Extension of Bogoliubov theory to quasicondensates

Christophe Mora

Laboratoire de Physique Statistique, École Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 5, France

Yvan Castin

Laboratoire Kastler Brossel, École Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 5, France

We present an extension of the well-known Bogoliubov theory to treat low-dimensional degenerate Bose gases in the limit of weak interactions and low density fluctuations. We use a density-phase representation and show that a precise definition of the phase operator requires a space discretization in cells of size l. We perform a systematic expansion of the Hamiltonian in terms of two small parameters, the relative density fluctuations inside a cell and the phase change over a cell. The resulting macroscopic observables can be computed in one, two, and three dimensions with no ultraviolet or infrared divergence. Furthermore, this approach exactly matches Bogoliubov's approach when there is a true condensate. We give the resulting expressions for the equation of state of the gas, the ground state energy, and the first order and second order correlation functions of the field. Explicit calculations are done for homogeneous systems.

INTRODUCTION

Recent progress in the realization of low-dimensional Bose gases in the quantum degenerate regime offers new perspectives for comparison with theoretical treatments. In atomic Bose gases, low-dimensional systems are achieved by creating anisotropic trapping potentials. Bose-Einstein condensates of reduced dimensionality, that is, with the atomic motion frozen in the harmonic oscillator ground state along one or two directions, have been produced [1,2].

Low-dimensional Bose gases with repulsive interactions were the subject of early theoretical studies. In the thermodynamic limit for spatially homogeneous systems, the Mermin-Wagner-Hohenberg theorem [3,4] excludes the formation of a Bose-Einstein condensate at finite temperature. This is physically due to large phase fluctuations which restrict the coherence length of the bosonic field to a finite value. One expects, however, that strong enough repulsive interactions between the particles strongly reduce the density fluctuations of the gas in contrast to the noninteracting case [5,6]. In this context, Popov introduced a long time ago the concept of a *quasicondensate* [7]. This concept has been extended to trapped gases [8–10]. The recent observation of large phase fluctuations for a degenerate Bose gas in a highly anisotropic cigar shaped trap has brought qualitative experimental confirmation of the theory in a quasi-one-dimensional geometry [11,12].

It turns out that the theory of quasicondensates has not yet reached the maturity of the theory for condensates. In the case of regular three-dimensional (3D) Bose-Einstein condensation in the weakly interacting regime, the Bogoliubov theory [13], based on a systematic expansion in a small parameter, gives a very precise description of the state of the gas. The intuitive idea of the Bogoliubov theory is to use the existence of a single macroscopically occupied mode ϕ_0 of the field, the mode of the condensate. We recall here the U(1)-symmetry preserving version of the theory [14,15]. One first splits the bosonic field operator as $\hat{\psi} = \phi_0 \hat{a}_0 + \delta\hat{\psi}$,

where $\hat{a}_0$ annihilates a particle in the condensate mode and $\delta\hat{\psi}$ accounts for quantum and thermal fluctuations in the other modes. Then one uses the assumption $|\delta\hat{\psi}| \ll |\hat{a}_0|$ to solve the field equations of motion perturbatively. This approach is not suitable for a quasicondensate as there is no single macroscopically occupied field mode. Fortunately, in the case of weak density fluctuations, the Bogoliubov idea can still be adapted in a quantum phase-density representation of the field operator. One writes the field operator $\hat{\psi}$ as $\exp(i\hat{\theta})\hat{\rho}^{1/2}$ where $\hat{\theta}$ and $\hat{\rho}$ are position dependent operators, giving the phase and the density. One then splits the operator giving the density as $\rho_0 + \delta\hat{\rho}$, where ρ_0 is a c number and $\delta\hat{\rho}$ are fluctuations, and one uses the fact that $|\delta\hat{\rho}| \ll \rho_0$. This idea has already been used in the literature [16] but to our knowledge without a precise definition of the phase operator, a well-known delicate point of quantum field theory [17,18]. As a consequence of the nonrigorous definition of the phase, divergences appear in the theory [16]: one has to introduce an arbitrary energy cutoff, so that predictions in 1D at zero temperature are made within a logarithmic accuracy only, and in 3D there is no full equivalence with the Bogoliubov theory. Another approach based on the current-density operator rather than on the phase operator was given by Schwartz [19]: an expansion of the Hamiltonian in terms of weak density and current fluctuations is performed relating the correlation function of the field to the static structure factor. It is subject to the same divergence problem in 2D and 3D in the absence of an energy cutoff if one calculates the structure factor in the Bogoliubov approximation.

A possibility to circumvent these difficulties is to rely on the path integral formulation of quantum field theory, which involves a functional integral over a classical field, for which the phase is perfectly well defined. This is the approach used by Popov, but with the introduction of an energy cutoff much smaller than the chemical potential of the gas, so that the physics at length scales smaller than the healing length is not

accurately described. This functional integral has recently been considered: it can lead to a cutoff independent formalism for quasicondensates [20,21] and it reproduces the same results as the Bogoliubov theory for a 3D condensate [22].

In this paper, we propose an improved Bogoliubov approach to treat quasicondensates in the phase-density formalism for a weakly interacting Bose gas. This approach is based on a lattice model, that is, with discrete spatial modes, which allows us to give a careful definition of the phase operator of the field and to introduce from the start an energy cutoff. It uses a systematic expansion in powers of the density fluctuations and of the spatial phase gradient and leads to simple expressions for the first and second order spatial correlation functions of the bosonic field that do not depend on the energy cutoff and that exactly reproduce in 3D the predictions of the Bogoliubov theory. We also use this formalism to determine the equation of state of the gas to the lowest nonvanishing order in the thermal and quantum excitations.

In Sec. I, we construct a discretized space model in order to define in a precise way the operators giving the phase and the density. We give the physical implications of the space discretization, restricting this approach to highly degenerate and weakly interacting Bose systems. In Sec. II, we derive a quadratic approximation to the Hamiltonian, that is, we derive approximate linear equations of motion for the density fluctuations and the phase operators. We recover to the lowest order the Gross-Pitaevskii equation for the quasicondensate density and we recover the Bogoliubov spectrum for the excitations. We also push the expansion to the next order, by producing a cubic correction to the quadratic Hamiltonian, including the interaction between the quasicondensate and the excitations. We show that inclusion of this correction is necessary to get a consistent theory and to establish the full equivalence between our approach and the number conserving Bogoliubov theory. In Sec. III, we present a few applications of our formalism: we give general formulas for the equation of state and the ground state of the gas, and for the first and second order correlation functions g_1 and g_2 of the field operator. In Sec. IV, we apply our formal results to the homogeneous Bose gas in various dimensions of space. This allows us to derive simply the validity condition of the method and to compare our results with existing results in the literature.

I. CONSTRUCTION OF A DISCRETE PHASE-DENSITY REPRESENTATION

A. Why discretize the real space?

In previous studies of quasicondensates the basic tools of the theory are an operator

$$\hat{\rho}(\mathbf{r}) = \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r}) \qquad (1)$$

giving the density in $\mathbf{r}$ and an operator $\hat{\theta}(\mathbf{r})$ giving the phase of $\hat{\psi}(\mathbf{r})$, the field operator in $\mathbf{r}$, the position $\mathbf{r}$ being a continuous variable [23]. A small parameter of the theory characterizing the regime of quasicondensates is then that the density fluctuations, that is, the fluctuations of $\hat{\rho}(\mathbf{r})$, are small in relative values:

$$\mathrm{var}[\hat{\rho}(\mathbf{r})] \equiv \langle \hat{\rho}(\mathbf{r})^2 \rangle - \langle \hat{\rho}(\mathbf{r}) \rangle^2 \ll \langle \hat{\rho}(\mathbf{r}) \rangle^2. \qquad (2)$$

However, one finds that the expectation value of $\hat{\rho}(\mathbf{r})^2$ is infinite at every point with a nonvanishing mean density $\rho(\mathbf{r}) = \langle \hat{\rho}(\mathbf{r}) \rangle$:

$$\langle \hat{\rho}(\mathbf{r})^2 \rangle = \delta(\mathbf{0})\rho(\mathbf{r}) + \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}) \rangle, \qquad (3)$$

where $\delta(\mathbf{0})$, the value of the Dirac distribution at the origin, is infinite, and the second term on the right-hand side, giving the probability density of finding two atoms at the same point of space, is finite in any realistic model. Mathematically, this divergence is due to the use of the bosonic commutation relations of the field operators $\hat{\psi}(\mathbf{r})$ and $\hat{\psi}^\dagger(\mathbf{r})$ at the same point of space to put the atomic field product in normal order.

In order to have small, and therefore finite, density fluctuations, one is forced to discretize the space, that is, to collect the particles in little boxes at the nodes of a spatial grid. Each little box has equal length l along each dimension of space and is parametrized by the position $\mathbf{r}$ of its center. The field operator $\hat{\psi}(\mathbf{r})$ has the effect of removing a particle in the box at position $\mathbf{r}$ and it now satisfies the bosonic commutation relations

$$[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')] = \frac{\delta_{\mathbf{r},\mathbf{r}'}}{l^D}, \qquad (4)$$

where $\delta_{\mathbf{r},\mathbf{r}'}$ is the discrete Kronecker delta function and D is the dimension of space. The variance of the operator giving the density is now finite:

$$\mathrm{var}[\hat{\rho}(\mathbf{r})] = \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}) \rangle - \rho^2(\mathbf{r}) + \frac{\rho(\mathbf{r})}{l^D}. \quad (5)$$

In the validity domain of the theoretical approach of this paper, this variance will be much smaller than $\rho^2(\mathbf{r})$ because both the sum of the first two terms and the last term in the right-hand side are small:

$$|\langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}) \rangle - \rho^2(\mathbf{r})| \ll \rho^2(\mathbf{r}), \qquad (6)$$

$$\rho(\mathbf{r})l^D \gg 1. \qquad (7)$$

B. The phase operator

In the usual continuous space theories, a Hermitian field phase operator $\hat{\theta}(\mathbf{r})$ is introduced subject to the following commutation relation with the operator giving the density:

$$[\hat{\rho}(\mathbf{r}), \hat{\theta}(\mathbf{r}')] = i\delta(\mathbf{r} - \mathbf{r}'). \qquad (8)$$

In our discrete model the desired commutation relation is modified into

$$[\hat{\rho}(\mathbf{r}), \hat{\theta}(\mathbf{r}')] = i\frac{\delta_{\mathbf{r},\mathbf{r}'}}{l^D}. \tag{9}$$

First we recall briefly that there actually exists no Hermitian operator $\hat{\theta}(\mathbf{r})$ satisfying strictly the above commutation relation. From the identity (9) one can indeed show that the operator

$$T(\alpha) \equiv e^{-i\alpha\hat{\theta}(\mathbf{r})}, \tag{10}$$

where α is any real number, is a translation operator for the density [24]:

$$T(\alpha)^{\dagger}\hat{\rho}(\mathbf{r})T(\alpha) = \hat{\rho}(\mathbf{r}) + \frac{\alpha}{l^D}. \tag{11}$$

This identity contradicts two fundamental properties of $\hat{\rho}(\mathbf{r})$, the positiveness and the discreteness of its spectrum [25].

We now proceed with the construction of a phase operator $\hat{\theta}(\mathbf{r})$ approximately satisfying the commutation relation (9). The key ingredients allowing such an approximate construction are (i) to be in the limit of a large occupation number of the considered box of the lattice, and (ii) to construct the operator $e^{i\theta}$ first, which, according to Eq. (11) taken with $\alpha = -1$, simply reduces the number of particles in the considered box by 1.

In each spatial box we introduce the basis of Fock states $|n,\mathbf{r}\rangle$ with exactly n particles in the box. In this basis the field operators have the following matrix elements:

$$\hat{\psi}(\mathbf{r}')|n,\mathbf{r}\rangle = \frac{\delta_{\mathbf{r},\mathbf{r}'}}{l^{D/2}}\sqrt{n}\,|n-1,\mathbf{r}\rangle,$$

$$\hat{\psi}^{\dagger}(\mathbf{r}')|n,\mathbf{r}\rangle = \frac{\delta_{\mathbf{r},\mathbf{r}'}}{l^{D/2}}\sqrt{n+1}\,|n+1,\mathbf{r}\rangle \tag{12}$$

as a consequence of the commutation relation (4). The atomic density $\hat{\rho}$ defined by $\hat{\rho}(\mathbf{r}) = \hat{\psi}^{\dagger}(\mathbf{r})\hat{\psi}(\mathbf{r})$ is diagonal in the Fock state basis:

$$\hat{\rho}(\mathbf{r}')|n,\mathbf{r}\rangle = \delta_{\mathbf{r},\mathbf{r}'}\frac{n}{l^D}|n,\mathbf{r}\rangle. \tag{13}$$

We then introduce the operator $\hat{A}$ defined by

$$\hat{\psi}(\mathbf{r}) \equiv \hat{A}(\mathbf{r})\sqrt{\hat{\rho}(\mathbf{r})}. \tag{14}$$

In the Fock space $\hat{A}(\mathbf{r})$ reduces by 1 the number of particles n in the box $\mathbf{r}$:

$$\hat{A}(\mathbf{r}')|n,\mathbf{r}\rangle = (1 - \delta_{n,0})\delta_{\mathbf{r},\mathbf{r}'}|n-1,\mathbf{r}\rangle. \tag{15}$$

Note that its action on the vacuum state of the box gives zero. For each box $\mathbf{r}$, the definition of $\hat{A}$ leads to the exact relations

$$\hat{A}\hat{A}^{\dagger} = I, \quad \hat{A}^{\dagger}\hat{A} = I - |0\rangle\langle0|, \quad \text{and} \quad [\hat{A},\hat{A}^{\dagger}] = |0\rangle\langle0|, \tag{16}$$

where I is the identity operator and $|0\rangle$ is the zero-particle state or vacuum state in the box of center $\mathbf{r}$. We find that the operator $\hat{A}$ is almost unitary, i.e., it is effectively unitary for a physical state of the system with a negligible probability of having an empty box. In what follows, we assume that this condition is satisfied, so that the projector $|0\rangle\langle0|$ can be neglected:

$$\text{occupation probability of } |n=0,\mathbf{r}\rangle \ll 1. \tag{17}$$

In this case, we write the approximately unitary operator $\hat{A}$ as

$$\hat{A}(\mathbf{r}) \simeq e^{i\hat{\theta}(\mathbf{r})} \quad \text{with} \quad \hat{\theta}^{\dagger}(\mathbf{r}) \simeq \hat{\theta}(\mathbf{r}), \tag{18}$$

which amounts to writing the field operator as

$$\hat{\psi}(\mathbf{r}) \simeq e^{i\hat{\theta}(\mathbf{r})}\sqrt{\hat{\rho}(\mathbf{r})}. \tag{19}$$

This should be understood as a formal writing, allowing us, for example, to recover the matrix elements of $\hat{A}$ and therefore of the field operator $\hat{\psi}$ from the commutation relation (9). We summarize below all the commutation relations of our phase-density representation:

$$[\hat{\rho}(\mathbf{r}), \hat{\theta}(\mathbf{r}')] \simeq \frac{i\delta_{\mathbf{r},\mathbf{r}'}}{l^D}, \quad [\hat{\rho}(\mathbf{r}), \hat{\rho}(\mathbf{r}')] = 0,$$

$$[\hat{\theta}(\mathbf{r}), \hat{\theta}(\mathbf{r}')] \simeq 0. \tag{20}$$

We come back to the constraint (17) at the basis of the construction of $\exp(i\hat{\theta})$. A sufficient condition to have a low probability for zero particle occupation in a box is obtained for a large mean number of particles in the box and with small relative particle number fluctuations. This is the regime that we wish to consider in this paper. We are therefore back to the discussion of the previous subsection and to the conditions (6),(7) for weak density fluctuations. In particular, the construction of the operator $\exp(i\hat{\theta})$ becomes problematic in the limit $l \to 0$, that is, in the continuous model.

C. How to choose the grid spacing l

Working on a grid can also be seen as performing a coarse-grain average over all physical quantities on a scale l. This averaging suppresses the short wavelength modes (shorter than l) and thus introduces an energy cutoff:

$$E_{\text{cut}} \simeq \frac{\hbar^2}{ml^2}. \tag{21}$$

This cutoff is of no physical consequence if all characteristic energies (μ, k_BT) are smaller, i.e., l is smaller than the corresponding characteristic lengths. This leads, for instance, to the following restrictions for l:

$$l < \xi \quad \text{and} \quad l < \lambda, \tag{22}$$

where

$$\xi = \frac{\hbar}{\sqrt{m\mu}} \tag{23}$$

is the healing length, and

$$\lambda = \sqrt{\frac{2\pi\hbar^2}{mk_BT}} \tag{24}$$

is the thermal de Broglie wavelength. These two restrictions, combined with Eq. (7), impose

$$\rho\lambda^D \gg 1, \tag{25}$$

$$\rho\xi^D \gg 1. \tag{26}$$

These are conditions of validity for our discrete model.

The first one, Eq. (25), is the quantum degeneracy regime occurring at sufficiently low temperatures. The second restriction, Eq. (26), corresponds to the regime of weakly interacting systems. Its dependence on the density varies according to the dimension of space. In 1D and 3D, the mean field prediction for the chemical potential is $\mu \simeq g\rho$, where g is a constant characterizing the interaction potential, the so-called coupling constant. In 1D, Eq. (26) is the high-density limit where a mean field theory is valid; we recall that the small density limit $\rho\xi = \hbar\sqrt{\rho/mg} \ll 1$ corresponds to the strongly interacting (or strongly correlated) Tonks gas regime. In 3D, the effective coupling constant g is related to the s-wave scattering length a of the interaction potential, $g = 4\pi\hbar^2 a/m$, so that $\rho\xi^3 \propto 1/\sqrt{\rho a^3} \gg 1$: one recovers the usual small gaseous parameter $\sqrt{\rho a^3}$. In 2D, the chemical potential scales as $\hbar^2\rho/[m\ln(1/\rho a^2)]$ where a is the scattering length of the 2D interaction potential, so that the condition $\rho\xi^2 \gg 1$ results in a low-density condition, $\ln(1/\rho a^2) \gg 1$.

