an approach to the solution of Hamiltonian (3) is based on perturbation theory for nonlinear algebras [15]. This approach requires, however, the existence of a small parameter. In the TC model, such a parameter was found in the case where the atoms are in completely symmetric states ($r = \mathcal{N}/2$) [16,17].

The basic idea of the present work is to combine these two approaches through algebraic methods as applied, e.g., in Refs. [18–21]. To this end we reformulate the Hamiltonian in terms of an algebra that better allows the diagonalization of the Hamiltonian. This idea was already used by Holstein and Primakoff [22]. They expressed the generators $S_3, S_\pm$ of the su(2) algebra in terms of boson operators $b, b^\dagger$,

$$S_3 = r - b^\dagger b, \quad S_+ = \sqrt{2r}\sqrt{1 - \frac{b^\dagger b}{2r}}\, b, \quad S_- = (S_+)^\dagger. \quad (8)$$

Here r is an index that characterizes the irrep of su(2). However, in Ref. [22] the square root in the transformation Eq. (8) was in the end replaced by unity, which amounts to applying the so-called "weak-field" approximation ($\langle b^\dagger b \rangle \ll 2r$). Obviously, this approximation corresponds to zeroth order in the expansion of the problem with respect to parameter $1/2r$. Transformation (8) has also been applied [23] with expansion up to second order.

We consider here the case when the operators $A_0, A_\pm$ in Eq. (3) are generators of a polynomial deformation $su_n(2)$ of the Lie algebra su(2) [21,24,25]. Numerous physical applications exist for polynomially deformed algebras [22,26,28–37]. A particularly interesting, in view of the present problem, application of deformed algebras was developed by Karassiov (see in Ref. [26] and references therein). The method to be introduced below is an extension of Karassiov's method.

We introduce here the notion of a polynomial algebra of excitations (PAE). In this algebra the coefficients of the structure polynomials are *c* numbers, rather than the Casimir operators as is typical of polynomial deformations. We derive an exact mapping between isomorphic representations of two arbitrary PAE. We also provide a classification of isomorphic representations of polynomial algebras of excitations. In our approach, classes of isomorphic representations are specified by the multiplicity of the maximal and minimal eigenvalues of the operator A_0 in the given representation. We formulate then an analytical approach that allows us to expand the Hamiltonian, when expressed in terms of PAE, as a perturbation series.

For completely symmetric states of atoms, our results agree with those reported in Refs. [16,17]. Our formalism provides, however, a solution of the problem for any value of r, which allows us to discuss new physical effects in the Dicke model.

The paper is organized as follows. In Sec. II, we discuss the irreducible representations of PAE. In Sec. III, we apply the general approach to the Tavis-Cummings model, and in Sec. IV, we construct the perturbation theory for the TC Hamiltonian and solve its eigenvalue problem up to third order. The generalized $\mathcal{N}$-atom quantum Rabi frequency is defined for arbitrary quantum states of the system. In Sec. V, we use the zero-order approximation for the TC Hamiltonian to calculate the intensity of spontaneous emission of atoms prepared in the state of thermal equilibrium with the resonator mode. We show that the correlation of the atoms due to interaction with the field gives rise to the enhancement of spontaneous emission as compared to the atoms in the absence of resonator. In conclusion, we discuss possible further applications of the methods developed here. Technical details of the algebraic manipulations are given in the Appendix.

II. IRREDUCIBLE REPRESENTATIONS OF THE POLYNOMIAL ALGEBRA OF COLLECTIVE EXCITATIONS

The coefficients of the structure polynomial of a polynomially deformed algebra are usually expressed through the Casimir operators of the algebra. In this section, we discuss representations of a special class (PAE) of polynomially deformed algebras when the coefficients of the structure polynomial are *c* numbers. We denote a PAE with a structure polynomial of order κ as $\mathfrak{U}_\kappa$. Formally, $\mathfrak{U}_\kappa$ is an associative algebra with unity, defined by three generators $A_\pm, A_0$. These generators satisfy two basic commutation relations, Eq. (4). As can be readily seen from these commutation relations, $[A_0, A_+ A_-] = 0$. We can thus assume that

$$A_+ A_- = p_\kappa(A_0) = c_0 \prod_{i=1}^{\kappa}(A_0 - q_i). \quad (9)$$

Here $p_\kappa(x)$ is a structure polynomial of order κ, whose coefficients are generally complex numbers. The terminology is chosen in analogy to the structure functions of quantum algebras (q-deformed algebras) [27], and the structure constants of the linear Lie algebras. The set of n real roots of the structure polynomial is denoted by $\{q_i\}_{i=1}^{\kappa}$. In physical applications, the operators $A_\pm$ of Eq. (4) often play the role of creation and annihilation operators of collective excitations. Therefore, hereafter the algebra $\mathfrak{U}_\kappa$ will be referred to as the PAE of order κ. Notice that this algebra is different from the

algebra $su_n(2)$. All PAE's are completely defined by $\kappa+1$ c numbers, the coefficient c_0, and the κ roots $\{q_i\}$ of the structure polynomial. In the case of $su_n(2)$, however, the structure polynomial has some coefficients in the form of the Casimir operators.

Below we consider two elementary, but important for the following, examples of PAE. First we assume, without loss of generality, that $c_0=\pm1$. Indeed, in the case $|c_0|\neq1$, it is always possible to renormalize the generators of $\mathfrak{U}_\kappa$,

$$A_\pm\rightarrow|c_0|^{-1/2}A_\pm, \qquad (10)$$

such that commutation relations (4) remain intact. Using these commutation relations, it can also be readily seen that

$$A_0A_\pm=A_\pm(A_0\pm1), \quad A_-A_+=p_\kappa(A_0+1). \qquad (11)$$

As indicated above, A_+ and A_- have the physical meaning of creation and annihilation operators of collective excitations (quasippapers), while A_0 is the operator for the number of excitations. The most simple and important example of a PAE of first order, $\mathfrak{U}_1$, is provided by the well-known Heisenberg-Weil Lie algebra, viz.,

$$b^\dagger\rightarrow A_+, \quad b\rightarrow A_-, \quad b^\dagger b\rightarrow A_0,$$

$$c_0=1, \quad q_1=0. \qquad (12)$$

Here $b,b^\dagger$ are the usual boson operators. For the sake of simplicity, in what follows we will denote the generators of $\mathfrak{U}_1$ by $b,b^\dagger$. The algebra $\mathfrak{U}_1$ allows us to construct the irrep of any other PAE of higher order $\kappa>1$ as a multiple tensor product of $\mathfrak{U}_1$.

An example of a PAE of second order, denoted here by $\mathbb{S}_r$, that is relevant to the algebra su(2), is given by the commutation relations

$$[S_3,S_\pm]=\pm S_\pm, \quad [S_+,S_-]=2S_3. \qquad (13)$$

It is plain that

$$S_+S_-=S^2-S_3^2+S_3. \qquad (14)$$

For every irrep the Casimir operator S^2 is equal to $r(r+1)I$, where I is the identity operator. The corresponding PAE of second order, $\mathbb{S}_r$, is constructed such that

$$S_+\rightarrow A_+, \quad S_-\rightarrow A_-, \quad S_3\rightarrow A_0;$$

$$c_0=-1, \quad q_1=-r, \quad q_2=r+1. \qquad (15)$$

Obviously, $\mathbb{S}_r$ has a matrix irrep isomorphic to the irrep of su(2) with the same r. Notice that *only* in this representation of $\mathbb{S}_r$ condition (5) is fulfilled. This gives us a motivation to denote the three generators of $\mathbb{S}_r$ as $\widetilde{S}_3,\widetilde{S}_\pm$.

Previously [38] we have proposed a general method to construct a realization of any PAE $\mathfrak{U}_\kappa$ through the algebra $\mathfrak{U}_1$. Choosing a root q_j of the structure polynomial we can construct this realization in the form

$$A_0=b_j^\dagger b_j+q_j, \quad A_+=\sqrt{c_0\prod_{i=1,i\neq j}^n(b_j^\dagger b_j+q_j-q_i)}\,b_j^\dagger,$$

$$A_-=b_j\sqrt{c_0\prod_{i=1,i\neq j}^n(b_j^\dagger b_j+q_j-q_i)}. \qquad (16)$$

Here $b_j,b_j^\dagger$ are the usual boson operators associated with the chosen root q_j of the structure polynomial $p_\kappa(x)$. The corresponding Fock vectors, i.e., the eigenvectors of $b_j^\dagger b_j$, are denoted as $|n\rangle_j$. In what follows, the chosen root q_j of the structure polynomial will be referred to as the *pivotal* root. Notice that the product in Eq. (16) always contains exactly κ multipliers regardless of the multiplicity of the root. Since $b_j^\dagger b_j$ has a well-defined discrete spectrum, the square-root function is defined in the form of spectral decomposition. We specify the branch of the square root by choosing $\sqrt{-1}=i$. It is easy to see that condition (9) and commutation relations (4) for generators of $\mathfrak{U}_\kappa$ are satisfied.