II. PERTURBATIVE TREATMENT OF A MODEL HAMILTONIAN

A. Model Hamiltonian

In our lattice model, we represent the binary interaction potential among the particles by a discrete δ potential:

$$V(\mathbf{r}_1 - \mathbf{r}_2) = \frac{g_0}{l^D}\delta_{\mathbf{r}_1,\mathbf{r}_2}, \tag{27}$$

where g_0 is the bare coupling constant. Note that g_0 in general differs from the effective coupling constant g, and we shall come back to this point in Sec. IV A. With this model potential, the grand canonical Hamiltonian is

$$H = \sum_{\mathbf{r}} l^D \left[-\frac{\hbar^2}{2m}\hat{\psi}^\dagger(\mathbf{r})\Delta\hat{\psi}(\mathbf{r}) + [U(\mathbf{r}) - \mu]\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r}) \right.$$
$$\left. + \frac{g_0}{2}\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}) \right], \tag{28}$$

where $U(\mathbf{r})$ is an external trapping potential and where the Laplacian is a symmetric operator coupling the different neighboring boxes:

$$\Delta f(\mathbf{r}) = \sum_j \frac{f(\mathbf{r} + l\mathbf{e}_j) + f(\mathbf{r} - l\mathbf{e}_j) - 2f(\mathbf{r})}{l^2}. \tag{29}$$

The $\mathbf{e}_j$ are the unitary vectors and j the different orthogonal space directions (for example, $j = x,y,z$ in 3D). As usual we take periodic boundary conditions inside a rectangular box with lengths integer multiples of l.

We now rewrite the Hamiltonian in the density-phase representation, that is, in term of the operators $\hat{\rho}$ and $\hat{\theta}$ giving the density and the phase as defined in the previous section. The contributions of the trapping potential and of the interaction potential to the Hamiltonian are local in real space and therefore include the operator $\hat{\rho}$ only:

$$H_{\text{pot}} = \sum_{\mathbf{r}} l^D \hat{\rho}(\mathbf{r}) \left[U(\mathbf{r}) - \mu + \frac{g_0}{2}\left(\hat{\rho}(\mathbf{r}) - \frac{1}{l^D}\right) \right], \tag{30}$$

where we have used the bosonic commutation relation (4) to exchange one of the $\hat{\psi}^\dagger$ with $\hat{\psi}$ in the interaction term. The kinetic energy term involves the phase operator explicitly:

$$H_{\text{kin}} = -\frac{\hbar^2}{2ml^2}\sum_{\mathbf{r}} l^D \sum_j \sqrt{\hat{\rho}}\left(e^{i(\hat{\theta}_{+j} - \hat{\theta})}\sqrt{\hat{\rho}_{+j}} \right.$$
$$\left. + e^{i(\hat{\theta}_{-j} - \hat{\theta})}\sqrt{\hat{\rho}_{-j}} - 2\sqrt{\hat{\rho}} \right), \tag{31}$$

where we have introduced the notation $\hat{\theta}_{\pm j} = \hat{\theta}(\mathbf{r} \pm l\mathbf{e}_j)$ and $\hat{\rho}_{\pm j} = \hat{\rho}(\mathbf{r} \pm l\mathbf{e}_j)$. A remarkable property of this formulation, to be used below, is that it involves only the difference of two phase operators between two neighboring points of the lattice.

B. Hamiltonian quadratization and cubization

We now expand the Hamiltonian to third order in powers of two small parameters. As already discussed in Sec. I A the regime of quasicondensates that we are interested in corresponds to small relative fluctuations $\delta\hat{\rho}$ of the density. In the zeroth order approach totally neglecting the density fluctuations, the density is set to a deterministic value ρ_0, as we shall see. The second order expansion allows us to describe the density fluctuations:

$$\hat{\rho}(\mathbf{r}) = \rho_0(\mathbf{r}) + \delta\hat{\rho}(\mathbf{r}). \tag{32}$$

The third order expansion allows us to calculate the mean value of $\delta\hat{\rho}(\mathbf{r})$.

The first small parameter of the systematic expansion used in this paper is therefore given by

$$\epsilon_1 = \frac{|\delta\hat{\rho}|}{\rho_0} \ll 1, \tag{33}$$

where $|\delta\hat{\rho}|$ is the typical value of the operator $\delta\hat{\rho}$ in the physical state of the system. Mathematically, this allows us to expand $\sqrt{\hat{\rho}}$ as

$$\hat{\rho}^{1/2} = \rho_0^{1/2} + \frac{1}{2}\frac{\delta\hat{\rho}}{\rho_0^{1/2}} - \frac{1}{8}\frac{\delta\hat{\rho}^2}{\rho_0^{3/2}} + \frac{1}{16}\frac{\delta\hat{\rho}^3}{\rho_0^{5/2}} \cdots. \tag{34}$$

The second small parameter of the expansion is given by

$$\epsilon_2 = |l\boldsymbol{\nabla}\hat{\theta}| \ll 1. \tag{35}$$

Here $\boldsymbol{\nabla}$ represents the gradient on the lattice:

$$\boldsymbol{\nabla}f(\mathbf{r}) = \sum_j \frac{f(\mathbf{r}+l\mathbf{e}_j)-f(\mathbf{r}-l\mathbf{e}_j)}{2l}\mathbf{e}_j, \tag{36}$$

where f is an arbitrary function. Physically, the existence of the small parameter ϵ_2 is reasonable: it is at the basis of our

hypothesis that the continuous quantum field problem can be well approximated by a discrete lattice model, provided that l is small enough (see Sec. I C). Mathematically, this second small parameter allows us to expand the exponentials of the phase differences in Eq. (31):

$$e^{i(\hat{\theta}_{+j}-\hat{\theta})} = 1 + i(\hat{\theta}_{+j}-\hat{\theta}) - \frac{1}{2}(\hat{\theta}_{+j}-\hat{\theta})^2 \cdots. \tag{37}$$

From the fact that the discretization length l is on the order of the smaller of the two macroscopic length scales ξ and λ [see Eq. (22)], it will be checked later that the parameters ϵ_1 and ϵ_2, though of apparently different physical origin, can be chosen to be of the same order of magnitude,

$$\epsilon_1 \sim \epsilon_2 \sim \frac{1}{\sqrt{\rho_0 l^D}}, \tag{38}$$

and can therefore be treated mathematically as infinitesimals of the same order. The mathematical details of the expansion

$$H = H_0 + H_1 + H_2 + H_3 + \cdots \tag{39}$$

are given in Appendix A; we present here only the results:

$$H_0 = \sum_{\mathbf{r}} l^D\left[-\frac{\hbar^2}{2m}\sqrt{\rho_0}\Delta\sqrt{\rho_0} + \frac{g_0}{2}\rho_0^2 + [U(\mathbf{r})-\mu]\rho_0 \right],$$

$$H_1 = \sum_{\mathbf{r}} l^D\left[-\frac{\hbar^2}{2m\sqrt{\rho_0}}\Delta\sqrt{\rho_0} + U(\mathbf{r})-\mu+g_0\rho_0 \right]\delta\hat{\rho},$$

$$H_2 = E_2[\rho_0] + \sum_{\mathbf{r}} l^D\left[-\frac{\hbar^2}{2m}\frac{\delta\hat{\rho}}{2\sqrt{\rho_0}}\Delta\left(\frac{\delta\hat{\rho}}{2\sqrt{\rho_0}}\right) + \frac{\hbar^2\delta\hat{\rho}^2}{8m\rho_0^{3/2}}\Delta\sqrt{\rho_0} + \frac{g_0}{2}\delta\hat{\rho}^2 + \frac{\hbar^2}{2m}\sum_j \sqrt{\rho_0(\mathbf{r})\rho_0(\mathbf{r}+l\mathbf{e}_j)}\frac{[\hat{\theta}(\mathbf{r}+l\mathbf{e}_j)-\hat{\theta}(\mathbf{r})]^2}{l^2} \right],$$

$$H_3 = -\frac{g_0}{2}\sum_{\mathbf{r}}\delta\hat{\rho} + \frac{\hbar^2}{4ml^2}\sum_{\mathbf{r},j} l^D(\hat{\theta}_{+j}-\hat{\theta})\left(\frac{\rho_{0,+j}^{1/2}}{\rho_0^{1/2}}\delta\hat{\rho} + \frac{\rho_0^{1/2}}{\rho_{0,+j}^{1/2}}\delta\hat{\rho}_{+j}\right)(\hat{\theta}_{+j}-\hat{\theta}) + \frac{\hbar^2}{8m}\sum_{\mathbf{r}}\frac{\delta\hat{\rho}}{\rho_0}(\rho_0^{-1/2}\Delta\rho_0^{1/2}-\rho_0^{1/2}\Delta\rho_0^{-1/2})$$

$$- \frac{\hbar^2}{16m}\sum_{\mathbf{r}} l^D\left[\frac{\delta\hat{\rho}^3}{\rho_0^{5/2}}\Delta\sqrt{\rho_0} - \frac{\delta\hat{\rho}^2}{\rho_0^{3/2}}\Delta\left(\frac{\delta\hat{\rho}}{\sqrt{\rho_0}}\right)\right]. \tag{40}$$

The quantity E_2 in H_2 is a c-number functional of the density ρ_0, given in Appendix A, which therefore has no contribution to the dynamics of the quantum field.

C. Iterative solution for the quadratic Hamiltonian

We now solve perturbatively, order by order, the Hamiltonian problems defined by H_0, H_0+H_1, and $H_0+H_1+H_2$. To zeroth order in $\epsilon_{1,2}$, the Hamiltonian is a c number.

As the chemical potential is fixed in our approach, H_0 is minimized for a density profile $\rho_0(\mathbf{r})$ such that $\sqrt{\rho_0}$ solves the discrete version of the Gross-Pitaevskii equation:

$$\left[-\frac{\hbar^2}{2m}\Delta + U(\mathbf{r}) - \mu + g_0\rho_0 \right]\sqrt{\rho_0} = 0. \tag{41}$$

This density profile constitutes the zeroth order approxima-

tion to the density ρ. It contains a number of particles that we call N_0:

$$N_0 \equiv \sum_{\mathbf{r}} l^D \rho_0(\mathbf{r}). \tag{42}$$

Note that N_0 coincides with the mean total number of particles N only to lowest order in the theory. Equation (41) defines ρ_0 and therefore N_0 as functions of the chemical potential μ. It will, however, turn out to be more convenient to parametrize the theory in terms of N_0 rather than in terms of μ. We will therefore consider μ and ρ_0 as functions of N_0:

$$\mu = \mu_0(N_0), \tag{43}$$

$$\rho_0(\mathbf{r}) = \rho_0(\mathbf{r};N_0). \tag{44}$$

μ_0 is therefore the Gross-Pitaevskii prediction for the chemical potential of a gas of N_0 particles.

For the choice of density profile (41), the first order correction H_1 to the Hamiltonian vanishes. We therefore now have to solve the Hamiltonian problem defined by H_2, in order to determine the lowest-order approximation to the density fluctuation $\delta\hat{\rho}$ and the phase $\hat{\theta}$. It is instructive to write the corresponding Heisenberg equations of motion, which are linear (and therefore trivially solvable) since H_2 is quadratic. As $\hat{\theta}$ and $\delta\hat{\rho}$ are two canonically conjugate variables, the equations of motion are

$$\hbar\partial_t\hat{\theta} \simeq -\frac{1}{l^D}\frac{\partial H_2}{\partial\delta\hat{\rho}(\mathbf{r})} = \frac{\hbar^2}{2m\sqrt{\rho_0}}\left[\Delta\left(\frac{\delta\hat{\rho}}{2\sqrt{\rho_0}}\right) - \delta\hat{\rho}\frac{\Delta\sqrt{\rho_0}}{2\rho_0}\right]$$

$$- g_0\delta\hat{\rho},$$

$$\hbar\partial_t\delta\hat{\rho}(\mathbf{r}) \simeq \frac{1}{l^D}\frac{\partial H_2}{\partial\hat{\theta}(\mathbf{r})} = -\frac{\hbar^2}{m}\sqrt{\rho_0}[\Delta(\hat{\theta}\sqrt{\rho_0}) - \hat{\theta}\Delta\sqrt{\rho_0}]. \tag{45}$$

An important difference of these equations from the so-called quantum hydrodynamics equations for $\delta\hat{\rho}$ and $\hat{\theta}$ is that our formalism keeps the so-called quantum pressure term for $\partial_t\hat{\theta}$, whereas it is usually neglected in the literature [23]. This allows our treatment to have a cutoff energy larger than μ, whereas the usual treatment is restricted to energy modes much below μ.

Furthermore, one can simplify these equations using the Gross-Pitaevskii equation (41) to eliminate $\Delta\sqrt{\rho_0}$:

$$\hbar\partial_t\hat{\theta} = -\frac{1}{2\sqrt{\rho_0}}\left[-\frac{\hbar^2}{2m}\Delta + U + 3g_0\rho_0 - \mu\right]\left(\frac{\delta\hat{\rho}}{\sqrt{\rho_0}}\right), \tag{46}$$

$$\hbar\partial_t\delta\hat{\rho}(\mathbf{r}) = 2\sqrt{\rho_0}\left[-\frac{\hbar^2}{2m}\Delta + U + g_0\rho_0 - \mu\right](\hat{\theta}\sqrt{\rho_0}). \tag{47}$$

This gives the idea of a very simple canonical transformation which, remarkably, maps our equations for a quasicondensate (46),(47) into the equations for the Bogoliubov modes of a condensate: the field

$$\hat{B} = \frac{\delta\hat{\rho}}{2\sqrt{\rho_0}} + i\sqrt{\rho_0}\hat{\theta} \tag{48}$$

has bosonic commutation relations

$$[\hat{B}(\mathbf{r}),\hat{B}^\dagger(\mathbf{r}')] = \frac{\delta_{\mathbf{r},\mathbf{r}'}}{l^D} \tag{49}$$

and it obeys the standard Bogoliubov equations

$$i\hbar\partial_t\begin{pmatrix}\hat{B}\\\hat{B}^\dagger\end{pmatrix} = \mathcal{L}_{GP}\begin{pmatrix}\hat{B}\\\hat{B}^\dagger\end{pmatrix} \equiv \begin{pmatrix}-\frac{\hbar^2}{2m}\Delta + U - \mu + 2g_0\rho_0 & g_0\rho_0 \\ -g_0\rho_0 & -\left(-\frac{\hbar^2}{2m}\Delta + U - \mu + 2g_0\rho_0\right)\end{pmatrix}\begin{pmatrix}\hat{B}\\\hat{B}^\dagger\end{pmatrix}. \tag{50}$$

This mapping can be readily extended to the Hamiltonian H_2, which is expected to be canonically equivalent to the Bogoliubov Hamiltonian:

$$H_2 = l^D\sum_{\mathbf{r}}\hat{B}^\dagger\left(-\frac{\hbar^2}{2m}\Delta + U + g_0\rho_0 - \mu\right)\hat{B}$$

$$+ g_0\rho_0\left[\hat{B}^\dagger\hat{B} + \frac{1}{2}(\hat{B}^2 + \hat{B}^{\dagger 2})\right]. \tag{51}$$

We have checked that the identity (51) indeed holds by replacing $\hat{B}$ by its expression (48) in terms of $\delta\hat{\rho}$ and $\hat{\theta}$, and by using the value of the commutators (20) and the fact that $\sqrt{\rho_0}$ solves the Gross-Pitaevskii equation. Remarkably, the energy functional $E_2[\rho_0]$ is exactly compensated by the contribution of the commutators.

This mapping therefore allows us to reuse the standard diagonalization of the Bogoliubov Hamiltonian. We recall here briefly the procedure described in [15,27]. One intro-

duces the normal eigenmodes (u_s, v_s) of the Bogoliubov operator $\mathcal{L}_{GP}$ with an energy ϵ_s, normalizable as

$$\sum_{\mathbf{r}} l^D [|u_s(\mathbf{r})|^2 - |v_s(\mathbf{r})|^2] = 1. \tag{52}$$

Then (v_s^*, u_s^*) is an eigenmode of $\mathcal{L}_{GP}$ with the energy $-\epsilon_s$. To form a complete family of vectors one has to further introduce the zero-energy mode of $\mathcal{L}_{GP}$, given by $(\phi_0, -\phi_0)$, and the anomalous mode (ϕ_a, ϕ_a) with

$$\phi_0 = \sqrt{\rho_0/N_0} \quad \text{and} \quad \phi_a = \sqrt{N_0} \partial_{N_0} \sqrt{\rho_0}. \tag{53}$$

The corresponding normalization of the anomalous mode is such that the scalar product of ϕ_0 and ϕ_a is 1/2. With these definitions, one introduces the components of $(\hat{B}, \hat{B}^\dagger)$ on the zero-energy mode, on the anomalous mode and on the regular (u_s, v_s) modes:

$$\begin{pmatrix} \hat{B} \\ \hat{B}^\dagger \end{pmatrix} = -i\sqrt{N_0} \hat{Q} \begin{pmatrix} \phi_0 \\ -\phi_0 \end{pmatrix} + \frac{\hat{P}}{\sqrt{N_0}} \begin{pmatrix} \phi_a \\ \phi_a \end{pmatrix}$$
$$+ \sum_s \hat{b}_s \begin{pmatrix} u_s \\ v_s \end{pmatrix} + \hat{b}_s^\dagger \begin{pmatrix} v_s^* \\ u_s^* \end{pmatrix}. \tag{54}$$

$\hat{Q}$ is a collective coordinate representing the quantum phase of the field and $\hat{P}$ is its conjugate momentum

$$[\hat{P}, \hat{Q}] = -i. \tag{55}$$

Physically $\hat{P}$ corresponds to fluctuations in the total number of particles, as expected, and as shown in more detail later [see Eq. (67)]. The operators $\hat{b}_s$ are bosonic annihilation operators with the usual commutation relations $[\hat{b}_s, \hat{b}_{s'}] = \delta_{s,s'}$. They commute with $\hat{P}$ and $\hat{Q}$. The inverse formulas giving $\hat{b}_s$, $\hat{Q}$, and $\hat{P}$ in terms of $\hat{B}$ can be found, for example, in [15]. Equation (54) results in the following modal expansion for the density fluctuations and the phase operators:

$$\hat{\theta}(\mathbf{r}) = \sum_s \theta_s(\mathbf{r}) \hat{b}_s + \theta_s^*(\mathbf{r}) \hat{b}_s^\dagger - \hat{Q},$$

$$\delta\hat{\rho}(\mathbf{r}) = \sum_s \delta\rho_s(\mathbf{r}) \hat{b}_s + \delta\rho_s^*(\mathbf{r}) \hat{b}_s^\dagger + \hat{P} \partial_{N_0} \rho_0, \tag{56}$$

where

$$\theta_s(\mathbf{r}) = \frac{u_s(\mathbf{r}) - v_s(\mathbf{r})}{2i\sqrt{\rho_0(\mathbf{r})}},$$

$$\delta\rho_s(\mathbf{r}) = \sqrt{\rho_0(\mathbf{r})} [u_s(\mathbf{r}) + v_s(\mathbf{r})]. \tag{57}$$

By construction, this modal expansion, when inserted into the quadratic Hamiltonian H_2, results in

$$H = \sum_s \epsilon_s \hat{b}_s^\dagger \hat{b}_s + \frac{1}{2}\hat{P}^2 \mu_0' + \tilde{E}_2[\rho_0], \tag{58}$$

where $\mu_0' = d\mu_0/dN_0$. This is the sum of uncoupled harmonic oscillators, plus a massive free degree of freedom corresponding to the unbound phase variable $\hat{Q}$. The effective mass of the phase variable is given by $1/\mu_0'$. The energy functional $\tilde{E}_2[\rho_0]$ will be calculated in Sec. III B, where it will be shown that it leads to exactly the same ground state energy as the number conserving Bogoliubov theory. This shows that the Bogoliubov theory can be used to calculate the ground state energy even for, e.g., 1D quasicondensates, a fact commonly used in the literature [28,29] but which looks rather heuristic in the absence of justification.

D. Effect of cubic Hamiltonian corrections on the density

The physics contained in the cubic term H_3 of the Hamiltonian is very rich. It includes interaction effects between the Bogoliubov modes of the previous section, allowing a generalization to quasicondensates of the theory of energy shifts and Beliaev-Landau damping usually put forward for Bose-Einstein condensates [30–32].

We are more modest here. Our motivation to include the cubic corrections is that the quadratic Hamiltonian H_2 brings actually no correction to the zeroth order approximation ρ_0 to the mean density, since the mean value of $\delta\hat{\rho}$ vanishes at the level of the second order theory. This is highly nonsatisfactory as it brings some inconsistency into the calculation of an observable like g_1, the first order correlation function of the field: to get a nontrivial prediction for g_1 one has to include terms quadratic in the phase operator, which are second order in ϵ_2, which forces us to also include second order corrections to the mean density, as will become very explicit in Sec. III.

We therefore calculate the first order correction to the equations of motion of $\delta\hat{\rho}$ and $\hat{\theta}$ due to the cubic Hamiltonian term H_3, and we take the average over the quantum state corresponding to the density operator at thermal equilibrium for the Hamiltonian H_2. This gives source terms to add to the equations for the mean density and phase derived from H_2. We leave the details of the calculations to Appendix B and give the result directly:

$$\hbar \partial_t \langle \delta\hat{\rho} \rangle_3 = \rho_0^{1/2} \left[-\frac{\hbar^2}{2m}\Delta + U + g_0\rho_0 - \mu \right] (2\rho_0^{1/2} \langle \hat{\theta} \rangle_3), \tag{59}$$

$$-2\hbar\sqrt{\rho_0} \partial_t \langle \hat{\theta} \rangle_3 = \left[-\frac{\hbar^2}{2m}\Delta + U + 3g_0\rho_0 - \mu \right]$$
$$\times \left(\frac{\langle \delta\hat{\rho} \rangle_3 - \langle \hat{B}^\dagger \hat{B} \rangle_2}{\rho_0^{1/2}} \right)$$
$$+ g_0\rho_0^{1/2} \langle 4\hat{B}^\dagger \hat{B} + \hat{B}^2 + \hat{B}^{\dagger 2} \rangle_2$$
$$-2\langle \hat{P}^2 \rangle_2 \mu_0' \partial_{N_0} \sqrt{\rho_0}, \tag{60}$$

where the thermal average $\langle \cdots \rangle_2$ is taken with the unperturbed Hamiltonian H_2 and $\langle \cdots \rangle_3$ is taken with the perturbed Hamiltonian $H_2 + H_3$ to first order in H_3. The expectation value of the "kinetic energy" of the unbound phase variable in Eq. (58) is equal to $k_B T/2$ according to the equipartition theorem so that

$$\langle \hat{P}^2 \rangle_2 = \frac{k_B T}{\mu_0'}. \tag{61}$$

At equilibrium the expectation values of $\partial_t \delta \hat{\rho}$ and $\partial_t \hat{\theta}$ vanish. This fact is obvious for $\partial_t \delta \hat{\rho}$; it is less obvious for $\partial_t \hat{\theta}$ because of the presence of the unbound variable $\hat{Q}$; we therefore produce a proof of that in Appendix C. We therefore have to solve Eqs. (59) and (60) with the left-hand side set to zero. The first equation (59) imposes the condition that the mean value of $\hat{\theta}$ is position independent, a trivial result. In the second equation, the operator acting on $\langle \delta \hat{\rho} \rangle_3$ is strictly positive so that it is invertible, and Eq. (60) determines the correction to the mean density in a unique way.

We now go through a sequence of transformations allowing us to get a physical understanding of the value of $\langle \delta \hat{\rho} \rangle_3$. The first step is to pull out the contribution of the "anomalous" terms $\hat{P}$, $\hat{Q}$ in the modal expansion (54):

$$\hat{B}(\mathbf{r}) \equiv -i\sqrt{N_0} \hat{Q} \phi_0(\mathbf{r}) + \frac{1}{\sqrt{N_0}} \hat{P} \phi_a(\mathbf{r}) + \hat{B}_n. \tag{62}$$

We calculate the expectation values of Eq. (60) involving the operator $\hat{B}$, using the fact that all the crossed terms between the anomalous part and the operators b_s have a vanishing expectation value:

$$\langle \hat{B}^\dagger \hat{B} \rangle_2 = \frac{\phi_a^2}{N_0} \langle \hat{P}^2 \rangle_2 + N_0 \phi_0^2 \langle \hat{Q}^2 \rangle_2 + \langle \hat{B}_n^\dagger \hat{B}_n \rangle_2 - \phi_a \phi_0, \tag{63}$$

$$\langle \hat{B}^2 \rangle_2 + \langle \hat{B}^{\dagger 2} \rangle_2 = 2\frac{\phi_a^2}{N_0} \langle \hat{P}^2 \rangle_2 - 2N_0 \phi_0^2 \langle \hat{Q}^2 \rangle_2$$

$$+ \langle \hat{B}_n^2 \rangle_2 + \langle \hat{B}_n^{\dagger 2} \rangle_2. \tag{64}$$

The term $\phi_0 \phi_a$ in Eq. (63) comes from the noncommutation of $\hat{P}$ and $\hat{Q}$ [see Eq. (55)]. The contributions of $\langle \hat{Q}^2 \rangle_2$ in Eqs. (63) and (64), when inserted into Eq. (60), are shown to compensate exactly when one uses the fact that ϕ_0 solves the Gross-Pitaevskii equation. This was expected from the U(1) symmetry of the Hamiltonian: only differences of the phase operator at two points appear in the Hamiltonian, so that H does not depend on $\hat{Q}$ and the mean density does not depend on $\langle \hat{Q}^2 \rangle_2$.

We therefore get an equation for $\langle \delta \hat{\rho} \rangle_3$ involving the expectation value of $\hat{P}^2$ as a source term, and which looks rather involved:

$$0 = \left[-\frac{\hbar^2}{2m} \Delta + U + 3g_0 \rho_0 - \mu \right]$$

$$\times \left(\frac{\langle \delta \hat{\rho} \rangle_3 - \phi_a^2 N_0^{-1} \langle \hat{P}^2 \rangle_2 - (\langle \hat{B}_n^\dagger \hat{B}_n \rangle_2 - \phi_0 \phi_a)}{\rho_0^{1/2}} \right)$$

$$+ g_0 \rho_0^{1/2} \langle 4(\hat{B}_n^\dagger \hat{B}_n - \phi_0 \phi_a) + \hat{B}_n^2 + B_n^{\dagger 2} \rangle_2$$

$$+ \langle \hat{P}^2 \rangle_2 [6g_0 \rho_0^{1/2} \partial_{N_0} \sqrt{\rho_0} - 2\mu_0'] \partial_{N_0} \sqrt{\rho_0}. \tag{65}$$

Fortunately the underlying physics is very simple and allows us to predict the effect of this source term on the mean density. One first identifies the physical meaning of $\hat{P}$ in Eq. (56). Using the well-known fact that the eigenmodes of $\mathcal{L}_{\text{GP}}$ are orthogonal for the modified scalar product of signature $(1, -1)$, one has [15]

$$\langle \phi_0 | u_s \rangle + \langle \phi_0 | v_s \rangle \equiv \sum_{\mathbf{r}} l^D \phi_0(\mathbf{r}) [u_s(\mathbf{r}) + v_s(\mathbf{r})] = 0, \tag{66}$$

so that the sum of $\delta \rho_s$ over all spatial nodes vanishes. As a consequence, the operator $\hat{N}$ giving the total number of particles in the gas is simply

$$\hat{N} \equiv \sum_{\mathbf{r}} l^D \hat{\rho}(\mathbf{r}) = N_0 + \hat{P}, \tag{67}$$

where we have used the identity

$$\sum_{\mathbf{r}} l^D \partial_{N_0} \rho_0(\mathbf{r}) = \frac{d}{dN_0} \sum_{\mathbf{r}} l^D \rho_0(\mathbf{r}) = \frac{dN_0}{dN_0} = 1. \tag{68}$$

The source terms involving $\hat{P}$ therefore correspond to fluctuations in the total number of particles in the gas, due to the fact that we consider the grand canonical ensemble. The effect of these grand canonical fluctuations can be considered for the case of a pure quasicondensate at the order of the present calculation so it is easy to calculate it directly. In the grand canonical ensemble the probability that the quasicondensate has n particles is

$$\Pi_n \propto \exp[-\beta(E_0(n) - \mu n)], \tag{69}$$

where $E_0(n)$ is the Gross-Pitaevskii energy for the density profile $\rho_0(\mathbf{r}; n)$:

$$E_0(n) = \sum_{\mathbf{r}} l^D \left[-\frac{\hbar^2}{2m} \sqrt{\rho_0(\mathbf{r}; n)} \Delta \sqrt{\rho_0(\mathbf{r}; n)} + U(\mathbf{r}) \rho_0(\mathbf{r}; n) \right.$$

$$\left. + \frac{g_0}{2} \rho_0^2(\mathbf{r}; n) \right]. \tag{70}$$

The corresponding mean grand canonical density is

$$\rho_{\text{GC}}(\mathbf{r}) = \int dn \, \Pi_n \rho_0(\mathbf{r}; n), \tag{71}$$

where we treat n as a continuous variable. The zeroth order approximation $n = N_0$ for the number of particles in the quasicondensate is such that $E_0(n) - \mu n$ has a minimum:

$$\frac{d}{dn}[E_0(n) - \mu n] = \mu_0(n) - \mu = 0 \quad \text{for} \quad n = N_0 \quad (72)$$

as shown in Eq. (43). The corresponding density profile is $\rho_0(\mathbf{r}; N_0)$. The next order correction to that is obtained by expanding the n dependent density profile to second order in $n - N_0$ and by averaging over n:

$$\delta\rho_{GC}(\mathbf{r}) = \langle(n - N_0)\rangle \partial_{N_0} \rho_0(\mathbf{r}; N_0)$$

$$+ \frac{1}{2}\langle(n - N_0)^2\rangle \partial_{N_0}^2 \rho_0(\mathbf{r}; N_0). \quad (73)$$

The second moment of $n - N_0$ is calculated to lowest nonvanishing order by a Gaussian approximation to Π_n:

$$E_0(n) - \mu n \simeq E_0(N_0) - \mu N_0 + \frac{1}{2}\frac{d^2 E_0}{dN_0^2}(n - N_0)^2$$

$$= \text{const} + \frac{1}{2}\mu_0'(n - N_0)^2. \quad (74)$$

This leads to

$$\langle(n - N_0)^2\rangle_{\text{Gauss}} = \frac{k_B T}{\mu_0'} = \langle\hat{P}^2\rangle_2. \quad (75)$$

More care has to be taken in the calculation of the mean of $n - N_0$: the Gaussian approximation to Π_n gives a vanishing contribution, so that the cubic distortion to it has to be included:

$$E_0(n) - \mu n \simeq E_0(N_0) - \mu N_0 + \frac{1}{2}\mu_0'(n - N_0)^2$$

$$+ \frac{1}{6}\mu_0''(n - N_0)^3, \quad (76)$$

$$\Pi_n \propto \exp\left[-\frac{1}{2}\beta\mu_0'(n - N_0)^2\right]\left[1 - \frac{1}{6}\beta\mu_0''(n - N_0)^3\right]. \quad (77)$$

We then get a nonvanishing mean value for $n - N_0$:

$$\langle(n - N_0)\rangle_{\text{distor}} = -\frac{1}{6}\beta\mu_0''\langle(n - N_0)^4\rangle_{\text{Gauss}}$$

$$= -\frac{1}{2}\beta\mu_0''(\langle\hat{P}^2\rangle_2)^2. \quad (78)$$

We have therefore predicted in a very simple way the correction to the mean density due to grand canonical fluctuations:

$$\delta\rho_{GC}(\mathbf{r}) = \frac{1}{2}\langle\hat{P}^2\rangle_2\left[\partial_{N_0}^2 \rho_0(\mathbf{r}; N_0) - \frac{\mu_0''}{\mu_0'}\partial_{N_0}\rho_0(\mathbf{r}; N_0)\right]. \quad (79)$$

How does this compare to the general formalism (65)? We need to obtain a partial differential equation for $\delta\rho_{GC}$. We just take the second order derivative of the Gross-Pitaevskii equation (41) with respect to N_0 and we replace ρ_0 by $\sqrt{\rho_0}^2$ in the resulting equation and in Eq. (79). This leads to the remarkable identity

$$\left[-\frac{\hbar^2}{2m}\Delta + U + 3g_0\rho_0 - \mu\right]\left(\frac{\delta\rho_{GC} - N_0^{-1}\phi_a^2\langle\hat{P}^2\rangle_2}{\rho_0^{1/2}}\right)$$

$$= -\langle\hat{P}^2\rangle_2[6g_0\rho_0^{1/2}\partial_{N_0}\sqrt{\rho_0} - 2\mu_0']\partial_{N_0}\sqrt{\rho_0}. \quad (80)$$

The right-hand side of this identity coincides with the source term of Eq. (65) involving $\hat{P}$. We have therefore successfully identified $\delta\rho_{GC}$ as a piece of $\langle\delta\hat{\rho}\rangle_3$ and we are left with the simpler equation

$$0 = \left[-\frac{\hbar^2}{2m}\Delta + U + 3g_0\rho_0 - \mu\right]$$

$$\times\left(\frac{\langle\delta\hat{\rho}\rangle_3 - \delta\rho_{GC} - (\langle\hat{B}_n^\dagger\hat{B}_n\rangle_2 - \phi_0\phi_a)}{\rho_0^{1/2}}\right)$$

$$+ g_0\rho_0^{1/2}\langle 4(\hat{B}_n^\dagger\hat{B}_n - \phi_0\phi_a) + \hat{B}_n^2 + B_n^{\dagger 2}\rangle_2. \quad (81)$$

We are not totally satisfied yet since the operator $\hat{B}_n$ does not obey bosonic commutation relations when the system is not spatially homogeneous; in particular, the field $\hat{B}_n$ does not commute with itself when taken at two different points:

$$[\hat{B}_n(\mathbf{r}), \hat{B}_n(\mathbf{r}')] = \phi_a(\mathbf{r})\phi_0(\mathbf{r}') - \phi_a(\mathbf{r}')\phi_0(\mathbf{r}), \quad (82)$$

$$[\hat{B}_n(\mathbf{r}), \hat{B}_n^\dagger(\mathbf{r}')] = \frac{1}{l^D}\delta_{\mathbf{r},\mathbf{r}'} - \phi_0(\mathbf{r})\phi_a(\mathbf{r}')$$

$$- \phi_0(\mathbf{r}')\phi_a(\mathbf{r}). \quad (83)$$

To circumvent this difficulty we split the field $\hat{B}_n$ into its component along the quasicondensate mode ϕ_0 and its orthogonal component:

$$\hat{B}_n(\mathbf{r}) = \hat{\alpha}\phi_0(\mathbf{r}) + \hat{\Lambda}(\mathbf{r}). \quad (84)$$

The Bogoliubov functions $u_s(\mathbf{r})$ and $v_s(\mathbf{r})$ can be chosen here to be real. The operator $\hat{\alpha}$ can then be written as

$$\hat{\alpha} = \sum_s \langle\phi_0|u_s\rangle(\hat{b}_s - \hat{b}_s^\dagger), \quad (85)$$

where we have used the property (66). This clearly shows that the operator $\hat{\alpha}$ is anti-Hermitian:

$$\hat{\alpha}^\dagger = -\hat{\alpha}. \quad (86)$$

The field $\hat{\Lambda}$ has the following expansion on $\hat{b}_s$:

$$\hat{\Lambda}(\mathbf{r}) = \sum_s u_{s\perp}(\mathbf{r})\hat{b}_s + v_{s\perp}(\mathbf{r})\hat{b}_s^\dagger, \qquad (87)$$

where the index $\perp$ indicates projection orthogonally to ϕ_0. This field now has the desired bosonic commutation relations

$$[\hat{\Lambda}(\mathbf{r}),\hat{\Lambda}(\mathbf{r}')] = 0, \qquad (88)$$

$$[\hat{\Lambda}(\mathbf{r}),\hat{\Lambda}^\dagger(\mathbf{r}')] = \frac{1}{l^D}\delta_{\mathbf{r},\mathbf{r}'} - \phi_0(\mathbf{r})\phi_0(\mathbf{r}'). \qquad (89)$$

Note that $\hat{\alpha}$ does not commute with $\hat{\Lambda}$:

$$[\hat{\alpha},\hat{\Lambda}(\mathbf{r})] = \frac{1}{2}\phi_0(\mathbf{r}) - \phi_a(\mathbf{r}). \qquad (90)$$

We insert the splitting of $\hat{B}_n$ in Eq. (81). The terms quadratic in $\hat{\alpha}$ cancel exactly, in the same way the terms in $\hat{Q}^2$ canceled. The terms linear in $\hat{\alpha}$ can all be expressed in terms of the expectation value of an anticommutator $\langle\{\hat{\alpha},\hat{\Lambda}\}\rangle_2$ using the commutation relation (90) and the fact that $\langle\hat{\alpha}\hat{\Lambda}\rangle_2$ is a real quantity. Furthermore, using the techniques of Appendix E of [26], as shown here in Appendix D, one obtains a simple partial differential equation for the anticommutator:

$$\left[-\frac{\hbar^2}{2m}\Delta + U + g_0\rho_0 - \mu\right]\langle\{\hat{\alpha},\hat{\Lambda}(\mathbf{r})\}\rangle_2$$

$$= -\sum_{\mathbf{r}'} l^D g_0\rho_0(\mathbf{r}')\phi_0(\mathbf{r}')\langle\{\hat{\Lambda}(\mathbf{r}')+\hat{\Lambda}^\dagger(\mathbf{r}'),\hat{\Lambda}(\mathbf{r})\}\rangle_2. \qquad (91)$$