A useful automorphism $\hat{T}$ [38],

$$\hat{T}b^\dagger=ib, \quad \hat{T}b=ib^\dagger\Rightarrow\hat{T}b^\dagger b=-(b^\dagger b+1), \qquad (17)$$

of $\mathfrak{U}_1$ allows us to construct another realization of $\mathfrak{U}_\kappa$ through $\mathfrak{U}_1$,

$$A_0=q_j-1-b_j^\dagger b_j,$$

$$A_+=b_j\sqrt{(-c_0)\prod_{i=1,i\neq j}^\kappa(q_j-q_i-b_j^\dagger b_j)},$$

$$A_-=\sqrt{(-c_0)\prod_{i=1,i\neq j}^\kappa(q_j-q_i-b_j^\dagger b_j)}\,b_j^\dagger. \qquad (18)$$

It is worth mentioning that transformations similar to Eqs. (16) and (18) have been introduced earlier under the name of multiboson realizations of the Bose operators. These multiboson realizations satisfy the usual boson commutation relations $[A,A^\dagger]=1$ [28–30].

Applying the realizations (16) and (18) in any representation of $\mathfrak{U}_1$, we can construct a representation of $\mathfrak{U}_\kappa$. In the case of realization (16), an irreducible representation of $\mathfrak{U}_\kappa$ is constructed through the application of the operator A_+ to the vacuum vector $|0\rangle_j$. The finite-dimensional representation can be constructed in the case when the root q_{j+1} is separated from q_j by a natural number d. Then it can be readily seen that $A_+^d|0\rangle_j=0$. Indeed, provided that $A_-=A_+^\dagger$, it is not difficult to show that the norm of the vector $A_+^d|0\rangle_j$ vanishes, i.e., $_j\langle0|A_-^dA_+^d|0\rangle_j=0$. The general construction can be exemplified by $\mathbb{S}_r$. When r is an integer or half integer, $d=q_1-q_2=2r+1$ is an integer, and $\mathbb{S}_r$ has a finite-dimensional irrep which is isomorphic to the corresponding irrep of su(2).

In the case of realization (18), the corresponding irrep is constructed through the application of the operator A_- to the vacuum vector $|0\rangle_j$. Using an argumentation similar to that given above, it can be seen that the finite-dimensional repre-

sentation can now be constructed, provided that the root q_{j-1} is separated from q_j by a natural number.

The meaning of transformations (16) and (18) becomes now transparent. The realization (16) corresponds to the case when the operator A_+ is a creation operator, while realization (18) corresponds to the case when A_+ is an annihilation operator.

Since the spectrum of operator $b_j^\dagger b_j$ is a set of natural numbers and zero, the operator $A_0 - q_j$ in Eq. (16) has a non-negative spectrum. Therefore, the argument of the square-root function is a positive operator in the finite-dimensional subspace, where the structure polynomial $p_\kappa(A_0)$ has non-negative spectrum. In this case, the operators A_+ and A_- are Hermitian conjugated. In the subspace corresponding to the negative values of the spectrum of the structure polynomial, the argument of the square-root function in Eq. (16) has a negative spectrum. The operators will be anticonjugated, i.e., $(A_-)^\dagger = -A_+$, which is not plausible. Thus, in a physical problem [see Eq. (5)], we should only consider those irrep for which the spectrum of $p_\kappa(A_0)$ is non-negative. The same is true for Eq. (18).

The two relations, Eqs. (16) and (18), between any two algebras $\mathfrak{U}_\kappa$ and $\mathfrak{U}_1$, show that there is no principal difference as to how exactly the meanings of the creation and annihilation operators of collective excitations are prescribed to the pair $A_\pm$. The important point is that the pair exists. It is the physical problem in question which prescribes the meaning of operators $A_\pm$ and determines the location of the equidistant spectrum of A_0 on the real axis. Should one be interested in the eigenvalues of A_0, to the right of the pivotal root q_j, it is necessary to choose the transformation Eq. (16), while for the region to the left of q_j, it is necessary to use realization (18). As was explained above, we choose the region such that the structure polynomial is non-negative.

The general considerations given above can be illustrated by the $\mathbb{S}_r$ algebra. In the case of the realization of $\mathbb{S}_r$ connected to the pivotal root $q_1 = -r$, it is necessary to use Eq. (16). This leads to the conventional Holstein-Primakoff representation,

$$\tilde{S}_3 = b_1^\dagger b_1 - r, \quad \tilde{S}_+ = \sqrt{2r+1-b_1^\dagger b_1}\, b_1^\dagger, \quad \tilde{S}_- = (\tilde{S}_+)^\dagger. \tag{19}$$

In the case of $q_2 = r+1$, we use the realization Eq. (18),

$$\tilde{S}_3 = r - b_2^\dagger b_2, \quad \tilde{S}_+ = b_2\sqrt{2r+1-b_2^\dagger b_2}, \quad \tilde{S}_- = (\tilde{S}_+)^\dagger. \tag{20}$$

The spectra of $b_1^\dagger b_1$ and $b_2^\dagger b_2$ are limited from above by the value $2r$, while the subspace spanned by eigenvectors $|n\rangle_{1,2}(n=0,1,\ldots 2r)$ of operator A_0 is irreducible [cf. Eq. (8)].

Obviously, all the realizations of PAE constructed through Eqs. (16) and (18) are fully characterized by the dimension d of the invariant subspace and by the order k_- of the left and k_+ of the right roots defining the corresponding irreducible representation. Thus an irrep of PAE is characterized by a set of parameters $\{k_-,k_+,d\}$. Such irrep we will denote by

$R(k_-,k_+,d)$. For instance, $R(1,0,\infty)$ means the representation of $\mathfrak{U}_1$, while the irrep of $\mathbb{S}_r$ is $R(1,1,2r+1)$.

An isomorphism between irreps of $\mathfrak{U}_\kappa$ and $\mathfrak{U}'_{\kappa'}$ that belong to the same class $R(k_-,k_+,d)$, is given by

$$A_0 = A_0' + (q_j - q_{j'}'),$$

$$A_+ = \sqrt{\dfrac{c_0\prod_{i=1}^{\kappa}(A_0'+q_j-q_{j'}'-q_i)}{c_0'\prod_{i'=1}^{\kappa'}(A_0'-q_{i'}')}}\, A_+',$$

$$A_- = A_-'\sqrt{\dfrac{c_0\prod_{i=1}^{\kappa}(A_0'+q_j-q_{j'}'-q_i)}{c_0'\prod_{i'=1}^{\kappa'}(A_0'-q_{i'}')}}, \tag{21}$$

or by

$$A_0 = (q_{j'}'+q_j-1) - A_0',$$

$$A_+ = A_-'\sqrt{\dfrac{c_0\prod_{i=1}^{\kappa}(q_{j'}'+q_j-A_0'-q_i)}{c_0'\prod_{i'=1}^{\kappa'}(A_0'-q_{i'}')}},$$

$$A_- = \sqrt{\dfrac{c_0\prod_{i=1}^{\kappa}(q_{j'}'+q_j-A_0'-q_i)}{c_0'\prod_{i'=1}^{\kappa'}(A_0'-q_{i'}')}}\, A_+'. \tag{22}$$

In Eqs. (21) and (22), the operator argument of the square-root function should be taken after identical multipliers in the nominator and denominator are canceled. The pivotal roots q_j and $q_{j'}'$ define a vacuum vector of the irrep. Compare these expressions with Eqs. (16) and (18).

To recapitulate: in the general case two irreducible representations $R(k_-,k_+,d)$ and $R'(k_-',k_+',d')$ of $\mathfrak{U}_\kappa$ and $\mathfrak{U}'_{\kappa'}$ are isomorphic provided that $d=d'$, and

$$k_- = k_-', \quad k_+ = k_+' \text{ or } k_- = k_+', \quad k_+ = k_-'. \tag{23}$$

For symmetric irreps, i.e., when $k_+ = k_-$, these conditions coincide. The isomorphism is a consequence of the fact that, under conditions (23), one can choose $q_j(q_{j'}')$ in such a way that the functions under the square root in Eqs. (21) and (22) do not have zeros in the spectrum of operator A_0, and therefore we can consider the square root as a single-valued analytic function. This means that transformations are invertible, analytic, and therefore define an isomorphism of irreps.

The constructed transformations give us a tool to realize $\mathfrak{U}_\kappa$ in terms of a simpler PAE with the same type of irrep. This procedure will be applied below to the Tavis-Cummings Hamiltonian.

III. THE TAVIS-CUMMINGS HAMILTONIAN IN TERMS OF THIRD-ORDER PAE

The interaction part of the Hamiltonian Eq. (1) can be expressed in terms of third-order PAE. The generators $M_0, M_\pm$ of this algebra are realized as

$$M_- = aS_+, \quad M_+ = a^\dagger S_-, \quad M_0 = \frac{a^\dagger a - S_3}{2}. \quad (24)$$

It is plain that these generators satisfy commutation relations (4). The generators of the algebra $M_0, M_\pm$ commute with the operators

$$M = a^\dagger a + S_3 + r, \quad S^2 = S_3^2 + \frac{1}{2}(S_+ S_- + S_- S_+). \quad (25)$$

Hereafter, we use the same notation M both for the Casimir operator and its eigenvalue, if no confusion arises. We show below that the eigenvalues $M, r(r+1)$ of the operators of Eq. (25) parametrize the PAE in question. We thus denote this PAE as $\mathbb{M}_{M,r}$. The structure polynomial of $\mathbb{M}_{M,r}$ can be expressed in the form

$$p_3(M_0) = M_+ M_- = a^\dagger a(S^2 - S_3^2 - S_3)$$

$$= a^\dagger a(r - S_3)(r + S_3 + 1)$$

$$= -\left(M_0 + \frac{M-r}{2}\right)\left(M_0 - \frac{M-3r}{2}\right)$$

$$\times \left(M_0 - \frac{M+r+2}{2}\right). \quad (26)$$

The parameters of this structure polynomial are

$$c_0 = -1, \quad q_1 = -\frac{M-r}{2},$$

$$q_2 = \frac{M-r}{2} - r, \quad q_3 = \frac{M-r}{2} + r + 1, \quad (27)$$

and its behavior as a function of M_0 is given in Fig. 1.

We turn next to the description of finite-dimensional irrep of $\mathbb{M}_{M,r}$. In physical applications, the parameter r has the meaning of collective Dicke index. This index runs from $\varepsilon(\mathcal{N}) = 1 + (-1)^{\mathcal{N}}/4$ to $\mathcal{N}/2$ with unit steps, while M can be any natural number including zero. Thus, q_3 is the biggest positive root of first order. If $M < 2r$, then $q_1 > q_2$; if $M > 2r$ then $q_1 < q_2$; the case $M = 2r \Rightarrow q_1 = q_2$ corresponds to a root of second order. A typical plot of the structure polynomial is shown in Fig. 1 for these three cases is given. A finite-dimensional representation of $\mathbb{M}_{M,r}$ [where $(M_+)^\dagger = M_-$] corresponds to the positive spectrum of $p_3(M_0)$]. The spectrum is limited from the right by q_3 and from the
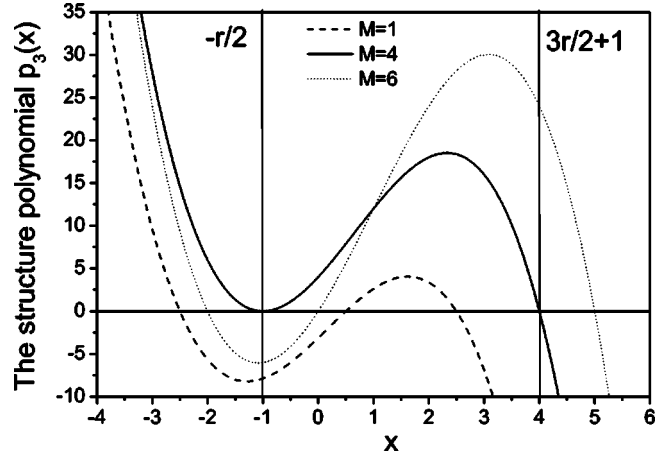

FIG. 1. The structure polynomial $p_3(x)$ of the $\mathbb{M}_{M,r}$ algebra for $r = 2$.

left by q_1 or q_2. Notice that the number M is conserved. Therefore, this number is determined by the initial state. The different values of M and r define different algebras $\mathbb{M}_{M,r}$, whose single physical finite-dimensional representation we will call a *zone*. The case $M < 2r$ corresponds to *nearby* zones. The two largest roots are q_1 and q_3 and the irrep has the type $R(1,1,M+1)$. Consequently, the well-known weak-field limit corresponds to nearby zones.

The case $M > 2r$ corresponds to *remote* zones. The two largest roots are q_2 and q_3, and the corresponding irrep is of the type $R(1,1,2r+1)$. Notice that the region $M \gg 2r$ is usually called the strong-field limit.

In the special case $2r = M$, called as *intermediate* zone, the algebra $\mathbb{M}_{M,r}$ possesses an irrep of the type $R(2,1,2r+1)$. It is the only irreducible representation that principally differs from all the others.

As indicated above, the simplest PAE with irrep of the type $R(1,1,d)$ is $\mathbb{S}_{\tilde{r}}^-$ (we use here $\tilde{r}$ to distinguish it from the (physical) collective index r). It would be convenient to solve the eigenvalue problem in terms of the simplest algebra $\mathbb{S}_{\tilde{r}}^-$. Notice that in any finite-dimensional irrep of PAE characterized by $R(1,1,d)$, the structure polynomial of the algebra can be approximated by a parabolic curve. This is shown in Fig. 2 (for $M \neq 2r$) for the structure polynomial of $\mathbb{M}_{M,r}$. The larger (smaller) M is in comparison with $2r$, the better is the approximation. However, for $M \approx 2r$ the approximation is not satisfactory [see Fig. 2(b)]. In the regions where the approximation is adequate, it is then not difficult to diagonalize the operator $V = S_+ + S_-$, defined in terms of generators of the conventional su(2) algebra. The latter has a parabolic structure polynomial. In Fig. 2 the two roots of $p_2(x)$ are chosen to be equal to the corresponding two roots of $p_3(x)$. The choice of c_0 in $p_2(x)$ will be explained below. Thus in the cases of nearby or remote zones, it is convenient to study the problem in terms of algebra $\mathbb{S}_{\tilde{r}}^-$. The approximation illustrated in Fig. 2 indicates that the TC problem can be solved via an appropriate perturbation theory.

To begin with, we consider the transformation of $\mathbb{M}_{M,r}$ to $\mathbb{S}_{\tilde{r}}^-$ for the case of remote zones. The dimension of a remote zone is $2r+1$, and the algebra $\mathbb{S}_{\tilde{r}}^-$ should be characterized by
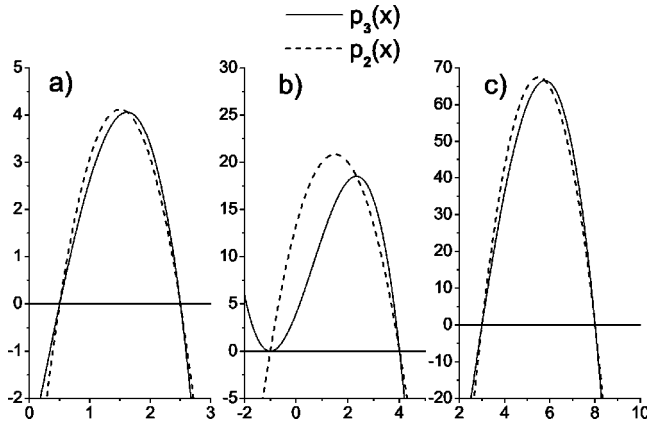
FIG. 2. Approximation of the structure polynomial $p_3(x)$ by the parabolic $p_2(x)$ for $r=2$. (a) $M=1$; (b) $M=4$; (c) $M=12$.

$\tilde{r}=r$. The finite-dimensional irrep of $\mathbb{S}_r^-$ is isomorphic to the corresponding irreducible representation of the atomic subsystem. For the pivotal root q_j, we choose the largest root that bounds the irrep of $\mathbb{M}_{M,r}$ from the right [the q_3 in Eq. (27)], while as the root $q'_{j'}$, we take the q_1 of $\mathbb{S}_r$ from Eq. (15). Applying mapping (22) we obtain

$$M_0 = \frac{M-r}{2} - \tilde{S}_3, \quad M_+ = \tilde{S}_- \sqrt{(M-r+1-\tilde{S}_3)},$$

$$M_- = \sqrt{(M-r+1-\tilde{S}_3)}\tilde{S}_+. \tag{28}$$

The spectrum $\{\tilde{m}\}$ of the operator $\tilde{S}_3$ belongs to the region $-r \leq \tilde{m} \leq r$, consequently the argument of the square-root function in Eq. (28) is positive in the remote zones ($M-r > r$). The relations, Eqs. (28), express the generators of algebra $\mathbb{M}_{M,r}$ as analytic function of the generators of the $\mathbb{S}_r$ algebra. They thus allow us to approximate the more complex algebra $\mathbb{M}_{M,r}$ of third order by a simpler algebra of second order.