Remarkably, this allows us to eliminate completely the operator $\hat{\alpha}$ in Eq. (81). We finally get an equation for the correction to the mean density involving the operator $\hat{\Lambda}$ only:

$$0 = \left[-\frac{\hbar^2}{2m}\Delta + U + 3g_0\rho_0 - \mu\right]\left(\frac{\langle\delta\hat{\rho}\rangle_3 - \delta\rho_{\mathrm{GC}} - \langle\hat{\Lambda}^\dagger\hat{\Lambda}\rangle_2}{\phi_0}\right)$$

$$+ S(\mathbf{r}), \qquad (92)$$

where we have introduced the source term

$$S(\mathbf{r}) \equiv g_0 N_0\phi_0(\mathbf{r})\langle 4\hat{\Lambda}^\dagger(\mathbf{r})\hat{\Lambda}(\mathbf{r}) + \hat{\Lambda}^2(\mathbf{r}) + \hat{\Lambda}^{\dagger 2}(\mathbf{r}) - \phi_0^2(\mathbf{r})\rangle_2$$

$$- \sum_{\mathbf{r}'} l^D g_0\rho_0(\mathbf{r}')\phi_0(\mathbf{r}')\langle\{\hat{\Lambda}(\mathbf{r}')+\hat{\Lambda}^\dagger(\mathbf{r}'),\hat{\Lambda}(\mathbf{r})\}\rangle_2. \qquad (93)$$

It will be convenient to introduce the function $\chi(\mathbf{r})$ defined in a unique way by

$$0 = \left[-\frac{\hbar^2}{2m}\Delta + U + 3g_0\rho_0 - \mu\right]\chi(\mathbf{r}) + \frac{1}{2}S(\mathbf{r}). \qquad (94)$$

We then obtain the following final expression for the correction to the mean density due to the cubic Hamiltonian terms H_3:

$$\langle\delta\hat{\rho}\rangle_3(\mathbf{r}) = \delta\rho_{\mathrm{GC}}(\mathbf{r}) + 2\phi_0(\mathbf{r})\chi(\mathbf{r}) + \langle\hat{\Lambda}^\dagger(\mathbf{r})\hat{\Lambda}(\mathbf{r})\rangle_2. \qquad (95)$$

In the particular case where the gas is Bose condensed, our general theory for quasicondensates also applies, of course. One then expects that the result (95) has already been obtained for the condensate and can be given a clear physical interpretation. This expectation is totally justified: as shown in Appendix E, the component of $\chi(\mathbf{r})/N_0$ orthogonal to ϕ_0 is the correction given in [15] to the Gross-Pitaevskii condensate wave function ϕ_0 due to the interaction with the noncondensed particles; the component of χ along ϕ_0 describes the condensate depletion, and $\langle\hat{\Lambda}^\dagger\hat{\Lambda}\rangle_2$ is the mean density of noncondensed particles.

III. APPLICATIONS OF THE FORMALISM: GENERAL FORMULAS

A. Equation of state

What is referred to as the *equation of state* of the gas is the expression of the chemical potential as a function of the mean total number of particles N and the temperature T. It is useful in particular to predict properties of an inhomogeneous gas within the local density approximation.

We therefore now have to calculate μ for the quasicondensate. This is equivalent to a calculation of N_0 as μ and N_0 are by definition related through Eq. (43). To lowest order of the theory one assumes a pure quasicondensate with a density profile $\rho(\mathbf{r})=\rho_0(\mathbf{r})$, where $\sqrt{\rho_0}$ solves the Gross-Pitaevskii equation (41). One therefore gets $N=N_0$ so that $\mu = \mu_0(N)$.

The first nonvanishing correction to the density profile is given by Eq. (95). By integrating Eq. (95) over space we get the corresponding correction for the mean total number of particles:

$$N \equiv N_0 + \delta N, \qquad (96)$$

$$\delta N \simeq \delta N_{\mathrm{GC}} + l^D\sum_{\mathbf{r}} 2\phi_0(\mathbf{r})\chi(\mathbf{r}) + l^D\sum_{\mathbf{r}} \langle\hat{\Lambda}^\dagger(\mathbf{r})\hat{\Lambda}(\mathbf{r})\rangle_2. \qquad (97)$$

The contribution to δN due to our use of the grand canonical ensemble can be calculated exactly from a spatial integration of Eq. (79), using the same technique as in Eq. (68):

$$\delta N_{\mathrm{GC}} = -k_B T\frac{\mu_0''}{2\mu_0'^2}. \qquad (98)$$

The contribution of the term involving χ can also be made explicit by multiplication of Eq. (94) by $\phi_a(\mathbf{r})$ defined in Eq. (53) and by spatial integration. The function $\phi_a(\mathbf{r})$ is indeed known [27] to solve the partial differential equation

$$\left[-\frac{\hbar^2}{2m}\Delta+U+3g_0\rho_0-\mu\right]\phi_a=N_0\mu_0'\phi_0(\mathbf{r}), \quad (99)$$

which can be checked easily, just by taking the derivative of Eq. (41) with respect to N_0. This leads to

$$l^D\sum_{\mathbf{r}}2\phi_0(\mathbf{r})\chi(\mathbf{r})=-\frac{1}{N_0\mu_0'}\sum_{\mathbf{r}}l^D\phi_a(\mathbf{r})S(\mathbf{r}), \quad (100)$$

where the source term S is known explicitly [see Eq. (93)]. We now just have to replace N_0 by $N-\delta N$ in Eq. (43) and expand to first order in δN:

$$\mu=\mu_0(N-\delta N)\simeq\mu_0(N)-\delta N\mu_0'(N_0). \quad (101)$$

We obtain the following expression for μ:

$$\mu\simeq\mu_0(N)+k_BT\frac{\mu_0''}{2\mu_0'}-\mu_0'(N_0)\sum_{\mathbf{r}}l^D\langle\hat{\Lambda}^\dagger(\mathbf{r})\hat{\Lambda}(\mathbf{r})\rangle_2$$

$$+\frac{1}{N_0}\sum_{\mathbf{r}}l^D\phi_a(\mathbf{r})S(\mathbf{r}). \quad (102)$$

Equivalently, we can replace the source term by its explicit expression to get

$$\mu\simeq\mu_0(N)+k_BT\frac{\mu_0''}{2\mu_0'}-\mu_0'(N_0)\left(\frac{1}{2}+\sum_{\mathbf{r}}l^D\langle\hat{\Lambda}^\dagger(\mathbf{r})\hat{\Lambda}(\mathbf{r})\rangle_2\right)$$

$$+\sum_{\mathbf{r}}l^Dg_0[\partial_{N_0}\rho_0(\mathbf{r})](2\langle\hat{\Lambda}^\dagger\hat{\Lambda}\rangle_2+\mathrm{Re}\langle\hat{\Lambda}^2\rangle_2)$$

$$-\sum_{\mathbf{r}}l^Dg_0\phi_0^3(\mathbf{r})\langle\{\hat{\Lambda}(\mathbf{r})+\hat{\Lambda}^\dagger(\mathbf{r}),\hat{\gamma}\}\rangle_2, \quad (103)$$

where we have introduced the operator

$$\hat{\gamma}=\sum_{\mathbf{r}}l^D\phi_a(\mathbf{r})\hat{\Lambda}(\mathbf{r}), \quad (104)$$

and we have used the identity

$$\mu_0'=\sum_{\mathbf{r}}l^Dg_0\phi_0^2(\mathbf{r})\partial_{N_0}\rho_0(\mathbf{r};N_0) \quad (105)$$

obtained by performing the scalar product of both sides of Eq. (99) with ϕ_0. The application to spatially homogeneous systems will be given in Sec. IV; in this case both the operator $\hat{\gamma}$ and μ_0'' vanish.

B. Ground state energy

We now show that the ground state energy of a quasicondensate can be calculated with exactly the same Bogoliubov formula as for the ground state energy of a condensate.

We have to determine the ground state energy of H_2. We write it as the expectation value of Eq. (51) at zero temperature, that is here in the vacuum of the $\hat{b}_s$ and of $\hat{P}$:

$$\mathcal{E}_{\mathrm{ground}}(H_2)=l^D\sum_{\mathbf{r}}\left\langle\hat{B}^\dagger\left(-\frac{\hbar^2}{2m}\Delta+U+g_0\rho_0-\mu\right)\hat{B}\right\rangle_2$$

$$+g_0\rho_0\left\langle\left[\hat{B}^\dagger\hat{B}+\frac{1}{2}(\hat{B}^2+\hat{B}^{\dagger2})\right]\right\rangle_2. \quad (106)$$

We reproduce the transformation of Sec. II D. We split $\hat{B}$ into an anomalous part involving $\hat{P},\hat{Q}$, plus the contributions of the anti-Hermitian operator $\hat{\alpha}$ and of $\hat{\Lambda}$, the orthogonal component of the normal part. In the first expectation value of the right-hand side of Eq. (106) the operators $\hat{Q}$ and $\hat{\alpha}$ disappear as they come with the factor $\phi_0(\mathbf{r})$ in $\hat{B}$, and ϕ_0 solves the Gross-Pitaevskii equation (41). The expectation value of $\hat{P}^2$ in the ground state of H_2 also vanishes, so that

$$\left\langle\hat{B}^\dagger\left(-\frac{\hbar^2}{2m}\Delta+U+g_0\rho_0-\mu\right)\hat{B}\right\rangle_2$$

$$=\left\langle\hat{\Lambda}^\dagger\left(-\frac{\hbar^2}{2m}\Delta+U+g_0\rho_0-\mu\right)\hat{\Lambda}\right\rangle_2. \quad (107)$$

The same transformation is applied to the last expectation value in Eq. (106). Remarkably, the terms involving $\hat{\alpha}$ exactly cancel when one uses the relations (63), (64), (90), and (D5). This leads to

$$\left\langle\left[\hat{B}^\dagger\hat{B}+\frac{1}{2}(\hat{B}^2+\hat{B}^{\dagger2})\right]\right\rangle_2$$

$$=-\frac{1}{2}\phi_0^2+\left\langle\left[\hat{\Lambda}^\dagger\hat{\Lambda}+\frac{1}{2}(\hat{\Lambda}^2+\hat{\Lambda}^{\dagger2})\right]\right\rangle_2. \quad (108)$$

The expectation values involving $\hat{\Lambda}$ are readily calculated from the modal expansion (87):

$$\mathcal{E}_{\mathrm{ground}}(H_2)=-\frac{1}{2}\sum_{\mathbf{r}}l^Dg_0\rho_0\phi_0^2$$

$$+\sum_s\langle v_{s\perp}|\left[\left(-\frac{\hbar^2}{2m}\Delta+U+2g_0\rho_0-\mu\right)|v_{s\perp}\rangle\right.$$

$$+g_0\rho_0|u_{s\perp}\rangle\bigg]. \quad (109)$$

As (u_s,v_s) is an eigenvector of $\mathcal{L}_{\mathrm{GP}},(u_{s\perp},v_{s\perp})$ is an eigenvector of the operator $\mathcal{L}$ defined in [15] and this expression can be further simplified to

$$\mathcal{E}_{\mathrm{ground}}(H_2)=-\frac{1}{2}\sum_{\mathbf{r}}l^Dg_0\rho_0\phi_0^2-\sum_s\epsilon_s\langle v_{s\perp}|v_{s\perp}\rangle. \quad (110)$$

The last step is to include the contribution of H_0 and to remove the $-\mu\hat{N}$ term from the grand canonical Hamiltonian. The ground state energy of the canonical Hamiltonian for N particles is therefore

$$E_{\text{ground}}(N) \simeq \mu N + E_0(N_0) - \mu N_0 + \mathcal{E}_{\text{ground}}(H_2), \quad (111)$$

where E_0 is the Gross-Pitaevskii energy (70). As we did in Sec. III A we replace N_0 by $N - \delta N$, where δN is calculated from H_3, and we expand $E_0(N - \delta N)$ to first order in δN:

$$\mu N + E_0(N_0) - \mu N_0 \simeq E_0(N) - \delta N(\mu_0(N) - \mu) \simeq E_0(N).$$
$$(112)$$

We recall that by definition $\mu = \mu_0(N_0)$. The first term in the right-hand side of Eq. (110) amounts to performing a small change in the Gross-Pitaevskii energy functional, expressing the fact that a given particle interacts in the gas with $N-1$ particles so that the mean field term should be proportional to $N-1$ rather than to N. The final expression for the ground state energy is

$$E_{\text{ground}}(N) \simeq N \sum_{\mathbf{r}} l^D \left[-\frac{\hbar^2}{2m} \phi_0(\mathbf{r};N) \Delta \phi_0(\mathbf{r};N) \right.$$
$$\left. + U(\mathbf{r}) \phi_0^2(\mathbf{r};N) + \frac{1}{2} g_0(N-1) \phi_0^4(\mathbf{r};N) \right]$$
$$- \sum_s \epsilon_s \langle v_{s\perp} | v_{s\perp} \rangle. \quad (113)$$

This exactly coincides with the Bogoliubov result; see, e.g., Eq. (71) of [15].

C. Second order correlation function

The second order correlation function of the atomic field is defined as

$$g_2(\mathbf{r}) \equiv \langle \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{0}) \hat{\psi}(\mathbf{0}) \hat{\psi}(\mathbf{r}) \rangle, \quad (114)$$

where we have taken for simplicity one of the two points as the origin of the coordinates. To calculate g_2 with the formalism of this paper we have to express g_2 in terms of the operator $\hat{\rho}$ giving the density. This is achieved using the commutation relation (4) of the bosonic field $\hat{\psi}$:

$$g_2(\mathbf{r}) = \langle \hat{\rho}(\mathbf{r}) \hat{\rho}(\mathbf{0}) \rangle - \frac{\delta_{\mathbf{r},\mathbf{0}}}{l^D} \langle \hat{\rho}(\mathbf{0}) \rangle. \quad (115)$$

We then insert the splitting (32) of $\hat{\rho}$ in terms of the quasicondensate density ρ_0 and the fluctuations $\delta\hat{\rho}$:

$$g_2(\mathbf{r}) = \rho_0(\mathbf{r}) \rho_0(\mathbf{0}) + \rho_0(\mathbf{0}) \langle \delta\hat{\rho}(\mathbf{r}) \rangle + \rho_0(\mathbf{r}) \langle \delta\hat{\rho}(\mathbf{0}) \rangle$$
$$+ \langle \delta\hat{\rho}(\mathbf{r}) \delta\hat{\rho}(\mathbf{0}) \rangle - \frac{\delta_{\mathbf{r},\mathbf{0}}}{l^D} [\rho_0(\mathbf{0}) + \langle \delta\hat{\rho}(\mathbf{0}) \rangle].$$
$$(116)$$

This expression for g_2 is still exact. We now perform approximations consistent with an expansion of g_2 up to second order in the small parameters $\epsilon_{1,2}$. The expectation value of the term quadratic in $\delta\hat{\rho}$ is calculated within the thermal equilibrium for the quadratic Hamiltonian H_2. The expectation value of $\delta\hat{\rho}$ is evaluated in Sec. II D by inclusion of the cubic perturbation H_3. The contribution of $\delta\hat{\rho}$ in the last term of Eq. (116) is negligible as it is ϵ_1^4 times smaller than the leading term in g_2. We therefore obtain the explicit expression

$$g_2(\mathbf{r}) \simeq \rho_0(\mathbf{r}) \rho_0(\mathbf{0}) + \rho_0(\mathbf{0}) \langle \delta\hat{\rho}(\mathbf{r}) \rangle_3 + \rho_0(\mathbf{r}) \langle \delta\hat{\rho}(\mathbf{0}) \rangle_3$$
$$+ \langle \delta\hat{\rho}(\mathbf{r}) \delta\hat{\rho}(\mathbf{0}) \rangle_2 - \frac{\delta_{\mathbf{r},\mathbf{0}}}{l^D} \rho_0(\mathbf{0}). \quad (117)$$

This formulation, however, is not the optimal one as the last term in $1/l^D$ gives the wrong impression that $g_2(0)$ strongly depends on the discretization length l in the continuous limit $l \to 0$. In fact, this strong dependence exactly compensates a term in $1/l^D$ in the density fluctuations $\langle \delta\hat{\rho}^2(\mathbf{0}) \rangle$ coming from the fact that $\delta\hat{\rho}^2(\mathbf{0})$ is a product of field operators not in normal order. To reveal this fact we express $\delta\hat{\rho}$ in terms of the operator $\hat{\Lambda}$ of Eq. (87):

$$\delta\hat{\rho}(\mathbf{r}) = \sqrt{\rho_0(\mathbf{r})} [\hat{\Lambda}(\mathbf{r}) + \hat{\Lambda}^\dagger(\mathbf{r})] + \hat{P} \partial_{N_0} \rho_0(\mathbf{r};N_0),$$
$$(118)$$

and we put the resulting expression in normal order with respect to the field $\hat{\Lambda}$ using the bosonic commutation relation (89):

$$\delta\hat{\rho}(\mathbf{r}) \delta\hat{\rho}(\mathbf{0}) = \, :\delta\hat{\rho}(\mathbf{r}) \delta\hat{\rho}(\mathbf{0}): + \frac{\delta_{\mathbf{r},\mathbf{0}}}{l^D} \rho_0(\mathbf{0}) - N_0 \phi_0^2(\mathbf{r}) \phi_0^2(\mathbf{0}),$$
$$(119)$$

where $: :$ is the standard notation to represent normal order. The spurious term in $1/l^D$ is then exactly canceled:

$$g_2(\mathbf{r}) = N_0(N_0 - 1) \phi_0^2(\mathbf{r}) \phi_0^2(\mathbf{0}) + \rho_0(\mathbf{0}) \langle \delta\hat{\rho}(\mathbf{r}) \rangle_3$$
$$+ \rho_0(\mathbf{r}) \langle \delta\hat{\rho}(\mathbf{0}) \rangle_3 + \langle :\delta\hat{\rho}(\mathbf{r}) \delta\hat{\rho}(\mathbf{0}): \rangle_2. \quad (120)$$