Before we begin to study this approximation, we construct a realization of $\mathbb{S}_r$ in terms of boson and spin variables. From Eqs. (24), (25), and (28) it follows that in remote zones,

$$\tilde{S}_3 = S_3, \quad \tilde{S}_+ = \frac{1}{\sqrt{a^\dagger a + 1}} a S_+, \quad \tilde{S}_- = (\tilde{S}_+)^\dagger. \tag{29}$$

Notice that the subspaces that correspond to remote zones do not contain the vacuum state of the field. It is also worth mentioning that the matrix representation of the operator $1/\sqrt{a^\dagger a + 1}\,a$ is $\delta_{n,n+1}$ in any remote zone. This operator has been considered before as earlier in terms of phase operator [17–19].

We turn now to the nearby zones $M<2r$. For this region, the mapping of the algebra $\mathbb{M}_{M,r}$ to the algebra $\mathbb{S}_r^-$ is realized through procedure similar to that described above for remote zones. Notice that the dimension of nearby zones is $d=q_3 - q_1 = M+1$, and therefore $\tilde{r} = M/2$ [cf. Fig. 2(a)]. For the nearby zones there is no simple correspondence to a physical

representation like that of Eq. (29). The particular choice of the pivotal root $q'_{j'}$ is unimportant and we use the same choice as before. Applying Eq. (22) we obtain

$$M_0 = \frac{r}{2} - \tilde{S}_3, \quad M_+ = \tilde{S}_- \sqrt{\left(\frac{4r-M}{2} + 1 - \tilde{S}_3\right)},$$

$$M_+ = (M_-)^\dagger. \tag{30}$$

Since all the eigenvalues of the operator $\tilde{S}_3$ belong to the interval $-\tilde{r}$ to $\tilde{r}$, the argument of the square-root function does not have zero eigenvalues in the nearby zones. The realization of $\mathbb{S}_r^-$ through spin and boson variables is then given by

$$\tilde{S}_3 = \frac{M}{2} - a^\dagger a, \quad \tilde{S}_+ = \frac{1}{\sqrt{r+1-S_3}} S_+ a,$$

$$\tilde{S}_- = S_- \frac{1}{\sqrt{r+1-S_3}} a^\dagger. \tag{31}$$

Notice that the nearby zones do not contain the eigenvector $|r,r\rangle$ of S_3.

To clarify the structure of intermediate zone we choose the pivotal root as the third root of $p_3(x)$ of Eq. (27), and apply transformation (18). We thus obtain

$$M_0 = q_3 - b_3^\dagger b_3 - 1 = \frac{M+r}{2} - b_3^\dagger b_3,$$

$$M_+ = b_3 \sqrt{(M+1-b_3^\dagger b_3)(2r+1-b_3^\dagger b_3)},$$

$$M_- = \sqrt{(M+1-b_3^\dagger b_3)(2r+1-b_3^\dagger b_3)} b_3^\dagger. \tag{32}$$

Here $b_3, b_3^\dagger$ are generators of the algebra $\mathfrak{U}_1$.

In the region $M>2r$, the multiplier $(2r+1-b_3^\dagger b_3)$ in Eq. (32) vanishes first. This corresponds to remote zones. In the region $M<2r$, the multiplier that vanishes first is $(M+1-b_3^\dagger b_3)$. In the intermediate zone $M=2r$, we have

$$M_+ = b_3(2r+1-b_3^\dagger b_3), \quad M_- = (2r+1-b_3^\dagger b_3)b_3^\dagger. \tag{33}$$

In the intermediate zone we thus obtain a special realization of the TC Hamiltonian,

$$H_{\{M=2r\}} = \left(2r+\frac{1}{2}\right)(b_3+b_3^\dagger) - \frac{1}{2}[b_3^\dagger b_3(b_3+b_3^\dagger)$$

$$+ (b_3+b_3^\dagger)b_3^\dagger b_3]. \tag{34}$$

The domain of the quantum space for this Hamiltonian is specified by the condition $n_3 \leq 2r$.

IV. DIAGONALIZATION OF THE TAVIS-CUMMINGS HAMILTONIAN

Let us introduce an $\mathcal{N}$-atom generalization for arbitrary values r and M of the well-known quantum Rabi frequency such that

$$\Omega_R \equiv \begin{cases} 2\sqrt{M-r+\frac{1}{2}}, & M \geq 2r \\ 2\sqrt{\dfrac{4r-M+1}{2}}, & M < 2r. \end{cases} \tag{35}$$

For $r=\mathcal{N}/2$ our definition agrees with that used in Refs. [16,17]. Introducing a small parameter $\alpha \equiv (\frac{1}{2}\Omega_R)^{-2}$ we can rewrite the realizations of $M_\pm$ in nearby and remote zones [see Eqs. (28) and (30)] in the form

$$M_+ = \frac{\Omega_R}{2}\widetilde{S}_- \sqrt{1-\alpha\left(\widetilde{S}_3 - \frac{1}{2}\right)}, \quad M_- = M_+^\dagger. \tag{36}$$

The diagonalization problem for operator (1) can now be solved in each zone by means of perturbation theory with respect to the small parameter α. One can show that the eigenvalues of the argument of the square-root function in Eq. (36) are less than unity. Hence, we can expand the square with respect to α, and find thereby for the interaction part of the Hamiltonian

$$\begin{aligned} V = \frac{\Omega_R}{2}\Bigg\{ &(\widetilde{S}_+ + \widetilde{S}_-) - \frac{\alpha}{2}\left[\left(\widetilde{S}_3 - \frac{1}{2}\right)\widetilde{S}_+ + \widetilde{S}_-\left(\widetilde{S}_3 - \frac{1}{2}\right)\right] \\ &- \frac{1}{2!}\left(\frac{\alpha}{2}\right)^2\left[\left(\widetilde{S}_3 - \frac{1}{2}\right)^2\widetilde{S}_+ + \widetilde{S}_-\left(\widetilde{S}_3 - \frac{1}{2}\right)^2\right] \\ &- \left[\sum_{n=3}^{\infty} \frac{(2n-3)!!}{n!}\left(\frac{\alpha\left(\widetilde{S}_3 - \frac{1}{2}\right)}{2}\right)^n \widetilde{S}_+ + \widetilde{S}_- \right. \\ &\left. \times \sum_{n=3}^{\infty} \frac{(2n-3)!!}{n!}\left(\frac{\alpha\left(\widetilde{S}_3 - \frac{1}{2}\right)}{2}\right)^n\right]\Bigg\}. \end{aligned} \tag{37}$$

In the interaction representation the Hamiltonian coincides with V, we only need to diagonalize the latter. Up to third order in α we find that

$$V = \frac{\Omega_R}{2}(V^{(0)} + V^{(1)} + V^{(2)} + V^{(3)}),$$

where the $V^{(n)}$ are the terms of nth order in α, and are given in the Appendix. In the Appendix, we also show that unitary transformations $\widetilde{U}_k$, $k=0,1,2,3$, which bring the interaction operator into diagonal form:

$$\overline{V} \equiv \widetilde{U}V\widetilde{U}^{-1} = \Omega_R\widetilde{S}_3\left\{1 + \left(\frac{\alpha}{4}\right)^2[5\widetilde{S}_3^2 - 3\widetilde{r}(\widetilde{r}+1) + 1]\right\},$$

$$\widetilde{U} \equiv \widetilde{U}_3\widetilde{U}_2\widetilde{U}_1\widetilde{U}_0. \tag{38}$$

The spectrum of the operator V as given by Eq. (38) agrees with the results of Refs. [16,17] for the symmetric states of the atoms.

We compared third-order solution (38) with the exact numerical diagonalization of V and found that result (38) is very accurate, especially for increasing values of $|M-2r|$. The results of this comparison are shown in Fig. 3.

In their original paper [1], Tavis and Cummings also found an approximative analytical expression for the spectrum of the interaction operator V. We compare the results of Ref. [1] with our analytical and numerical solutions in Fig. 4. It is evident that the Tavis and Cummings solution is only accurate for very large and very small values of index j.

Figure 5 compares the energies calculated numerically and in accordance with analytical solution (38). In the intermediate region of M, the curves for nearby and remote zones overlap and coincide thus providing still satisfactory correspondence to the exact solution. However, evidently the expansion for the remote zone breaks down in the nearby zone and vice versa. This means that the classification of zones introduced in this paper is indeed adequate.

Finally, we consider the $\mathcal{N}$-atom *dressed states*. In other words, we introduce a representation in which the zero-order Hamiltonian is diagonal. This representation is given by transformation $\widetilde{U}_0$. For $\mathcal{N}=1$, all the higher-order terms in Eq. (37) (higher than zero order) vanish, while the eigenstates of the zero-order Hamiltonian coincide with the dressed states of the Jaynes-Cummings model. We can thus call the eigenstates of the zero-order Hamiltonian the *dressed states* of the $\mathcal{N}$-atom model. Notice also that in the remote zones the Rabi frequency (35) does not depend on the Dicke index r. If we consider only the zero-order terms of H, it is convenient to combine all the remote zones that have the same Rabi frequency into a remote *superzone*, whose dimension is $2^{\mathcal{N}}$. Introducing an operator for the total number of quanta in the atom-field system [cf. the definition of M in Eq. (25)],
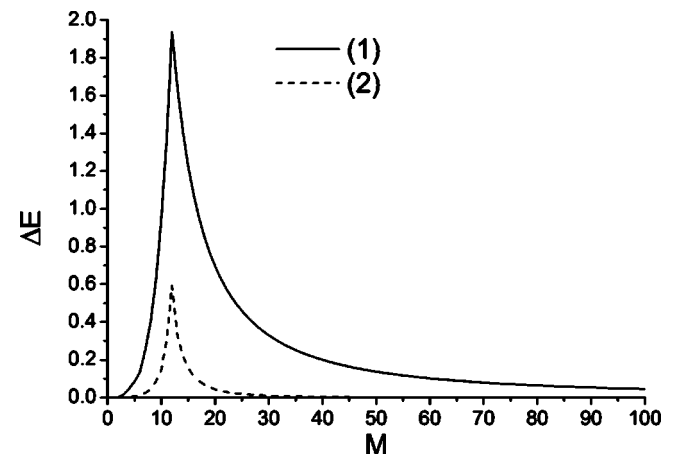

FIG. 3. Deviation of the eigenvalues of V from their numerical values in zero (solid line) and second (dashed line) order in α, for $r=6$.
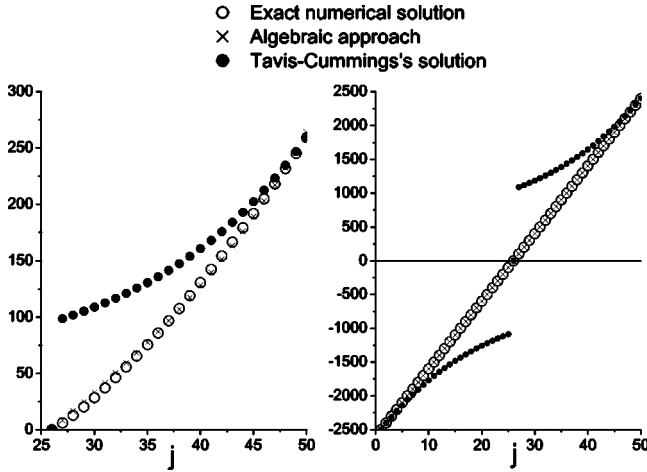
FIG. 4. Energy levels of V in the ascending order for $r=25$, as calculated numerically (circles), from Eq. (38) and (crosses), and as is given in Ref. [1] (squares). (a) $M=50$; (b) $M=2525$.

$$K = a^\dagger a + S_3 + \frac{\mathcal{N}}{2}, \tag{39}$$

we can define the remote superzone as follows. A remote superzone contains all eigenvectors of K that have the same eigenvalue, provided it is larger than $\mathcal{N}$. In Fig. 6, we show the definition of the zones in the Tavis-Cummings model for $\mathcal{N}=4$.

It can be readily seen that, in the Hilbert subspace corresponding to the remote superzone, the unitary transformation $\widetilde{U}_0$ can be factorized into a product of $\mathcal{N}$ single-particle transformations that are the dressing operators for the Jaynes-Cummings problem [cf. Eq. (29)],

$$\widetilde{U}_0 = \prod_{j=1}^{\mathcal{N}} \exp\left[\frac{\pi}{4}\left(\frac{1}{\sqrt{a^\dagger a+1}} a\,\sigma_+^j - \sigma_-^j\, a^\dagger \frac{1}{\sqrt{a^\dagger a+1}} \right) \right]. \tag{40}$$

From the discussion following Eq. (29), it follows that in each remote superzone, the field operators $1/\sqrt{a^\dagger a+1}\,a$ and
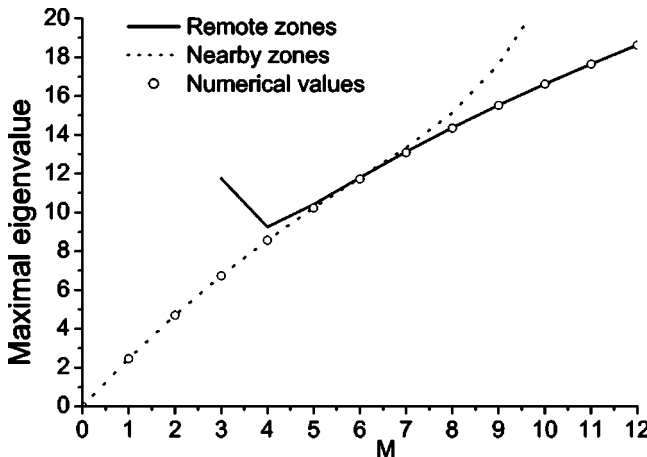

FIG. 5. The dependence on M of the maximal eigenvalue of V for $r=3$, as calculated numerically and from Eq. (38).
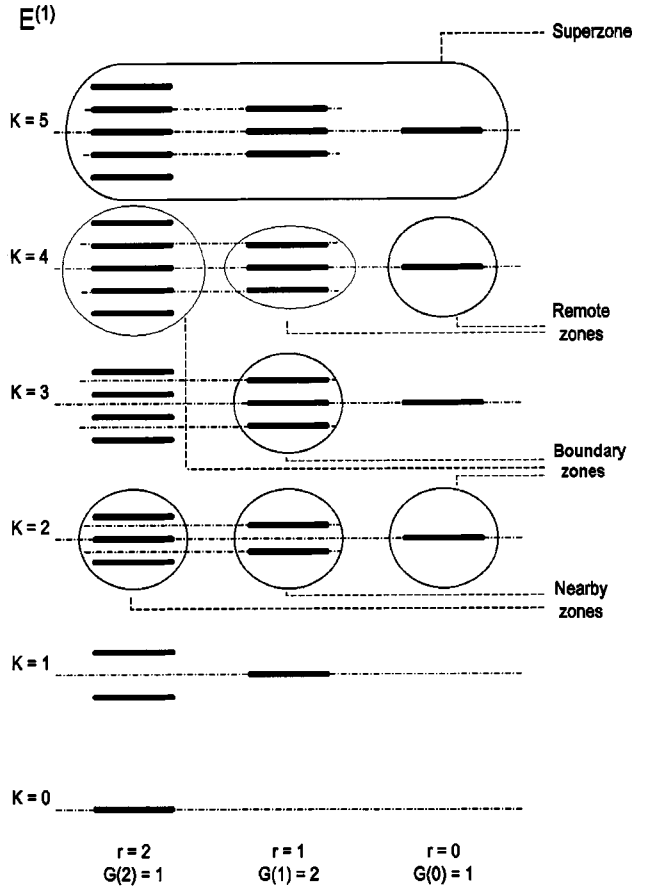

FIG. 6. Zones structure for $\mathcal{N}=4$. $E^{(1)}$ is the spectrum of the zero-order Hamiltonian. K is an eigenvalue of the corresponding operator (39), and $G(r)$ is a number of equivalent irreps of su(2) defined below.

$a^\dagger 1/\sqrt{a^\dagger a+1}$ commute. This means that the system behaves almost like a semiclassical one.