This expression allows a proof of the equivalence with the prediction for g_2 in the Bogoliubov theory. We do not present the calculations here, as they are a straightforward application of Appendix E. Finally, we give a last alternative expression for g_2 equivalent to Eq. (120) at the present order:

$$g_2(\mathbf{r}) = (1 - 1/N) \rho(\mathbf{0}) \rho(\mathbf{r}) + \langle :\delta\hat{\rho}(\mathbf{r}) \delta\hat{\rho}(\mathbf{0}): \rangle_2, \quad (121)$$

where N is the mean total number of particles and ρ is the mean total density:

$$\rho(\mathbf{r}) = \rho_0(\mathbf{r}) + \langle \delta\hat{\rho}(\mathbf{r}) \rangle_3. \quad (122)$$

D. First order correlation function

The first order correlation function of the field is defined as

$$g_1(\mathbf{r}) \equiv \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{0})\rangle = \langle \sqrt{\hat{\rho}(\mathbf{r})}\, e^{i[\hat{\theta}(\mathbf{0})-\hat{\theta}(\mathbf{r})]}\sqrt{\hat{\rho}(\mathbf{0})}\rangle.$$
(123)

As previously done we perform the calculation up to second order in the small parameters $\epsilon_{1,2}$. We therefore expand $\sqrt{\hat{\rho}}$ up to second order in $\delta\hat{\rho}$ using Eq. (34). Note that we do not expand the exponential in $\hat{\theta}(\mathbf{0})-\hat{\theta}(\mathbf{r})$, contrary to what we did in the Hamiltonian: as $\mathbf{r}$ and $\mathbf{0}$ are not neighboring points of the lattice anymore, the phase difference of the field can be arbitrarily large. The expansion in $\delta\hat{\rho}$ gives rise to six terms:

$$g_1(\mathbf{r}) = \rho_0^{1/2}(\mathbf{r})\rho_0^{1/2}(\mathbf{0})\Big[\langle e^{i\Delta\theta}\rangle + \frac{1}{2}\langle \delta\widetilde{\rho}(\mathbf{r})e^{i\Delta\theta} + e^{i\Delta\theta}\delta\widetilde{\rho}(\mathbf{0})\rangle$$

$$-\frac{1}{8}\langle \delta\widetilde{\rho}^2(\mathbf{r})e^{i\Delta\theta} + e^{i\Delta\theta}\delta\widetilde{\rho}^2(\mathbf{0})$$

$$-2\,\delta\widetilde{\rho}(\mathbf{r})e^{i\Delta\theta}\delta\widetilde{\rho}(\mathbf{0})\rangle\Big],$$
(124)

where we have introduced the following notation to simplify the writing:

$$\Delta\theta \equiv \hat{\theta}(\mathbf{0}) - \hat{\theta}(\mathbf{r})$$
(125)

$$\delta\widetilde{\rho}(\mathbf{r}) \equiv \frac{\delta\hat{\rho}(\mathbf{r})}{\rho_0(\mathbf{r})}.$$
(126)

We calculate the expectation values in this expression in two steps, first using the thermal equilibrium distribution for H_2, and then including the corrections due to H_3.

The thermal expectation values corresponding to the quadratic Hamiltonian H_2 are evaluated using Wick's theorem. One first expands the exponential in powers of $\Delta\theta$, one calculates the expectation value of each term, and then one performs an exact resummation of the resulting series. This leads to the simple identities

$$\langle e^{i\Delta\theta}\rangle_2 = e^{-\langle(\Delta\theta)^2\rangle_2/2},$$
(127)

$$\langle \delta\widetilde{\rho}(\mathbf{r})e^{i\Delta\theta}\rangle_2 = e^{-\langle(\Delta\theta)^2\rangle_2/2}\langle \delta\widetilde{\rho}(\mathbf{r})i\Delta\theta\rangle_2,$$
(128)

$$\langle \delta\widetilde{\rho}^2(\mathbf{r})e^{i\Delta\theta}\rangle_2 = e^{-\langle(\Delta\theta)^2\rangle_2/2}$$

$$\times[\langle \delta\widetilde{\rho}^2(\mathbf{r})\rangle_2 + (\langle \delta\widetilde{\rho}(\mathbf{r})i\Delta\theta\rangle_2)^2],$$
(129)

$$\langle \delta\widetilde{\rho}(\mathbf{r})e^{i\Delta\theta}\delta\widetilde{\rho}(\mathbf{0})\rangle_2 = e^{-\langle(\Delta\theta)^2\rangle_2/2}[\langle \delta\widetilde{\rho}(\mathbf{r})\delta\widetilde{\rho}(\mathbf{0})\rangle_2$$

$$+\langle \delta\widetilde{\rho}(\mathbf{r})i\Delta\theta\rangle_2\langle i\Delta\theta\delta\widetilde{\rho}(\mathbf{0})\rangle_2].$$
(130)

The expectation value of a product of the density fluctuation $\delta\hat{\rho}$ and of the phase variation $i\Delta\theta$ is particularly simple. In classical field theory this expectation value would obviously vanish, as there is no crossed term in H_2 between the density fluctuations and the phase. In the present quantum field theory this is not exactly the case as $\hat{\rho}$ and $\hat{\theta}$ do not commute. To show that, we use the fact that the Bogoliubov mode functions $u_s(\mathbf{r})$ and $v_s(\mathbf{r})$ can be chosen to be real, so that $\delta\hat{\rho}$ and $i\Delta\theta$ are linear combinations of $\hat{b}_s, \hat{b}_s^\dagger, \hat{P}$ with real coefficients. As a consequence,

$$\langle \delta\widetilde{\rho}(\mathbf{r})i\Delta\theta\rangle_2 = \langle \delta\widetilde{\rho}(\mathbf{r})i\Delta\theta\rangle_2^* = -\langle i\Delta\theta\delta\widetilde{\rho}(\mathbf{r})\rangle.$$
(131)

This leads to

$$\langle \delta\widetilde{\rho}(\mathbf{r})i\Delta\theta\rangle_2 = \frac{i}{2}\langle[\delta\widetilde{\rho}(\mathbf{r}),\Delta\theta]\rangle_2 = \frac{1-\delta_{\mathbf{r},\mathbf{0}}}{2\rho_0(\mathbf{r})l^D}.$$
(132)

The same reasoning can be applied for the other expectation value:

$$\langle i\Delta\theta\delta\widetilde{\rho}(\mathbf{0})\rangle_2 = \frac{1-\delta_{\mathbf{r},\mathbf{0}}}{2\rho_0(\mathbf{0})l^D}.$$
(133)

These expressions are second order in $\epsilon_{1,2}$. An important consequence is that the product of such crossed phase-density expectation values in Eqs. (129) and (130) is actually negligible at the present order of the calculation. The resulting form for g_1, at the level of H_2, is quite simple:

$$g_1(\mathbf{r})|_{H_2} = \rho_0^{1/2}(\mathbf{r})\rho_0^{1/2}(\mathbf{0})e^{-\langle(\Delta\theta)^2\rangle_2/2}\Big[1 - \frac{1}{8}\langle(\Delta\delta\widetilde{\rho})^2\rangle_2$$

$$+\frac{1}{4}\mathcal{E}_{\text{scoria}}(\mathbf{r})\Big].$$
(134)

The notation $\Delta\delta\widetilde{\rho}$ is similar to the one for the phase:

$$\Delta\delta\widetilde{\rho} \equiv \delta\widetilde{\rho}(\mathbf{0}) - \delta\widetilde{\rho}(\mathbf{r})$$
(135)

and $\mathcal{E}_{\text{scoria}}$ comes from the crossed expectation value of $\Delta\theta$ and $\delta\widetilde{\rho}$:

$$\mathcal{E}_{\text{scoria}}(\mathbf{r}) \equiv (1-\delta_{\mathbf{r},\mathbf{0}})\left(\frac{1}{\rho_0(\mathbf{r})l^D} + \frac{1}{\rho_0(\mathbf{0})l^D}\right).$$
(136)

At this point we face the same apparent problem as in the calculation of g_2: $\mathcal{E}_{\text{scoria}}$ scales as $1/l^D$ and gives the wrong impression that our expression for g_1 will depend dramatically on l in the continuous limit $l\to 0$. As in the case of g_2, we solve this problem by expressing $\delta\hat{\rho}$ and $\hat{\theta}$ in terms of the field $\hat{\Lambda}$ and putting the operators $\hat{\Lambda}$ and $\hat{\Lambda}^\dagger$ in normal order. We use Eq. (118) for the expression for $\delta\hat{\rho}$. For the difference of two phase operators, $\hat{Q}$ and the anti-Hermitian operator $\hat{\alpha}$ cancel so that

$$\Delta\theta = \frac{1}{2i}(\Delta\tilde{\Lambda} - \Delta\tilde{\Lambda}^\dagger), \tag{137}$$

where we have introduced the notation

$$\tilde{\Lambda}(\mathbf{r}) \equiv \frac{\hat{\Lambda}(\mathbf{r})}{\rho_0^{1/2}(\mathbf{r})}, \tag{138}$$

$$\Delta\tilde{\Lambda} \equiv \tilde{\Lambda}(\mathbf{0}) - \tilde{\Lambda}(\mathbf{r}). \tag{139}$$

After some calculations we arrive at

$$\langle(\Delta\,\delta\tilde{\rho})^2\rangle_2 = \langle:(\Delta\,\delta\tilde{\rho})^2:\rangle_2 + \mathcal{E}_{\text{scoria}}(\mathbf{r}), \tag{140}$$

$$\langle(\Delta\,\theta)^2\rangle_2 = \langle:(\Delta\,\theta)^2:\rangle_2 + \frac{1}{4}\mathcal{E}_{\text{scoria}}(\mathbf{r}). \tag{141}$$

As $\mathcal{E}_{\text{scoria}}$ is second order in $\epsilon_{1,2}$, its exponential function can be expanded to first order. We then find as expected that all the $1/l^D$ terms exactly cancel:

$$g_1(\mathbf{r})|_{H_2} = \rho_0^{1/2}(\mathbf{r})\rho_0^{1/2}(\mathbf{0})e^{-\langle:(\Delta\theta)^2:\rangle_2/2}\left[1 - \frac{1}{8}\langle:(\Delta\,\delta\tilde{\rho})^2:\rangle_2\right]. \tag{142}$$

The last step is to include the first order correction to g_1 coming from the cubic Hamiltonian H_3. One then has to calculate expectation values with the thermal equilibrium density operator $\exp[-\beta(H_2+H_3)]$ to first order in H_3. This thermal density operator can be viewed as the evolution operator during the imaginary time $-i\hbar\beta$ so that one can use first order time dependent perturbation theory to get

$$\langle\hat{O}\rangle_3 \simeq \langle\hat{O}\rangle_2 - \int_0^\beta d\tau\langle e^{\tau H_2}H_3 e^{-\tau H_2}\hat{O}\rangle_2, \tag{143}$$

where $\hat{O}$ is an arbitrary operator of the gas and where we have used the fact that H_3 has a vanishing expectation value in the thermal equilibrium state for H_2. One is back to the calculation of expectation values of some operators in the thermal state corresponding to H_2. Wick's theorem can be applied. The resulting calculations are very similar to the ones leading to Eq. (142), but more involved, and are detailed in Appendix F. The same phenomenon occurs, that terms of a higher order than the present calculation can be neglected. One then gets

$$\langle e^{i\Delta\theta}\rangle_3 \simeq e^{-\langle(\Delta\theta)^2\rangle_2/2}[1 + \langle i\Delta\theta\rangle_3], \tag{144}$$

$$\langle\delta\tilde{\rho}(\mathbf{r})e^{i\Delta\theta}\rangle_3 \simeq e^{-\langle(\Delta\theta)^2\rangle_2/2}[\langle\delta\tilde{\rho}(\mathbf{r})i\Delta\theta\rangle_2 + \langle\delta\tilde{\rho}(\mathbf{r})\rangle_3]. \tag{145}$$

The first terms in the right-hand sides of Eqs. (144) and (145) already appeared at the level of H_2, and the second terms are corrections due to H_3 that we now take into account. There is no need to include H_3 corrections to the other terms of Eq. (124) since they are quadratic in $\delta\hat{\rho}$ and are therefore already of second order. The expectation values of

the phase $\hat{\theta}$ and of the density fluctuations $\delta\hat{\rho}$ have been calculated in Sec. II D. It was found that the expectation value of the phase operator is space independent so that $\langle\Delta\theta\rangle_3$ vanishes. The expectation value of the density fluctuations including the effect of H_3 was given in Eq. (95) and is in general different from zero. Remarkably, the whole effect on the correlation function g_1 of the first order correction in H_3 is to replace $\rho_0(\mathbf{r})$ by the total mean density $\rho(\mathbf{r})$ defined in Eq. (122).

We write our final expression for the first order correlation function of the field, calculated consistently up to $\epsilon_{1,2}^2$:

$$g_1(\mathbf{r}) = \sqrt{\rho(\mathbf{r})\rho(\mathbf{0})}\exp\left[-\frac{1}{2}\langle:(\Delta\theta)^2:\rangle_2 - \frac{1}{8}\langle:(\Delta\,\delta\tilde{\rho})^2:\rangle_2\right]. \tag{146}$$

Note that we have inserted the contribution of the density fluctuations inside the exponential factor, which is allowed at the order of the present calculation since this contribution is of order $\epsilon_{1,2}^2$.

What happens in the regime where a true condensate is present? Both phase and density fluctuations are small, so that the exponential function in Eq. (146) can be expanded to first order. We then express $\Delta\theta$ and $\Delta\tilde{\rho}$ in terms of the operator $\hat{\Lambda}$ and the operator $\hat{P}$. Since the Bogoliubov theory is usually considered in the canonical ensemble we remove the terms corresponding to the grand canonical fluctuations of the particle number. We then recover exactly the Bogoliubov prediction:

$$g_1^{\text{Bog}}(\mathbf{r}) = \Psi_c(\mathbf{r})\Psi_c(\mathbf{0}) + \langle\hat{\Lambda}^\dagger(\mathbf{r})\hat{\Lambda}(\mathbf{0})\rangle, \tag{147}$$

where $\Psi_c(\mathbf{r}) = \sqrt{N_0}\phi_0(\mathbf{r}) + \chi(\mathbf{r})/\sqrt{N_0}$ is the condensate field. Amazingly, the general formula (146) for quasicondensates can be related to the Bogoliubov formula in the following very simple way:

$$g_1(\mathbf{r}) = \sqrt{\rho(\mathbf{r})\rho(\mathbf{0})}\exp\left[\frac{g_1^{\text{Bog}}(\mathbf{r})}{\sqrt{\rho(\mathbf{r})\rho(\mathbf{0})}} - 1\right]. \tag{148}$$

IV. EXPLICIT RESULTS FOR THE SPATIALLY HOMOGENEOUS CASE

In this section we apply our approach to a spatially homogeneous Bose gas. The quasicondensate density is then uniform:

$$\rho_0(\mathbf{r}) = \frac{N_0}{L^D} = \frac{\mu}{g_0}. \tag{149}$$

The Bogoliubov equations (50) can then be exactly solved for any dimension of space and lead to $u_k(r) = \bar{u}_k e^{ikr}/L^{D/2}$ and $v_k(r) = \bar{v}_k e^{ikr}/L^{D/2}$ with

$$\bar{u}_k - \bar{v}_k = \left[\frac{\hbar^2 k^2/2m + 2\mu}{\hbar^2 k^2/2m}\right]^{1/4}$$

and

$$\bar{u}_k + \bar{v}_k = \left[\frac{\hbar^2 k^2/2m}{\hbar^2 k^2/2m + 2\mu} \right]^{1/4}. \tag{150}$$

The corresponding eigenenergies are given by

$$\epsilon_k = \left[\frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} + 2\mu \right) \right]^{1/2}. \tag{151}$$

A. Equation of state

From the general expression (103) for the chemical potential of the gas we arrive in the thermodynamic limit at

$$\mu = \rho g_0 + g_0 \int_{\mathcal{D}} \frac{d\mathbf{k}}{(2\pi)^D} [(\bar{u}_k + \bar{v}_k)^2 n_k + \bar{v}_k(\bar{u}_k + \bar{v}_k)], \tag{152}$$

where $n_k = 1/[\exp(\beta \epsilon_k) - 1]$ is the mean occupation number of the Bogoliubov mode $\mathbf{k}$. $\mathcal{D} = [-\pi/l, \pi/l]^D$ is the square domain of integration in the $\mathbf{k}$ space. The integral over the wave vector $\mathbf{k}$ does not contain any infrared divergence for any dimension of space. However, the long wave vector behavior given by

$$\bar{v}_k(\bar{u}_k + \bar{v}_k) \simeq -\frac{m\mu}{\hbar^2 k^2} \tag{153}$$

gives rise to an integral convergent in 1D and divergent in 2D and 3D in the $l \to 0$ limit. This gives the impression that the result depends strongly on l. The solution of this paradox comes from the link between the bare coupling constant g_0 of the model potential in the discretized space and the low-energy two-body scattering properties of the exact potential in the continuous space. This gives to g_0 in two and three dimensions a dependence in l so that our expression for μ does not depend on l anymore in the $l \to 0$ limit. In one dimension, the bare coupling g_0 is simply equal to the actual coupling strength g for $l \to 0$ and there is no divergence. At $T = 0$, Eq. (152) leads to

$$\mu = g\rho \left(1 - \frac{1}{\pi\rho\xi} \right), \tag{154}$$

where ξ is the healing length defined in Eq. (23). This agrees with the result of Lieb and Liniger in the weak interaction limit [28]. In three dimensions, we refer to the Appendix of [26] where the calculation has been done. One finds

$$g_0 = \frac{g}{1 - g \int_{\mathcal{D}} [d\mathbf{k}/(2\pi)^3](m/\hbar^2 k^2)}. \tag{155}$$

g is the usual 3D coupling strength given by

$$g = \frac{4\pi\hbar^2 a}{m}, \tag{156}$$

where a is the exact potential scattering length. A more explicit form of Eq. (155) is

$$g_0 = \frac{g}{1 - \mathcal{K}a/l}, \tag{157}$$

where $\mathcal{K} = 2.442 \dots$. It has to be noted that the difference between g_0 and g is still small in the validity domain of our approach since it is a second order correction in $\epsilon_{1,2}$: taking $l \sim \xi$ one finds $a/l \sim 1/\rho l^3$. Replacing the first factor g_0 in Eq. (152) with the formula (155) expanded up to second order in $\epsilon_{1,2}$ gives

$$\mu = \rho g + g_0 \int_{\mathcal{D}} \frac{d\mathbf{k}}{(2\pi)^3} \left((\bar{u}_k + \bar{v}_k)^2 n_k + \bar{v}_k(\bar{u}_k + \bar{v}_k) + \frac{m\mu}{\hbar^2 k^2} \right). \tag{158}$$