V. ENHANCEMENT OF SPONTANEOUS EMISSION IN THE RESONATOR DUE TO COLLECTIVE EFFECTS

In the preceding section we developed an algebraic approach to the Tavis-Cummings model. We introduced the operators $\widetilde{S}_\pm$ describing collective excitations in the atom-field system. In terms of these operators we constructed a perturbation series for the Tavis-Cummings Hamiltonian Eq. (1). The derived perturbation series gives us a tool to distinguish and classify cooperative (multiparticle) effects of different orders that are involved in calculations of different physical observables characterizing the atom-field system. In the preceding section we constructed a Hamiltonian Eq. (37) corresponding to the zero-order approximation for the Hamiltonian Eq. (1). This simplified operator depends on multippaper Rabi frequency Eq. (35), which depends on the number of atoms in the cavity. Therefore, the simplified Hamiltonian Eq. (37) allows to account for cooperative effects in the system.

In this section, we study a contribution of cooperative effects into the rate of spontaneous emission generated by

the atom-field system. The atom-field system is assumed to be prepared in the state of thermal equilibrium. This state is described by the canonical Gibbs ensemble with the thermostat temperature T. This means that under the "system" we imply $\mathcal{N}$ atoms strongly coupled to the resonator mode. Under the "thermostat" we imply the surrounding environment, for instance, cavity walls taken at the temperature T. In the state of thermal equilibrium, the exact atom-field density matrix should be defined using the Hamiltonian Eq. (1). However, for the exact density matrix the analytical analysis of the transition probability, if possible at all, would be highly technically involved.

We demonstrate that nontrivial physical results for the intensity of spontaneous emission of $\mathcal{N}$ two-level atoms placed inside the cavity can be already obtained for the zero-order approximation of the exact Hamiltonian Eq. (37). The thermal state is given by

$$\rho_{th} = \frac{1}{Z} \exp\left[-\frac{H_0 + \frac{\Omega_R}{2}(\tilde{S}_+ + \tilde{S}_-)}{kT} \right]. \tag{41}$$

Here Z is a normalization factor.

We show here that the intensity of spontaneous emission of the system comprised of $\mathcal{N}$ atoms *strongly* coupled to the resonator mode and prepared in the thermal state can be greatly enhanced at a certain temperature. This amplification results from high correlations in the atomic subsystem. Similar effect exists for Dicke's super-radiant state [2]. This state is prepared by a short laser pulse. Therefore, the setting of Dicke's theory is quite different from our considerations of the stationary state of thermal equilibrium.

When calculating the rate of spontaneous emission (or the intensity proportional to this quantity), we merely follow the ideas of Dicke's paper (see, e.g., Ref. [2]). According to this theory, the rate of spontaneous emission in the system is proportional to the average of the square of the atomic dipole, viz.,

$$I = I_0 \langle S_+ S_- \rangle = I_0 \text{Tr}\{\rho_{th} S_+ S_-\}. \tag{42}$$

It is convenient to calculate the average in the dressed states basis, where

$$\rho_{th} \rightarrow \frac{1}{Z} \tilde{U}_0 \rho_{th} \tilde{U}_0^{-1} = \exp\left[-\frac{H_0 + \Omega_R \tilde{S}_3}{kT} \right]. \tag{43}$$

Taking into account Eq. (14) along with the fact that, for an arbitrary zone,

$$S_3 = (\tilde{r} - r) + \tilde{S}_3, \tag{44}$$

it can be shown that in the dressed states basis

$$\tilde{U}_0(S_+ S_-)\tilde{U}_0^{-1} = \tilde{r}\left(2r - \frac{3}{2}\tilde{r} + \frac{1}{2}\right) - \left(r + \frac{1}{2} - \tilde{r}\right)(\tilde{S}_+ \tilde{S}_-)$$
$$- \frac{1}{4}(\tilde{S}_+^2 - 2\tilde{S}_3^2 + \tilde{S}_-^2). \tag{45}$$

Finally, the intensity of spontaneous emission is given by

$$I = \frac{I_0}{Z} \sum_{M=0}^{\infty} \sum_{r=\varepsilon}^{\mathcal{N}/2} G(r) \sum_{\tilde{m}=-\tilde{r}}^{\tilde{r}} \exp\frac{-\omega\left(M - r + \frac{\mathcal{N}}{2}\right) + \Omega_R \tilde{m}}{kT}$$
$$\times \left[\tilde{r}\left(2r - \frac{3}{2}\tilde{r} + \frac{1}{2}\right) + \frac{1}{2}\tilde{m}^2\right]. \tag{46}$$

Here $\varepsilon \equiv [1 - (-1)^{\mathcal{N}}]/4$, and $G(r) = [\mathcal{N}!(2r+1)]/[(\mathcal{N}/2 + r + 1)!(\mathcal{N}/2 - r)!]$ is the number of equivalent representations with the same r.

Let us consider the intensity per atom, i.e., $I_1 \equiv I/\mathcal{N}$. This intensity consists of two terms, i.e., the first is given by a single-particle contribution I_{single} and the second one proportional to the two-particle correlation function F_{Cor}, $I_1 = I_{single} + I_0(\mathcal{N} - 1)F_{Cor}$. They are found to be

$$I_{single} \equiv I_0 \frac{1}{\mathcal{N}} \left\langle \sum_i \sigma_+^i \sigma_-^i \right\rangle = I_0\left(\frac{1}{2} + \frac{1}{\mathcal{N}}\langle S_3 \rangle\right),$$

$$F_{Cor} \equiv \frac{1}{\mathcal{N}(\mathcal{N} - 1)} \left\langle \sum_{i \neq j} \sigma_+^i \sigma_-^j \right\rangle. \tag{47}$$

It is plain that in the absence of the cavity, the correlation function vanishes and the only contribution to I_1 is given by the first term,

$$I_{single} = I_{cl} \equiv I_0(1 + e^{\omega/kT})^{-1}. \tag{48}$$

The contribution of any remote superzone can easily be found due to the factorization property of the operator Eq. (40). If we denote the trace over states that belong to the same remote superzone by subscript K, we obtain

$$\langle S_+ S_- \rangle_K = \frac{\mathcal{N}}{2Z} e^{-K\omega/kT} \left[2\cosh\left(\frac{\Omega_R}{2kT}\right)\right]^{\mathcal{N}}$$
$$+ \frac{\mathcal{N}(\mathcal{N}-1)}{Z} e^{-K\omega/kT} \left[2\cosh\left(\frac{\Omega_R}{2kT}\right)\right]^{\mathcal{N}-2}$$
$$\times \left[\sinh\left(\frac{\Omega_R}{2kT}\right)\right]^2. \tag{49}$$

In Fig. 7, we compare the spontaneous emission I_1 in the presence of a cavity against the intensity I_{cl} of $\mathcal{N}$ atoms in the absence of cavity.

Notice that if the number of atoms is big enough, the intensity of radiation exhibits a high maximum. In a cavity at low T, the cluster of $\mathcal{N}$ two-level atoms emits much more intensively than it does in the free space. It should be possible to drive the system to thermal equilibrium at the temperature where the spontaneous emission exhibits maximum. The marked amplification of spontaneous emission should be observed in cavity experiments.

Concluding this section, we recapitulate our main results. We consider spontaneous emission of the system comprised $\mathcal{N}$ two-level atoms strongly coupled to the cavity mode and prepared in the state of thermal equilibrium. In the absence
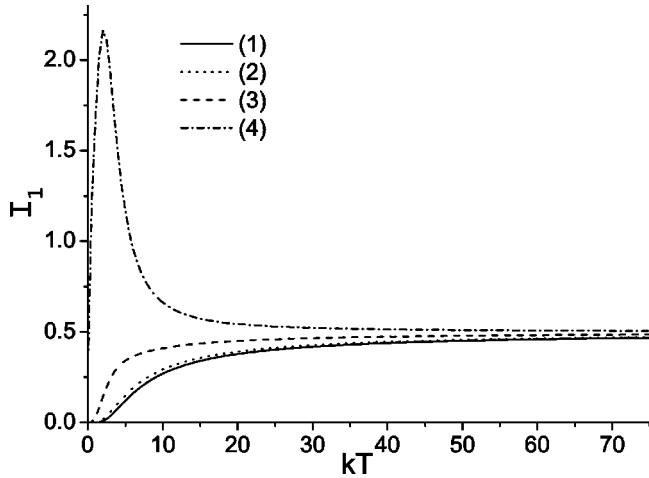
FIG. 7. The intensity of spontaneous emission per atom (in units of I_0) versus cavity temperature. The curve (1) is the classical result given by Eq. (48). The curves (2,3,4) correspond to $\mathcal{N}=10$, $\mathcal{N}=50$, $\mathcal{N}=100$, respectively, and $\omega/g=10$.

of the cavity, the atoms in the thermal equilibrium would be uncorrelated. In this case, the spontaneous emission would be described by conventional formula (48). For high-Q resonators, the strong coupling to the resonator mode should necessarily be taken into account. We demonstrate that in this case the intensity of spontaneous emission can be greatly enhanced. This phenomenon can be explained by additional correlation between atoms established by the cavity mode. To analyze the effect analytically we have replaced the exact Tavis-Cummings Hamiltonian Eq. (1) by its zero-order approximation derived in the previous sections. This allowed us to represent the intensity of spontaneous emission in simple analytical form (46). It is appropriate to emphasize once again that the zero-order approximation of the Hamiltonian contains strong coupling and, thus, describes cooperative effects in the atomic subsystem. This is the consequence of the fact that the operators $\widetilde{S}_\pm$ describe collective excitations in the atom-field system.

VI. CONCLUSION

In this work we solved the Tavis-Cummings problem by applying the technique of polynomially deformed algebras. We constructed the transformations that map one polynomial algebra of operators onto another. This allowed us to reformulate the problem in terms of a simpler algebra of second order, $\mathbb{S}_r$, and develop a specific perturbation theory. Our results have a significant advantage over the so-called linearization approximation, i.e., the case when the Hamiltonian is linearized in terms of the algebra $\mathfrak{U}_1$. In this latter approach, a structure polynomial of higher order is approximated by a polynomial of first order. This method allows only to calculate the lowest or highest eigenvalues, and the corresponding eigenstates. It does not take into account the finite- dimensionality of the representation. The parabolic approximation developed in this work provides in this respect a significant advantage because it allows to construct a finite-dimensional representation for the problem. We were able to find analyti-

cal expressions for all the eigenvalues of the Hamiltonian up to third order in the small parameter α. For the nearby zones we showed explicitly how the collective quantum Rabi frequency depends on the Dicke index r. Since this index characterizes the symmetry of atomic states, the result has significant physical implications. The dependence on atomic symmetry is revealed already in zeroth order in the perturbation expansion. Employing our methods, we found an interesting new effect, amplification of spontaneous emission of thermal $\mathcal{N}$-atom states due to collective effects. We expect that this phenomenon can be observed in cavity experiments. It is worth mentioning that the applicability of the method developed in this paper can be extended to many other problems, including the Bose-Einstein condensation, multiphoton interactions in the micromaser, and multimode interaction of an electromagnetic field with matter. Our methods allow, in particular, to address the problem of collective and dressed states in the mentioned physical systems.

APPENDIX: SIMILARITY TRANSFORMATIONS

We look for similarity transformations $\widetilde{U}_k$ that diagonalize the Hamiltonian in different orders of α,

$$V_k^{(n)} \equiv \widetilde{U}_k V_{k-1}^{(n)} \widetilde{U}_k^{-1}, \tag{A1}$$

where $k=0,1,2,3$. In Eq. (A1) only the terms of order n in the small parameter α are present. For $k=0$, the term $V_{k-1}^{(n)}$ should be replaced by the corresponding term in Eq. (37). Regrouping the terms we obtain

$$V^{(0)} = 2\widetilde{S}_x \left[1 - \frac{1}{2}\left(\frac{\alpha}{4}\right)^2 \right], \tag{A2}$$

$$V^{(1)} = -\frac{\alpha}{2}\left[1 + \frac{1}{4}\left(\frac{\alpha}{4}\right)^2 \right] B, \tag{A3}$$

$$V^{(2)} = -\left(\frac{\alpha}{2}\right)^2 \left[\widetilde{S}_3 \frac{\widetilde{S}_+ + \widetilde{S}_-}{2} \widetilde{S}_3 \right], \tag{A4}$$

$$V^{(3)} = -\frac{1}{2}\left(\frac{\alpha}{2}\right)^3 [\widetilde{S}_3 B \widetilde{S}_3], \tag{A5}$$

where

$$B \equiv [\widetilde{S}_3 \widetilde{S}_x + \widetilde{S}_x \widetilde{S}_3], \tag{A6}$$

and

$$\widetilde{S}_x \equiv \frac{\widetilde{S}_+ + \widetilde{S}_-}{2}. \tag{A7}$$

1. The zero-order transformation $\tilde{U}_0$

As known from the theory of su(2) algebra, the operator $\tilde{S}_x$ can be diagonalized by the transformation

$$\tilde{U}_0 = \exp\left[\frac{\pi}{4}(\tilde{S}_+ - \tilde{S}_-)\right] = \exp\left[-i\frac{\pi}{2}\tilde{S}_y\right]. \quad \text{(A8)}$$

Employing this transformation we obtain

$$V_0^{(0)} = 2\tilde{S}_3\left[1 - \frac{1}{2}\left(\frac{\alpha}{4}\right)^2\right], \quad V_0^{(1)} = -V^{(1)},$$

$$V_0^{(2)} = -\left(\frac{\alpha}{2}\right)^2[\tilde{S}_x\tilde{S}_3\tilde{S}_x],$$

$$V_0^{(3)} = \frac{1}{2}\left(\frac{\alpha}{2}\right)^3[\tilde{S}_x B\tilde{S}_x]. \quad \text{(A9)}$$

2. The first-order transformation $\tilde{U}_1$

It can readily be seen that the transformation

$$\tilde{U}_1 = \exp[\alpha x D1], \quad D1 \equiv -i[\tilde{S}_3\tilde{S}_y + \tilde{S}_y\tilde{S}_3] \quad \text{(A10)}$$

diagonalizes the operators in the first order. In the diagonalization, one needs the commutators

$$[\tilde{S}_3, D1] = B,$$

$$[[\tilde{S}_3, D1], D1] = 4\tilde{S}_3\left(\tilde{S}^2 - 2\tilde{S}_3^2 - \frac{1}{4}\right),$$

$$[[[\tilde{S}_3, D1], D1], D1] = (4\tilde{S}^2 - 1)B - 8[\tilde{S}_3 B + B\tilde{S}_3 \\ - \tilde{S}_3\tilde{S}_x\tilde{S}_3],$$

$$[\tilde{S}_3\tilde{S}_x\tilde{S}_3, D1] = 2\tilde{S}_3^2(\tilde{S}^2 - 2\tilde{S}_3^2) + (\tilde{S}_3\tilde{S}_x)^2 + (\tilde{S}_x\tilde{S}_3)^2. \quad \text{(A11)}$$

Then up to third order in α,

$$\tilde{U}_1 V_0^{(0)}(\tilde{U}_1)^{-1}$$

$$= V_0^{(0)} - 2\alpha x\left(1 - \frac{1}{2}\left(\frac{\alpha}{4}\right)^2\right)B + 4(\alpha x)^2\tilde{S}_3$$

$$\times\left(\tilde{S}^2 - 2\tilde{S}_3^2 - \frac{1}{4}\right) - \frac{(\alpha x)^3}{3}\{(4\tilde{S}^2 - 1)B - 8\tilde{J}\}, \quad \text{(A12)}$$

and

$$\tilde{U}_1 V_0^{(1)}(\tilde{U}_1)^{-1}$$

$$= \frac{\alpha}{2}\left[1 + \frac{1}{4}\left(\frac{\alpha}{4}\right)^2\right]B - \alpha^2 x 2\tilde{S}_3\left(\tilde{S}^2 - 2\tilde{S}_3^2 - \frac{1}{4}\right)$$

$$+ \alpha^3\left(\frac{x}{2}\right)^2\{(4\tilde{S}^2 - 1)B - 8\tilde{J}\}, \quad \text{(A13)}$$

where $\tilde{J} \equiv \tilde{S}_3 B + B\tilde{S}_3 - \tilde{S}_3\tilde{S}_x\tilde{S}_3$.

$$\tilde{U}_1 V_0^{(2)}(\tilde{U}_1)^{-1} = V_0^{(2)} + \alpha^3\left(\frac{x}{2}\right)^2\{\tilde{S}_x\tilde{S}_3(2\tilde{S}^2 - 4\tilde{S}_3^2 - \tilde{S}_x^2) \\ + (2\tilde{S}^2 - 4\tilde{S}_3^2 - \tilde{S}_x^2)\tilde{S}_3\tilde{S}_x\}. \quad \text{(A14)}$$