One can then safely take the $l \to 0$ limit. At $T = 0$, the integration gives:

$$\mu = g\rho \left(1 + \frac{32\sqrt{\pi}}{3} \sqrt{\rho a^3} \right) \tag{159}$$

which is the same result as Lee and Yang's [33]. In two dimensions, the low-energy two-body scattering of a general short range potential is described by a single length a also named the scattering length. In a continuous space, the T matrix can be calculated in the low-energy limit:

$$\langle \mathbf{k} | T(E + i\eta) | \mathbf{k}' \rangle \simeq -\frac{2\pi\hbar^2}{m[\ln(ak_0/2) + C - i\pi/2]}, \tag{160}$$

where $C = 0.57721 \dots$ is the Euler constant, a is the scattering length, $E = \hbar^2 k_0^2/m$, and $\eta \to 0^+$. We can also calculate the T matrix for the discrete δ potential defined by Eq. (27), which can also be expressed as

$$V = \frac{g_0}{l^2} |\mathbf{r} = 0\rangle \langle \mathbf{r} = 0|. \tag{161}$$

The general scattering theory gives the relations between the T matrix, the propagator G, and the free propagator G_0:

$$T = V + VGV, \tag{162}$$

$$G = G_0 + G_0 VG. \tag{163}$$

Using these relations and Eq. (161) for the potential, we find

$$\langle \mathbf{k} | T_{\text{grid}}(E + i\eta) | \mathbf{k}' \rangle = \frac{g_0}{1 - g_0 \langle \mathbf{r} = 0 | G_0(E + i\eta) | \mathbf{r} = 0 \rangle}. \tag{164}$$

The only term we need to calculate is the free propagator taken at the origin, which is conveniently performed with a Fourier transform:

$$\langle \mathbf{r}=0|G_0(E+i\eta)|\mathbf{r}=0\rangle = \int_{\mathcal{D}} \frac{d\mathbf{k}}{(2\pi)^2} \frac{1}{E+i\eta-\hbar^2 k^2/m}.$$
(165)

We split the square $\mathcal{D}$ into a disk of radius π/l and the complementary domain. Integration over the complementary domain gives simply a constant term in the low-energy limit $E \ll \hbar^2/ml^2$:

$$J \equiv \frac{2\pi\hbar^2}{m} \int_{\mathcal{D}-\text{disk}} \frac{d\mathbf{k}}{(2\pi)^2} \frac{1}{E+i\eta-\hbar^2 k^2/m}$$

$$\simeq -\frac{1}{2\pi} \int_{\mathcal{D}-\text{disk}} \frac{d\mathbf{k}}{k^2} = \frac{2G}{\pi} - \ln(2), \qquad (166)$$

where $G = 0.91596\ldots$ is the Catalan constant. The disk integration is straightforward and leads to the following expression for the T matrix:

$$\langle \mathbf{k}|T_{\text{grid}}(E+i\eta)|\mathbf{k}'\rangle$$

$$= \frac{1}{1/g_0 - (m/2\pi\hbar^2)\ln(lk_0/\pi) + im/4\hbar^2 - (m/2\pi\hbar^2)J}.$$
(167)

We now take $T_{\text{grid}}=T$, where T is approximated by Eq. (160), in order to reproduce the low-energy scattering properties of the exact potential. This leads to

$$\frac{1}{g_0} = \frac{m}{2\pi\hbar^2} \left[\ln\left(\frac{l}{\pi a}\right) - C + \frac{2G}{\pi} \right]. \qquad (168)$$

Note that the condition (26) has to be satisfied in our approach. In two dimensions, this gives $\hbar^2/mg_0 \gg 1$, or using Eq. (168)

$$\frac{1}{2\pi} \ln\left(\frac{l}{a}\right) \gg 1. \qquad (169)$$

We now show that the logarithmic dependence on l appearing in g_0, Eq. (168), exactly cancels the one appearing in the equation of state. Equation (152) can be rewritten as

$$\rho = \frac{\mu}{g_0} - \int_{\mathcal{D}} \frac{d\mathbf{k}}{(2\pi)^2} [(\bar{u}_k + \bar{v}_k)^2 n_k + \bar{v}_k(\bar{u}_k + \bar{v}_k)]. \qquad (170)$$

In the thermal part, one can immediately take the $l \to 0$ limit. In order to calculate the integral corresponding to the $T=0$ case, we use the same technique as for the calculation of g_0: the integration is done on a disk domain of radius π/l and we keep as a correction the integration over the complementary domain. The complementary domain integration is done by using the high-wave-vector behavior of $\bar{v}_k(\bar{u}_k + \bar{v}_k)$, Eq. (153). This leads to

$$-\int_{\mathcal{D}} \frac{d\mathbf{k}}{(2\pi)^2} \bar{v}_k(\bar{u}_k + \bar{v}_k) = \frac{m\mu}{4\pi\hbar^2} \left[\ln\left(\frac{\pi^2\hbar^2}{ml^2\mu}\right) - 1 - 2J \right].$$
(171)

Using Eqs. (168) and (171) in Eq. (170), we arrive at an implicit equation of state:

$$\rho = \frac{m\mu}{4\pi\hbar^2} \ln\left(\frac{4\hbar^2}{a^2 m\mu e^{2C+1}}\right) - \int \frac{d\mathbf{k}}{(2\pi)^2} (\bar{u}_k + \bar{v}_k)^2 n_k. \qquad (172)$$

Remarkably, this is identical to the result (20.45) obtained by the functional integral method in [7]. At $T=0$, one can show from the condition $\rho\xi^2 \gg 1$ [see Eq. (26)] that the validity condition of our approach is $\ln(1/\rho a^2) \gg 4\pi$. If one approximately inverts Eq. (172), neglecting constant terms and $\ln[\ln(1/\rho a^2)]$ with respect to $\ln(1/\rho a^2)$, one recovers Schick's formula [34].

B. Are density and gradient-of-phase fluctuations small?

As mentioned in Sec. II B, our approach relies in particular on two assumptions: the assumption that the relative density fluctuation ϵ_1 is small, and the assumption that the phase variation ϵ_2 between two neighboring points of the grid is small.

Let us consider first the relative density fluctuations. Because of Eq. (119), their mean square value can be separated into two parts:

$$\epsilon_1^2 = \frac{\langle \delta\hat{\rho}^2(\mathbf{0})\rangle_2}{\rho_0^2} = \frac{1}{\rho_0 l^D} + \frac{\langle :\delta\hat{\rho}^2(\mathbf{0}):\rangle_2}{\rho_0^2}, \qquad (173)$$

where we have neglected $1/L^D$ with respect to $1/l^D$ in the thermodynamic limit. The second term in Eq. (173), involving the normal order, is expressed in terms of the $\bar{u}_k, \bar{v}_k$ in the thermodynamic limit as

$$\frac{\langle :\delta\hat{\rho}^2(\mathbf{0}):\rangle_2}{\rho_0^2} = \frac{2}{\rho_0} \int_{\mathcal{D}} \frac{d\mathbf{k}}{(2\pi)^D} [(\bar{u}_k + \bar{v}_k)^2 n_k + \bar{v}_k(\bar{u}_k + \bar{v}_k)], \qquad (174)$$

where the integration domain is $\mathcal{D} = [-\pi/l, \pi/l]^D$. At zero temperature one introduces the change of variable $\mathbf{q} = \mathbf{k}\xi$ in the integral: one finds that Eq. (174) is of the order of $1/\rho_0\xi$ in 1D, of the order of $\ln(\xi/l)/\rho_0\xi^2$ in 2D, and of the order of $1/\rho_0\xi^2 l$ in 3D. Since $l < \xi$ the second term in Eq. (173) is dominated by the first term, and one has indeed

$$\epsilon_1^2 \simeq \frac{1}{\rho_0 l^D}. \qquad (175)$$

At finite temperature we have to calculate the thermal contribution to Eq. (174) involving the occupation number n_k.

At a temperature $k_B T < \mu$ we use the low-momentum expansion of $\bar{u}_k + \bar{v}_k$, and ϵ_k and we find that the thermal contribution is $(k_B T/\mu)^{D+1}(l/\xi)^D$ times smaller than $1/\rho_0 l^D$.

At a temperature $k_B T > \mu$, that is, $\lambda < \xi$, the treatment depends on the dimension of space. In 1D the main contribution to the integral comes from the domain $\epsilon_k \sim \mu$, over which one can approximate the Bose formula by its low-energy limit $k_B T / \epsilon_k$. This leads to a normal ordered fluctuation (174) of the order of $k_B T / (\mu \rho_0 \xi)$. This is larger than $1/\rho_0 \lambda$ so that the condition $l < \lambda$ then no longer implies that the first term $1/\rho_0 l$ in Eq. (173) is the dominant one. For convenience, one can however adjust l to a value such that

$$\frac{1}{\rho_0 l} \sim \frac{k_B T}{\mu} \frac{1}{\rho_0 \xi}. \tag{176}$$

The condition for weak density fluctuations then becomes

$$\epsilon_1^2 \sim \frac{k_B T}{\mu} \frac{1}{\rho_0 \xi} \ll 1. \tag{177}$$

Using $\rho_0 \simeq \rho$ and $\mu \simeq g\rho$ we recover the condition already obtained in [5] with a pure classical field approach. Note that this condition can be rewritten as $\xi \ll l_c$ where the coherence length of the field will be defined in Eq. (187). In 2D both the low-energy domain $\epsilon_k < k_B T$ and the high-energy domain $\epsilon_k > k_B T$ have important contributions. In the low-energy domain we approximate the Bose law by its low-energy limit. In the high-energy domain we keep the full Bose law but, ϵ_k being then larger than μ, we approximate $\bar{u}_k + \bar{v}_k$ by unity and ϵ_k by $\hbar^2 k^2 / 2m$. This leads to a normal ordered fluctuation (174) of the order of $\ln(k_B T / \mu) k_B T / (\mu \rho_0 \xi^2)$, a quantity that is larger than $1/\rho_0 \lambda^2$. As in 1D we therefore adjust l so that

$$\epsilon_1^2 \sim \frac{1}{\rho_0 l^2} \sim \frac{k_B T}{\mu} \ln\left(\frac{k_B T}{\mu}\right) \frac{1}{\rho_0 \xi^2}. \tag{178}$$

In 3D the high-energy domain $\epsilon_k > k_B T$ gives the dominant contribution so that the normal ordered expectation value (174) scales as $1/\rho_0 \lambda^3$. This is dominated by the first term in Eq. (173) so that the estimate (175) applies as soon as $l < \lambda, \xi$.

Let us consider now the condition that the mean squared phase change over a grid cell,

$$\epsilon_2^2 = \langle (l \nabla \hat{\theta})^2 \rangle_2 = \frac{l^2}{2\rho_0} \int_{\mathcal{D}} \frac{d\mathbf{k}}{(2\pi)^D} k^2 (\bar{u}_k - \bar{v}_k)^2 (n_k + 1/2), \tag{179}$$

is much smaller than unity. The presence of the factor k^2 inside the integral, coming from the action of ∇, has the consequence that the contribution to the integral is dominated by the high energy domain. At zero temperature one can replace $\bar{u}_k - \bar{v}_k$ by unity since the integral is dominated by wave vectors $k \sim 1/l > 1/\xi$. This leads to

$$\epsilon_2^2 \sim \frac{1}{\rho_0 l^D} \tag{180}$$

whatever the dimension D.

At a temperature $k_B T < \mu$ we estimate the thermal contribution by replacing $\bar{u}_k - \bar{v}_k$ and ϵ_k by their low-momentum approximations: the thermal contribution is then $(l/\lambda)^{2+D} (\xi/\lambda)^D$ times smaller than the zero-temperature result (180) and is therefore negligible since $l < \xi < \lambda$.

At a temperature $k_B T > \mu$ we use the high-energy approximation, replacing $\bar{u}_k - \bar{v}_k$ by unity and ϵ_k by $\hbar^2 k^2 / 2m$. Note that this works even in 1D because of the presence of the k^2 factor in the integral (179). This leads to a thermal contribution which is $(l/\lambda)^{2+D}$ times smaller than the zero-temperature contribution (180), and which is negligible since $l < \lambda$.

We conclude that the small parameter ϵ_2 of the theory, ensuring that there is a weak phase variation over a grid cell, is always given by Eq. (180) provided that the conditions (22), (25), and (26) are satisfied.

One may wonder if the corrections of the mean density due to the interaction H_3 between the Bogoliubov modes lead to an extra validity condition of our treatment. For the considered case of a spatially homogeneous gas it turns out that the answer to this question is no. One has indeed the remarkable identity in the thermodynamic limit

$$\frac{1}{\rho_0} \langle \delta \hat{\rho}(\mathbf{r}) \rangle_3 = -\frac{1}{2\rho_0^2} \langle : \delta \hat{\rho}(\mathbf{r})^2 : \rangle_2. \tag{181}$$

If the relative density fluctuations are weak, the relative correction to the density will also be weak.

To end this subsection we discuss briefly the second order correlation function of the field $g_2(\mathbf{r})$. Restricting the general formula (121) to the spatially homogeneous case in the thermodynamic limit, we obtain

$$g_2(\mathbf{r}) = \rho^2 + 2\rho \int_{\mathcal{D}} \frac{d\mathbf{k}}{(2\pi)^D} [(\bar{u}_k + \bar{v}_k)^2 n_k + \bar{v}_k(\bar{u}_k + \bar{v}_k)] \cos(\mathbf{k} \cdot \mathbf{r}). \tag{182}$$

Limiting cases of this general formula can be compared to existing results in the literature. At zero temperature for a 1D Bose gas one gets for $r = 0$,

$$g_2(0) = \rho^2 \left(1 - \frac{2}{\pi \rho \xi}\right). \tag{183}$$

This formula can be checked from [28]: the mean interaction energy per particle v is equal to $g_2(0)$ multiplied by $g/2\rho$, and v can be calculated in the weakly interacting regime by combining (3.29) of [28] (relating v to the derivative of the ground state energy with respect to g) and (4.2) of [28] (giving the ground state energy in the Bogoliubov approximation). This exactly leads to Eq. (183). This prediction for $g_2(0)$ also appears in [35].

C. First order correlation function

Because of the general formula (148) the first order correlation function of the field for the quasicondensate is immediately related to that of the Bogoliubov theory, here in the thermodynamic limit

$$\ln[g_1(\mathbf{r})/\rho] = \frac{g_1^{\text{Bog}}(\mathbf{r})}{\rho} - 1$$

$$= -\frac{1}{\rho} \int \frac{d^D k}{(2\pi)^D} [(\bar{u}_k^2 + \bar{v}_k^2) n_k + \bar{v}_k^2]$$

$$\times (1 - \cos \mathbf{k} \cdot \mathbf{r}). \tag{184}$$

We have also taken here the continuous limit $l \to 0$, which does not lead to any divergence.

We concentrate our analysis on the 1D case and we make the link with existing results in the literature. These existing results deal with the asymptotic behavior of g_1 for large r, where r is the absolute value of the spatial coordinate. At zero temperature, we find for $r \gg \xi$:

$$g_1(r) \simeq \rho \left(\frac{r_1}{r} \right)^{1/2\pi\rho\xi}, \tag{185}$$

with $r_1 = e^{2-C} \xi/4 \simeq 1.037\xi$ where $C = 0.577\,21 \ldots$ is Euler's constant [36]. This reproduces a result obtained in a nonexplicit way in [37]. At a finite temperature, g_1/ρ is the exponential of an integral of the form

$$\int_0^{+\infty} \frac{A(k)}{k^2} [1 - \cos(kr)], \tag{186}$$

where the function $A(k)$ is a regular and even function of k, therefore behaving quadratically with k around $k = 0$ [38]. Writing $A(k)$ as $[A(k) - A(0)] + A(0)$ and splitting the integral, accordingly one obtains for r much larger than both ξ and λ,

$$\ln[g_1(r)/\rho] = \frac{r}{l_c} + K + o(1/r^n), \tag{187}$$

where the coherence length $l_c = \rho\lambda^2/\pi$ coincides with the one of [39] and the constant K is given by

$$K = \int_0^{+\infty} \frac{A(k) - A(0)}{k^2}. \tag{188}$$

Since $A(k)$ is even one can show by repeated integration by parts that the remainder in Eq. (187) tends to 0 faster than any power law, contrary to what is stated in [39].

Of course our formula gives access to g_1 for any value of the distance. This is illustrated in Fig. 1 where we have plotted the logarithm of g_1 as function of r/ξ for various temperatures.

As a consequence, we can, for example, calculate the momentum distribution of the atoms:
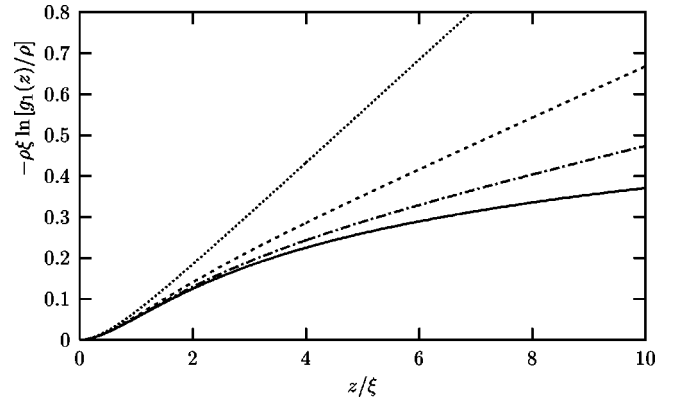

FIG. 1. First order correlation function of the field $g_1(z)$ for a repulsive 1D Bose gas in the thermodynamic limit. The different curves correspond to various ratios of the temperature to the chemical potential: $k_B T/\mu = 0$ (solid line), 1/15 (dot-dashed line), 1/8 (dashed line), 1/4 (dotted line). We plot the logarithm of $g_1(z)$ multiplied by the parameter $\rho\xi$, where ρ is the 1D spatial density and $\xi = \hbar/\sqrt{m\mu}$ is the healing length, so that we obtain a quantity depending only on z/ξ and $k_B T/\mu$ in the weakly interacting limit.

$$\Pi(p) = 2 \int_0^{+\infty} dr\, g_1(r) \cos(pr/\hbar) \tag{189}$$

normalized here as $\int dp\, \Pi(p) = 2\pi\hbar\rho$ so that $\Pi(p)$ is dimensionless. This is illustrated in Fig. 2 where we have plotted the momentum distribution for various temperatures and for $\rho\xi = 10$. Using integration by parts we can show that the behavior of Π for large p is related to the fact that the third order derivative of g_1 in $r = 0^+$ does not vanish:

$$\Pi(p) \sim \frac{2\hbar^4 g_1^{(3)}(0^+)}{p^4} \quad \text{with} \quad g_1^{(3)}(0^+) = \mu^2 m^2/(2\hbar^4). \tag{190}$$
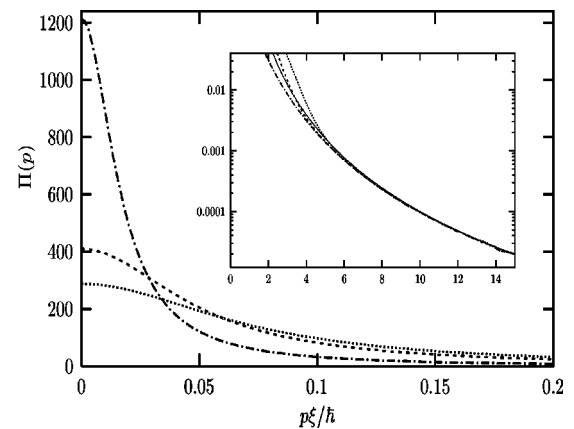

FIG. 2. Momentum distribution of a repulsive 1D Bose gas in the thermodynamic limit. $\Pi(p)$ is normalized as $\int dp\, \Pi(p) = 2\pi\hbar\rho$ where ρ is the 1D spatial density so that $\Pi(p)$ is dimensionless. We plot $\Pi(p)$ as a function of $p\xi/\hbar$ for various ratios of the temperature to the chemical potential: $k_B T/\mu = 1/3$ (dot-dashed lines), 1 (dashed lines), 10/7 (dotted lines). We have taken $\rho\xi = 10 \gg 1$ where $\xi = \hbar/\sqrt{m\mu}$ is the healing length. The solid line is the large p limit: $(\hbar/p\xi)^4$. The inset is a magnification.

This prediction, valid at zero or finite temperature, agrees with the weak interaction limit of a recently obtained exact result based on the Bethe ansatz [40]. At zero temperature we find that the momentum distribution diverges at $p=0$ as

$$\Pi(p) \sim \frac{\hbar \rho \pi \nu}{p} (r_1 p/\hbar)^\nu, \tag{191}$$

where $\nu = 1/(2\pi \rho \xi) \ll 1$.

CONCLUSION

We have studied the thermal equilibrium of weakly interacting degenerate Bose gases in the regime of weak density fluctuations, the so-called quasicondensate regime. The method can be considered as a Bogoliubov method in the density-phase representation of the field operator.

In the first step one discretizes the real space in cells of size l: l is small enough that the macroscopic properties of the gas are not affected by the discretization, and l large enough that each cell contains on the average a large number of particles. The macroscopic occupation of each cell allows one to give a precise definition of the phase operator, following the method of Girardeau and Arnowitt [18].

In a second step one performs a systematic expansion of the full Hamiltonian in terms of two small parameters, the relative density fluctuations inside a cell and the phase change over a grid cell. This procedure leads to an exact expansion of the observables of the gas in the regime of weak interactions and low density fluctuations, in 1D, 2D, and 3D. In particular, it is free of any ultraviolet or infrared divergences and exactly matches the usual Bogoliubov predictions when the gas contains a true Bose-Einstein condensate.

As a first application of the general formalism, we have given in this paper formulas for the equation of state of the gas, the ground state energy, and the first order and second order correlation functions of the field. We have applied these formulas to the spatially homogeneous case in 1D, 2D, and 3D, recovering in this way known results, but obtaining also other results, like the full position dependence of the first order correlation function of the field.

ACKNOWLEDGMENTS

APPENDIX A: EXPANSION OF THE HAMILTONIAN

As explained in Sec. II B we expand the Hamiltonian (28) up to third order in powers of the small parameters ϵ_1 and ϵ_2 defined in Eqs. (33) and (35). This will produce terms $H^{(n_1,n_2)}$ of order $\epsilon_1^{n_1} \epsilon_2^{n_2}$ with $n_1+n_2 \leq 3$. The expansion of the potential energy part H_{pot} defined in Eq. (30) is very simple as it involves only the operator giving the density.

The only point is to realize that the term $1/l^D$ is ϵ_1^2 times smaller than the zeroth order density ρ_0. This leads to

$$H_{\text{pot}}^{(0,0)} = \sum_{\mathbf{r}} l^D \rho_0 \left[U(\mathbf{r}) - \mu + \frac{g_0}{2} \rho_0 \right], \tag{A1}$$

$$H_{\text{pot}}^{(1,0)} = \sum_{\mathbf{r}} l^D \delta\hat{\rho} [U(\mathbf{r}) - \mu + g_0 \rho_0], \tag{A2}$$

$$H_{\text{pot}}^{(2,0)} = \sum_{\mathbf{r}} l^D \frac{g_0}{2} \left[\delta\hat{\rho}^2 - \frac{\rho_0}{l^D} \right], \tag{A3}$$

$$H_{\text{pot}}^{(3,0)} = -\sum_{\mathbf{r}} \frac{g_0}{2} \delta\hat{\rho}. \tag{A4}$$

The expansion of the kinetic energy part (31) is more complicated as it involves also the phase operator $\hat{\theta}$, which, furthermore, does not commute with $\delta\hat{\rho}$. An expression slightly more convenient than Eq. (31) can be given for the kinetic energy. Because of the periodic boundary conditions one can freely shift the summation variable in the term of Eq. (31) involving $\hat{\rho}_{-j}$, so that

$$H_{\text{kin}} = -\frac{\hbar^2}{2ml^2} \sum_{\mathbf{r},j} l^D \{ [\sqrt{\hat{\rho}} e^{i(\hat{\theta}_{+j} - \hat{\theta})} \sqrt{\hat{\rho}_{+j}} + \text{H.c.}] - 2\hat{\rho} \}. \tag{A5}$$

The calculation to zeroth order in ϵ_2 can be done first easily: using the expansion (37) to zeroth order, we get from Eq. (31) to all orders in ϵ_1,

$$H_{\text{kin}}^{(\leq +\infty,0)} = -\frac{\hbar^2}{2m} \sum_{\mathbf{r}} l^D \sqrt{\hat{\rho}} \Delta \sqrt{\hat{\rho}}. \tag{A6}$$

This involves a function of $\hat{\rho}$ only that it is easily expanded in powers of ϵ_1 using Eq. (34). A simplification occurs after summation over the lattice, as the matrix Δ is symmetric for the considered periodic boundary conditions:

$$\sum_{\mathbf{r}} u \Delta v = \sum_{\mathbf{r}} (\Delta u) v, \tag{A7}$$

where u and v are arbitrary functions on the lattice. This leads to

$$H_{\text{kin}}^{(0,0)} = -\frac{\hbar^2}{2m} l^D \sum_{\mathbf{r}} \sqrt{\rho_0} \Delta \sqrt{\rho_0}, \tag{A8}$$

$$H_{\text{kin}}^{(1,0)} = -\frac{\hbar^2}{2m} l^D \sum_{\mathbf{r}} \frac{\delta\hat{\rho}}{\sqrt{\rho_0}} \Delta \sqrt{\rho_0}, \tag{A9}$$

$$H_{\text{kin}}^{(2,0)} = -\frac{\hbar^2}{2m} l^D \sum_{\mathbf{r}} \left[\frac{\delta\hat{\rho}}{4\sqrt{\rho_0}} \Delta \frac{\delta\hat{\rho}}{\sqrt{\rho_0}} - \frac{\delta\hat{\rho}^2}{4\rho_0^{3/2}} \Delta \sqrt{\rho_0} \right], \tag{A10}$$

$$H_{\text{kin}}^{(3,0)} = -\frac{\hbar^2}{2m} l^D \sum_{\mathbf{r}} \left[\frac{1}{8} \frac{\delta\hat{\rho}^3}{\rho_0^{5/2}} \Delta \sqrt{\rho_0} - \frac{1}{8} \frac{\delta\hat{\rho}^2}{\rho_0^{3/2}} \Delta \frac{\delta\hat{\rho}}{\rho_0^{1/2}} \right].$$
(A11)

The second order term of vanishing order in ϵ_1 is also immediately obtained:

$$H_{\text{kin}}^{(0,2)} = \frac{\hbar^2}{2ml^2} l^D \sum_{\mathbf{r},j} \sqrt{\rho_0 \rho_{0,+j}} (\hat{\theta}_{+j} - \hat{\theta})^2. \quad \text{(A12)}$$

The last second order quantity to calculate is $H_{\text{kin}}^{(1,1)}$, which is first order in ϵ_1 and first order in ϵ_2. There are four terms, two involving $\hat{\theta}_{+j}$ and two being their Hermitian conjugates. One can then collect the terms to form commutators:

$$H_{\text{kin}}^{(1,1)} = -\frac{\hbar^2}{2ml^2} l^D \sum_{\mathbf{r},j} \frac{i}{2} \left(\frac{\rho_0}{\rho_{0,+j}} \right)^{1/2} [\hat{\theta}_{+j} - \hat{\theta}, \delta\hat{\rho}_{+j}]$$

$$- \frac{i}{2} \left(\frac{\rho_{0,+j}}{\rho_0} \right)^{1/2} [\hat{\theta}_{+j} - \hat{\theta}, \delta\hat{\rho}]. \quad \text{(A13)}$$

$$= -\frac{\hbar^2}{4ml^2} \sum_{\mathbf{r},j} \left[\left(\frac{\rho_{0,+j}}{\rho_0} \right)^{1/2} + \left(\frac{\rho_0}{\rho_{0,+j}} \right)^{1/2} \right],$$
(A14)

where we have used the commutation relation of $\hat{\rho}$ and $\hat{\theta}$ [see Eq. (20)].

We collect all the second order c-number contributions to the Hamiltonian H in a single energy functional of the density profile of the quasicondensate,

$$E_2[\rho_0] = -\frac{g_0}{2} \sum_{\mathbf{r}} \rho_0 - \frac{\hbar^2}{4ml^2} \sum_{\mathbf{r},j} \left[\left(\frac{\rho_{0,+j}}{\rho_0} \right)^{1/2} + \left(\frac{\rho_0}{\rho_{0,+j}} \right)^{1/2} \right].$$
(A15)

The technique used to calculate $H_{\text{kin}}^{(1,1)}$ can be extended to the calculation of $H_{\text{kin}}^{(2,1)}$. There are now three terms and their Hermitian conjugates. Two of these terms, when combined with their Hermitian conjugates, form a commutator that is calculated according to Eq. (20). The third term and its Hermitian conjugate involve the expression

$$\delta\hat{\rho}(\hat{\theta}_+ - \hat{\theta}) \delta\hat{\rho}_+ - \delta\hat{\rho}_+(\hat{\theta}_+ - \hat{\theta}) \delta\hat{\rho}$$

$$= \delta\hat{\rho}[\hat{\theta}_+, \delta\hat{\rho}_+] - [\delta\hat{\rho}, \hat{\theta}] \delta\hat{\rho}_+, \quad \text{(A16)}$$

which is a sum of two commutators, easy to evaluate. This leads to

$$H_{\text{kin}}^{(2,1)} = \frac{\hbar^2}{8m} \sum_{\mathbf{r}} \frac{\delta\hat{\rho}}{\rho_0} (\rho_0^{-1/2} \Delta \rho_0^{1/2} - \rho_0^{1/2} \Delta \rho_0^{-1/2}). \quad \text{(A17)}$$

To calculate $H_{\text{kin}}^{(1,2)}$ we first evaluate

$$H_{\text{kin}}^{(\leqslant+\infty,2)} = \frac{\hbar^2}{4ml^2} \sum_{\mathbf{r},j} l^D [\sqrt{\hat{\rho}} (\hat{\theta}_{+j} - \hat{\theta})^2 \sqrt{\hat{\rho}_{+j}} + \text{H.c.}]$$
(A18)

and we expand to first order in $\delta\hat{\rho}$, which leads to a sum of terms that are not individually Hermitian. We then use the commutation relation (20) to produce Hermitian terms, e.g.,

$$\delta\hat{\rho}(\hat{\theta}_{+j} - \hat{\theta})^2 = (\hat{\theta}_{+j} - \hat{\theta}) \delta\hat{\rho}(\hat{\theta}_{+j} - \hat{\theta}) - \frac{i}{l^D}(\hat{\theta}_{+j} - \hat{\theta}).$$
(A19)

The last term of the right-hand side of this expression is anti-Hermitian and does not contribute to the final result

$$H_{\text{kin}}^{(1,2)} = \frac{\hbar^2}{4ml^2} \sum_{\mathbf{r},j} l^D (\hat{\theta}_{+j} - \hat{\theta}) \left(\frac{\rho_{0,+j}^{1/2}}{\rho_0^{1/2}} \delta\hat{\rho} + \frac{\rho_0^{1/2}}{\rho_{0,+j}^{1/2}} \delta\hat{\rho}_{+j} \right)$$

$$\times (\hat{\theta}_{+j} - \hat{\theta}). \quad \text{(A20)}$$

Finally, $H_{\text{kin}}^{(0,1)}$ and $H_{\text{kin}}^{(0,3)}$ vanish as the odd order expansion of $\exp[i(\hat{\theta}_{+j} - \hat{\theta})]$ is anti-Hermitian.

APPENDIX B: CORRECTIONS TO THE EQUATIONS OF MOTION DUE TO H_3

The Hamiltonian H_3 gives rise to quadratic corrections to the equations of motion for $\delta\hat{\rho}$ and $\hat{\theta}$. In this appendix, these corrections are calculated explicitly and the thermal average is taken over the equations of motion with the Hamiltonian $H_2 + H_3$ for the linear part and the Hamiltonian H_2 for the quadratic corrections. This allows us to calculate the first correction to the mean density due to H_3.

The corrections to the equation of motion for the density fluctuations are given by

$$\hbar \partial_t \delta\hat{\rho}|_{H_3} = \frac{\hbar^2}{4ml^2} \sum_j \left[\left\{ \hat{\theta} - \hat{\theta}_{+j}, \left(\frac{\rho_{0,+j}}{\rho_0} \right)^{1/2} \delta\hat{\rho} + \left(\frac{\rho_0}{\rho_{0,+j}} \right)^{1/2} \delta\hat{\rho}_{+j} \right\} + (+j \leftrightarrow -j) \right], \quad \text{(B1)}$$

where $\{A, B\}$ stands for the anticommutator $AB + BA$ of two operators. When we take the average with the Hamiltonian H_2, we use the explicit modal expansion of $\delta\hat{\rho}$ and $\hat{\theta}$ given by Eq. (56). The operator $\hat{Q}$ disappears since Eq. (B1) involves only differences of $\hat{\theta}$. Terms with $\hat{P}$ also disappear since $\langle \hat{P} \rangle_2 = 0$. The expectation value of the product $\hat{\theta}(\mathbf{r}) \delta\hat{\rho}(\mathbf{r}')$, where $\hat{\theta}$ is written without $\hat{Q}$ and $\delta\hat{\rho}$ is written without $\hat{P}$, is actually purely imaginary: as u_s and v_s can be chosen to be real, $\theta_s^* = -\theta_s$ [see Eq. (57)]. Since $\partial_t \langle \delta\hat{\rho} \rangle$ is real, all imaginary contributions to it have to cancel so that the corrections to the motion of $\langle \delta\hat{\rho} \rangle$ due to H_3 finally vanish when we take the thermal average:

$$\hbar \partial_t \langle \delta\hat{\rho} \rangle|_{H_3} = 0. \quad \text{(B2)}$$

The corrections to the equation of motion for $\hat{\theta}$ are more involved:

$$\hbar\partial_t\hat{\theta}|_{H_3} = \frac{1}{2\sqrt{\rho_0}}\left[-\frac{\hbar^2}{4m\rho_0}\delta\hat{\rho}\Delta\left(\frac{\delta\hat{\rho}}{\sqrt{\rho_0}}\right) + \frac{3\hbar^2}{8m\rho_0^2}\delta\hat{\rho}^2\Delta(\sqrt{\rho_0})\right.$$

$$-\frac{\hbar^2}{8m}\Delta\left(\frac{\delta\hat{\rho}^2}{\rho_0^{3/2}}\right) - \frac{\hbar^2}{2ml^2}$$

$$\times\sum_j\left[\sqrt{\rho_{0,+j}}(\hat{\theta}_{+j}-\hat{\theta})^2\right.$$

$$+\sqrt{\rho_{0,-j}}(\hat{\theta}_{-j}-\hat{\theta})^2] + \frac{g_0\sqrt{\rho_0}}{l^D}$$

$$\left. -\frac{\hbar^2}{4ml^D\sqrt{\rho_0}}(\rho_0^{-1/2}\Delta\rho_0^{1/2}-\rho_0^{1/2}\Delta\rho_0^{-1/2})\right]. \quad (B3)$$

Fortunately, we can use the linear equations of motion (46),(47) to significantly simplify the above equation of motion. We rewrite the first term in square brackets of Eq. (B3) as

$$-\frac{\hbar^2}{4m\rho_0}\delta\hat{\rho}\Delta\left(\frac{\delta\hat{\rho}}{\sqrt{\rho_0}}\right)$$

$$=-\frac{\delta\hat{\rho}}{2\rho_0}\left[(U-\mu+3g_0\rho_0)\left(\frac{\delta\hat{\rho}}{\sqrt{\rho_0}}\right)+2\sqrt{\rho_0}\hbar\partial_t\hat{\theta}\right].$$

$$(B4)$$

The second term in square brackets of Eq. (B3) gives, as $\sqrt{\rho_0}$ solves the Gross-Pitaevskii equation,

$$\frac{3\hbar^2}{8m\rho_0^2}\delta\hat{\rho}^2\Delta(\sqrt{\rho_0}) = \frac{3\delta\hat{\rho}^2}{4\rho_0^{3/2}}(U-\mu+g_0\rho_0). \quad (B5)$$