In third order, the operator $V_0^{(3)}$ remains unchanged after the transformation, i.e., $V_1^{(3)} = V_0^{(3)}$. In order to calculate $V_1^{(2)}$ up to third order, we take into account that

$$x = \frac{1}{4}\frac{\left[1 + \left(\frac{\alpha}{8}\right)^2\right]}{\left[1 - 2\left(\frac{\alpha}{8}\right)^2\right]} \approx \frac{1}{4}\left[1 + 3\left(\frac{\alpha}{8}\right)^2\right],$$

and find then that

$$V_1^{(0)} = V_0^{(0)} - 4\left(\frac{\alpha}{4}\right)^2\tilde{S}_3\left(\tilde{S}^2 - 2\tilde{S}_3^2 - \frac{1}{4}\right),$$

$$V_1^{(1)} = 0, \quad V_1^{(2)} = V_0^{(2)},$$

$$V_1^{(3)} = V_0^{(3)} - \frac{1}{2}\left(\frac{\alpha}{4}\right)^3 B + \frac{2}{3}\left(\frac{\alpha}{4}\right)^3\{(4\tilde{S}^2 - 1)B - 8\tilde{J}\}$$

$$+ \left(\frac{\alpha}{4}\right)^3\{\tilde{S}_x\tilde{S}_3[2\tilde{S}^2 - 4\tilde{S}_3^2 - (\tilde{S}_x)^2] + [2\tilde{S}^2 - 4\tilde{S}_3^2 \\ - (\tilde{S}_x)^2]\tilde{S}_3\tilde{S}_x\}. \quad \text{(A15)}$$

3. The second- and the third-order transformations

To find the second-order transformation we rewrite $V_1^{(2)}$ in a symmetrized form

$$V_1^{(2)} = -\frac{1}{2}\left(\frac{\alpha}{2}\right)^2[\tilde{L}_x\tilde{S}_3 + \tilde{S}_3\tilde{L}_x + \tilde{S}_3(\tilde{S}^2 - \tilde{S}_3^2 - 1)], \quad \text{(A16)}$$

where $\tilde{L}_x \equiv (\tilde{S}_+^2 + \tilde{S}_-^2)/4$. The diagonalizing transformation is then given by

$$\tilde{U}_2 \equiv \exp\left\{i\frac{1}{2}\left(\frac{\alpha}{4}\right)^2[\tilde{L}_y\tilde{S}_3 + \tilde{S}_3\tilde{L}_y]\right\}, \quad \text{(A17)}$$

where $\tilde{L}_y \equiv (\tilde{S}_+^2 - \tilde{S}_-^2)/4$. Keeping the terms up to third order we obtain [see Eq. (A13)]

$$\tilde{U}_2 V_1^{(0)}(\tilde{U}_2)^{-1} = V_1^{(0)} + \frac{1}{2}\left(\frac{\alpha}{2}\right)^2[\tilde{L}_x\tilde{S}_3 + \tilde{S}_3\tilde{L}_x]. \quad \text{(A18)}$$

The transformation (A17) does not change the expressions, given above for $V_1^{(2)}$ and $V_1^{(3)}$, and we find that

$$V_2^{(0)} = V_1^{(0)} - \left(\frac{\alpha}{4}\right)^2 2\tilde{S}_3(\tilde{S}^2 - \tilde{S}_3^2 - 1),$$

$$V_2^{(1)} = V_2^{(2)} = 0, \quad V_2^{(3)} = V_1^{(3)}. \quad \text{(A19)}$$

Diagonalization of $V_2^{(3)}$ can be performed in a similar way with an operator $\tilde{U}_3 = \exp[-(\alpha/4)^3 O]$. Since there are no diagonal terms in $V_2^{(3)}$, which would contribute to the spectrum of the Hamiltonian, we do not give here the fairly complicated form of operator O. The final diagonal form for the interaction V is thus given by

$$\bar{V} = \Omega \tilde{S}_3 \left\{ 1 + \left(\frac{\alpha}{4} \right)^2 [5\tilde{S}_3^2 - 3\tilde{r}(\tilde{r}+1) + 1] \right\}. \quad \text{(A20)}$$

To recapitulate, we introduced four transformations $\tilde{U}_k$, $k = 0,1,2,3$, which successively diagonalize the interaction operator in the Tavis-Cummings Hamiltonian up to third order with respect to the small parameter $\alpha = (\frac{1}{2}\Omega_R)^{-2}$, with Ω_R being the generalized Rabi frequency of Eq. (35), such that

$$\bar{V} \equiv \tilde{U} V \tilde{U}^{-1}, \quad \tilde{U} \equiv \tilde{U}_3 \tilde{U}_2 \tilde{U}_1 \tilde{U}_0. \quad \text{(A21)}$$

[1] M. Tavis and F.W. Cummings, Phys. Rev. **170**, 379 (1968).

[2] R. Dicke, Phys. Rev. **93**, 99 (1954).

[3] E. Jaynes and F. Cummings, Proc. IEEE **51**, 89 (1963).

[4] D. Meschede, H. Walther, and G. Muller, Phys. Rev. Lett. **54**, 551 (1985).

[5] P. Filipowicz, J. Javanainen, and P. Meystre, Phys. Rev. A **34**, 3077 (1986).

[6] P. Elmfors, B. Lautrup, and B.-S. Skagerstam, Phys. Rev. A **54**, 5171 (1996).

[7] J.M. Raimond, M. Brune, and S. Haroche, Rev. Mod. Phys. **73**, 565 (2001).

[8] S. Yu, H. Rauch, and Y. Zhang, Phys. Rev. A **52**, 2585 (1995).

[9] X. Yang, Y. Wu, and Y. Li, Phys. Rev. A **55**, 4545 (1997).

[10] A. Rybin, G. Kastelewicz, J. Timonen, and N. Bogoliubov, J. Phys. A **31**, 4705 (1998); N.M. Bogoliubov, R.K. Bullough, and J. Timonen, *ibid.* **29**, 6305 (1996).

[11] S. Carusotto, Phys. Rev. A **40**, 1848 (1989).

[12] M.E. Smithers and E.Y.C. Lu, Phys. Rev. A **9**, 790 (1974).

[13] B.R. Mollow and R.J. Glauber, Phys. Rev. **160**, 1076 (1967).

[14] J. Tucker and D.F. Walls, Phys. Rev. **178**, 2036 (1969).

[15] A.B. Klimov and L.L. Sanchez-Soto, Phys. Rev. A **61**, 063802 (2000).

[16] M. Kozierowski, A.A. Mamedov, and S.M. Chumakov, Phys. Rev. A **42**, 1762 (1990).

[17] S.M. Chumakov, A.B. Klimov, and J.J. Sanchez-Mondragon, Phys. Rev. A **49**, 4972 (1994).

[18] C. Saavedra, A.B. Klimov, S.M. Chumakov, and J.C. Retamal, Phys. Rev. A **58**, 4078 (1998).

[19] J. Delgado, E.C. Yustas, L.L. Sanchez-Soto, and A.B. Klimov, Phys. Rev. A **63**, 063801 (2001).

[20] C. Brif, Phys. Rev. A **54**, 5253 (1996).

[21] V. Sunilkumar, B.A. Bambah, R. Jagannathan, P.K. Panigrahi, and V. Srinivasan, J. Opt. B: Quantum Semiclassical Opt. **2**, 126 (2000).

[22] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).

[23] F. Persico and G. Vetri, Phys. Rev. A **12**, 2083 (1975).

[24] D. Bonatsos, C. Daskaloyannis, and P. Kolokotronis, J. Phys. A **26**, L871 (1993).

[25] D. Ruan, Y. Jia, and H. Sun, e-print quant-ph/0111007.

[26] V.P. Karassiov, J. Phys. A **27**, 153 (1994); V.P. Karassiov, A.A. Gusev, and S.I. Vinitsky, e-print quant-ph/0112040 (2001).

[27] D. Bonatsos, C. Daskaloyannis, and G.A. Lalazissis, Phys. Rev. A **47**, 3448 (1993).

[28] R. Brandt and O.W. Greenberg, J. Math. Phys. **10**, 1168 (1969).

[29] M. Rasetti, Int. J. Theor. Phys. **5**, 377 (1972).

[30] J. Katriel, Phys. Lett. **72A**, 94 (1979).

[31] C. Gerry, J. Phys. A **16**, 313 (1971).

[32] J. Katriel, A.I. Solomon, G. DAriano, and M. Rasetti, Phys. Rev. D **34**, 2332 (1986).

[33] S. Carusotto, Phys. Rev. A **38**, 3249 (1971).

[34] D. Bonatsos, C. Daskaloyannis, P. Kolokotronis, and D. Lenis, e-print hep-th/9402099.

[35] P. Shanta, S. Chaturvedi, V. Srinivasan, G.S. Agarwal, and C.L. Mehta, Phys. Rev. Lett. **72**, 1447 (1994).

[36] R. Floreanini, L. Lapointe, and L. Vinet, Physica B **389**, 327 (1996).

[37] D. Ruan and W. Ruan, Phys. Lett. A **263**, 78 (1999).

[38] I.P. Vadeiko, G.P. Miroshnichenko, A.V. Rybin, and J. Timonen, J. Opt. B (to be published).