The sum of Eqs. (B4) and (B5) and the third term in square brackets of Eq. (B3) leads to

$$\frac{1}{4}\left(-\frac{\hbar^2\Delta}{2m}+U-\mu+g_0\rho_0\right)\left(\frac{\delta\hat{\rho}^2}{\rho_0^{3/2}}\right) - g_0\frac{\delta\hat{\rho}^2}{\sqrt{\rho_0}} - \hbar(\partial_t\hat{\theta})\frac{\delta\hat{\rho}}{\sqrt{\rho_0}}.$$

$$(B6)$$

To rewrite the fourth term in square brackets of Eq. (B3), it is convenient to use the following identity:

$$\sum_j\sqrt{\rho_{0,+j}}(\hat{\theta}_{+j}-\hat{\theta})^2 + \sqrt{\rho_{0,-j}}(\hat{\theta}_{-j}-\hat{\theta})^2$$

$$+2\hat{\theta}[\sqrt{\rho_{0,+j}}(\hat{\theta}_{+j}-\hat{\theta}) + \sqrt{\rho_{0,-j}}(\hat{\theta}_{-j}-\hat{\theta})]$$

$$=\sum_j\sqrt{\rho_{0,+j}}(\hat{\theta}_{+j}^2-\hat{\theta}^2) + \sqrt{\rho_{0,-j}}(\hat{\theta}_{-j}^2-\hat{\theta}^2)$$

$$=l^2[\Delta(\sqrt{\rho_0}\hat{\theta}^2) - \hat{\theta}^2\Delta(\sqrt{\rho_0})], \quad (B7)$$

leading to

$$\sum_j\sqrt{\rho_{0,+j}}(\hat{\theta}_{+j}-\hat{\theta})^2 + \sqrt{\rho_{0,-j}}(\hat{\theta}_{-j}-\hat{\theta})^2$$

$$=l^2[\hat{\theta}^2\Delta\sqrt{\rho_0} - 2\hat{\theta}\Delta(\sqrt{\rho_0}\hat{\theta}) + \Delta(\sqrt{\rho_0}\hat{\theta}^2)]. \quad (B8)$$

Using this equality, the Gross-Pitaevskii equation (41), and the equation of motion (47), the fourth term in square brackets of Eq. (B3) can be written as

$$-\frac{\hbar^2}{2ml^2}\sum_j[\sqrt{\rho_{0,+j}}(\hat{\theta}_{+j}-\hat{\theta})^2 + \sqrt{\rho_{0,-j}}(\hat{\theta}_{-j}-\hat{\theta})^2]$$

$$=\left(-\frac{\hbar^2}{2m}\Delta+U-\mu+g_0\rho_0\right)(\sqrt{\rho_0}\hat{\theta}^2) - \hbar\hat{\theta}\frac{\partial_t\delta\hat{\rho}}{\sqrt{\rho_0}}.$$

$$(B9)$$

The sixth (and last) term in square brackets of Eq. (B3) can also be transformed using the Gross-Pitaevskii equation (41):

$$-\frac{\hbar^2}{4ml^D\sqrt{\rho_0}}(\rho_0^{-1/2}\Delta\rho_0^{1/2}-\rho_0^{1/2}\Delta\rho_0^{-1/2})$$

$$=-\left(-\frac{\hbar^2}{2m}\Delta+U-\mu+g_0\rho_0\right)\left(\frac{1}{2l^D\sqrt{\rho_0}}\right). \quad (B10)$$

This leads finally to a rewriting of the thermal average of Eq. (B3) as

$$2\sqrt{\rho_0}\hbar\langle\partial_t\hat{\theta}\rangle|_{H_3} = \left(-\frac{\hbar^2}{2m}\Delta+U-\mu+g_0\rho_0\right)$$

$$\times\left(\frac{\langle\delta\hat{\rho}^2\rangle_2}{4\rho_0^{3/2}} + \sqrt{\rho_0}\langle\hat{\theta}^2\rangle_2 - \frac{1}{2l^D\sqrt{\rho_0}}\right)$$

$$-g_0\frac{\langle\delta\hat{\rho}^2\rangle_2-\rho_0/l^D}{\sqrt{\rho_0}} - \frac{\hbar\partial_t\langle\hat{\theta}\delta\hat{\rho}\rangle_2}{\sqrt{\rho_0}}.$$

$$(B11)$$

The last term of this expression can be calculated using Eq. (56). The harmonic modes do not contribute since the expectation value of products of $\hat{b}_s$ and $\hat{b}_s^\dagger$ is time independent. We are left with

$$\partial_t\langle\hat{Q}\hat{P}\rangle_2 = \partial_t\left\langle\hat{Q}(0)\hat{P} + t\frac{\mu_0'}{\hbar}\hat{P}^2\right\rangle_2 = \frac{\mu_0'}{\hbar}\langle\hat{P}^2\rangle_2, \quad (B12)$$

which gives

$$-\frac{\hbar\partial_t\langle\hat{\theta}\delta\hat{\rho}\rangle_2}{\sqrt{\rho_0}} = 2\mu_0'\langle\hat{P}^2\rangle_2\partial_{N_0}\sqrt{\rho_0}. \quad (B13)$$

As a conclusion, the quadratic correction to the first equation of motion can be written as in Eq. (60) if one uses the identities

$$\hat{B}^\dagger\hat{B} = \frac{\delta\hat{\rho}^2}{4\rho_0} + \rho_0\hat{\theta}^2 - \frac{1}{2l^D},$$

$$\frac{\delta\hat{\rho}^2}{\sqrt{\rho_0}} = \sqrt{\rho_0}(\hat{B}+\hat{B}^\dagger)^2$$

$$= \sqrt{\rho_0}\left(2\hat{B}^\dagger\hat{B}+\hat{B}^2+\hat{B}^{\dagger 2}+\frac{1}{l^D}\right). \quad (B14)$$

APPENDIX C: THE MEAN VALUE OF $\partial_t\hat{\theta}$ VANISHES AT EQUILIBRIUM

As the field degree of freedom $\hat{Q}$, that is, the global phase of the field, is not subject to a restoring force in H_2, it is not totally obvious that the perturbation H_3 cannot set it into permanent motion. We therefore check this point explicitly here.

The first step is to calculate the mean value of $\hat{P}$ to first order in H_3. We approximate the unnormalized density operator of the gas at thermal equilibrium to first order in H_3 using perturbation theory:

$$\sigma = e^{-\beta(H_2+H_3)} = e^{-\beta H_2} - \int_0^\beta d\tau e^{-(\beta-\tau)H_2}H_3 e^{-\tau H_2}+\cdots.$$
$$(C1)$$

$\hat{P}$ commutes with H_2 and has a vanishing mean value in the thermal state corresponding to H_2 so that, to first order in H_3,

$$\langle\hat{P}\rangle_3 = -\langle\beta\hat{P}H_3\rangle_2. \quad (C2)$$

The Hamiltonian H_3 is a polynomial of degree 3 in $\hat{P}$:

$$H_3 = A_0 + A_1\hat{P} + A_2\hat{P}^2 + A_3\hat{P}^3, \quad (C3)$$

where the A_i are still operators with respect to the harmonic oscillator variables b_s. This leads to

$$\langle\hat{P}\rangle_3 = -\beta[\langle A_1\rangle_2\langle\hat{P}^2\rangle_2 + \langle A_3\rangle_2\langle\hat{P}^4\rangle_2]. \quad (C4)$$

From Wick's theorem, $\langle\hat{P}^4\rangle_2 = 3\langle\hat{P}^2\rangle_2^2$.

In the second step we calculate $\langle d\hat{Q}/dt\rangle$ to first order in H_3:

$$\langle d\hat{Q}/dt\rangle \simeq \langle\partial_{\hat{P}}(H_2+H_3)\rangle_3 \simeq \mu_0'\langle\hat{P}\rangle_3 + \langle A_1\rangle_2$$
$$+ 3\langle A_3\rangle_2\langle\hat{P}^2\rangle_2, \quad (C5)$$

where the terms coming from $\partial_{\hat{P}}H_3$ are calculated in the thermal state for H_2 since they are already first order in the perturbation. From the value of $\langle\hat{P}\rangle_3$ obtained from Eq. (C4) and from Eq. (61) we obtain the desired result:

$$\langle d\hat{Q}/dt\rangle_3 = 0 \quad (C6)$$

to first order in H_3.

APPENDIX D: AN EQUATION FOR $\{\hat{\alpha},\hat{\Lambda}\}$

In this appendix, we derive the partial differential equation (91). We first note that $\hat{B}_n$, being a sum of eigenmodes of the operator $\mathcal{L}_{GP}$, obeys the differential equation for the evolution governed by H_2:

$$i\hbar\partial_t\begin{pmatrix}\hat{B}_n\\\hat{B}_n^\dagger\end{pmatrix} = \mathcal{L}_{GP}\begin{pmatrix}\hat{B}_n\\\hat{B}_n^\dagger\end{pmatrix}. \quad (D1)$$

We project this equation orthogonally to ϕ_0 and along ϕ_0, so that we get the quantum analog of Eqs. (E9) and (E10) of [26], with the simplification that $\phi_0(\mathbf{r})$ is real:

$$i\hbar\partial_t\begin{pmatrix}\hat{\Lambda}\\\hat{\Lambda}^\dagger\end{pmatrix} = \begin{pmatrix}\mathcal{Q}&0\\0&\mathcal{Q}\end{pmatrix}\mathcal{L}_{GP}\begin{pmatrix}\hat{\Lambda}\\\hat{\Lambda}^\dagger\end{pmatrix} + (\hat{\alpha}+\hat{\alpha}^\dagger)\begin{pmatrix}\mathcal{Q}g_0\rho_0\phi_0\\-\mathcal{Q}g_0\rho_0\phi_0\end{pmatrix},$$
$$(D2)$$

$$i\hbar\frac{d\hat{\alpha}}{dt} = l^D\sum_{\mathbf{r}} g_0\rho_0\phi_0(\hat{B}_n+\hat{B}_n^\dagger)$$

$$= l^D\sum_{\mathbf{r}} g_0\rho_0\phi_0(\hat{\Lambda}+\hat{\Lambda}^\dagger). \quad (D3)$$

We have introduced the projection matrix

$$\langle\mathbf{r}|\mathcal{Q}|\mathbf{r}'\rangle = \delta_{\mathbf{r},\mathbf{r}'} - l^D\phi_0(\mathbf{r})\phi_0(\mathbf{r}'). \quad (D4)$$

As $\hat{\alpha}$ is anti-Hermitian, the source term in Eq. (D2) vanishes and one can replace $\hat{B}_n$ by $\hat{\Lambda}$ in Eq. (D3).

We use these two equations of motion to calculate the first order time derivative of $A(\mathbf{r}) \equiv \langle\{\hat{\alpha},\hat{\Lambda}(\mathbf{r})\}\rangle_2$. We do not give the intermediate result. As A is real here, we have the property

$$\langle\{\hat{\alpha},\hat{\Lambda}^\dagger(\mathbf{r})\}\rangle_2 = -\langle\{\hat{\alpha},\hat{\Lambda}(\mathbf{r})\}\rangle_2. \quad (D5)$$

As $\hat{\Lambda}$ is orthogonal to ϕ_0 one has

$$\mathcal{Q}A = A. \quad (D6)$$

All this leads to Eq. (91).

APPENDIX E: INTERPRETATION OF χ IN THE NUMBER CONSERVING BOGOLIUBOV APPROACH

We assume here that the gas is a quasipure condensate so that ϕ_0 is now the condensate wave function in the Gross-Pitaevskii approximation. We then show that $\chi(\mathbf{r})/N_0$, where χ is defined in Eq. (94), essentially coincides with the lowest order deviation of the exact condensate wave function from the Gross-Pitaevskii prediction ϕ_0. This deviation was calculated in [15].

We split χ into a component orthogonal to ϕ_0 and a component collinear to ϕ_0:

$$\chi(\mathbf{r}) = \gamma\phi_0(\mathbf{r}) + \chi_\perp(\mathbf{r}). \quad (E1)$$

The component γ has a simple physical interpretation: we sum Eq. (95) over $\mathbf{r}$ after multiplication by l^D. If we omit the grand canonical term (absent in the canonical treatment of [15]) we obtain

$$N = N_0 + 2\gamma + \delta N, \tag{E2}$$

where

$$\delta N \equiv l^D \sum_{\mathbf{r}} \langle \hat{\Lambda}^\dagger \hat{\Lambda} \rangle_2 \tag{E3}$$

exactly coincides with the mean number of noncondensed particles predicted in [15]. The physical interpretation of 2γ is then simple:

$$\delta N_0 \equiv 2\gamma \tag{E4}$$

is the correction to apply to the pure condensate prediction for the number of condensate particles in order to recover the correct Bogoliubov prediction. Applying to Eq. (94) the matrix $\mathcal{Q}$ (D4) projecting orthogonally to ϕ_0 we obtain

$$\left[-\frac{\hbar^2}{2m}\Delta + U + g_0\rho_0 - \mu \right]\chi_\perp + 2\mathcal{Q}g_0\rho_0\chi_\perp$$
$$+ \mathcal{Q}\left(2g_0\rho_0\gamma\phi_0 + \frac{1}{2}S \right)$$
$$= 0. \tag{E5}$$

We modify slightly the form of the source term S, eliminating the anticommutator:

$$\{\hat{\Lambda}^\dagger(\mathbf{r}'), \hat{\Lambda}(\mathbf{r})\} = 2\hat{\Lambda}^\dagger(\mathbf{r}')\hat{\Lambda}(\mathbf{r}) + \frac{1}{l^D}\langle \mathbf{r}|\mathcal{Q}|\mathbf{r}'\rangle. \tag{E6}$$

This leads to the system

$$\begin{pmatrix} \mathcal{Q} & 0 \\ 0 & \mathcal{Q} \end{pmatrix}\mathcal{L}_{\text{GP}}\begin{pmatrix} \chi_\perp \\ \chi_\perp \end{pmatrix} + \begin{pmatrix} \mathcal{Q}S_{\text{eff}} \\ -\mathcal{Q}S_{\text{eff}} \end{pmatrix} = 0 \tag{E7}$$

with the effective source term

$$S_{\text{eff}}(\mathbf{r}) = g_0\rho_0(\mathbf{r})\phi_0(\mathbf{r})(\delta N_0 - 1) + g_0 N_0\phi_0(\mathbf{r})$$
$$\times [2\langle \hat{\Lambda}^\dagger(\mathbf{r})\hat{\Lambda}(\mathbf{r})\rangle_2 + \langle \hat{\Lambda}^2(\mathbf{r})\rangle_2]$$
$$- l^D \sum_{\mathbf{r}'} g_0\rho_0(\mathbf{r}')\phi_0(\mathbf{r}')\langle [\hat{\Lambda}(\mathbf{r}') + \hat{\Lambda}^\dagger(\mathbf{r}')]\hat{\Lambda}(\mathbf{r})\rangle_2, \tag{E8}$$

where we used the fact that here $\langle \hat{\Lambda}^2\rangle_2 = \langle \hat{\Lambda}^{\dagger 2}\rangle_2$ since the condensate wave function is real. Equation (E7) is the steady version of Eq. (95) of [15], which gives N times the correction to the condensate wave function, and the source term (E8) exactly coincides with the one of Eq. (96) of [15] if one realizes that $N = N_0$, so that $\delta N_0 = -\delta N$, in the systematic expansion used in [15].

APPENDIX F: CORRECTIONS TO g_1 DUE TO THE CUBIC HAMILTONIAN

We calculate the corrections to the first order correlation function due to H_3 using the perturbative formula (143). A first remark is that

$$H_3(\tau) \equiv e^{\tau H_2}H_3 e^{-\tau H_2} \tag{F1}$$

is still cubic in the operators $\hat{b}_s$, since one has

$$e^{\tau H_2}\hat{b}_s e^{-\tau H_2} = e^{-\tau\epsilon_s}\hat{b}_s, \tag{F2}$$

$$e^{\tau H_2}\hat{b}_s^\dagger e^{-\tau H_2} = e^{\tau\epsilon_s}\hat{b}_s^\dagger, \tag{F3}$$

where ϵ_s is the energy of the Bogoliubov mode s. The second step is to use Wick's theorem to calculate the expectation values in the thermal state corresponding to the Hamiltonian H_2. One can derive the general formulas

$$\langle A_1 A_2 A_3 e^{i\Delta\theta}\rangle_2 = [\langle A_1 A_2 A_3 i\Delta\theta\rangle_2$$
$$+ \langle A_1 i\Delta\theta\rangle_2\langle A_2 i\Delta\theta\rangle_2\langle A_3 i\Delta\theta\rangle_2]$$
$$\times e^{-\langle(\Delta\theta)^2\rangle_2/2}, \tag{F4}$$

$$\langle A_1 A_2 A_3 A_4 e^{i\Delta\theta}\rangle_2 = \left\{\langle A_1 A_2 A_3 A_4\rangle_2\left[1 + \frac{1}{2}\langle(\Delta\theta)^2\rangle_2\right]\right.$$
$$- \frac{1}{2}\langle A_1 A_2 A_3 A_4(\Delta\theta)^2\rangle_2$$
$$+ \langle A_1 i\Delta\theta\rangle_2\langle A_2 i\Delta\theta\rangle_2\langle A_3 i\Delta\theta\rangle_2$$
$$\left.\times \langle A_4 i\Delta\theta\rangle_2\right\}e^{-\langle(\Delta\theta)^2\rangle_2/2}, \tag{F5}$$

where the A_i are linear in $\delta\hat{\rho}$ and $\hat{\theta}$ and have a vanishing mean value. A last point is to realize that some of the terms obtained contain a larger number of factors equal to $\Delta\theta$ than other ones. Since $\Delta\theta$ scales as $1/\sqrt{\rho_0}$ [see, e.g., the expression of $\hat{\theta}$ in terms of the mode functions u_s, v_s in Eq. (57)], the terms with an excess of $\Delta\theta$ factors are higher order in the expansion and are therefore negligible. Note that, strictly speaking, this argument is correct provided that each factor $\langle A_i\Delta\theta\rangle_2$ remains bounded whatever the distance from $\mathbf{0}$ to $\mathbf{r}$. This can be checked to be indeed the case: from the form of H_3 one sees that A_i is either $\delta\hat{\rho}$ or the phase difference between two neighboring points of the grid. One can therefore use the approximate identities

$$\langle A_1 A_2 A_3 e^{i\Delta\theta}\rangle_2 \simeq \langle A_1 A_2 A_3 i\Delta\theta\rangle_2 e^{-\langle(\Delta\theta)^2\rangle_2/2}, \tag{F6}$$

$$\langle A_1 A_2 A_3 A_4 e^{i\Delta\theta}\rangle_2 \simeq \langle A_1 A_2 A_3 A_4\rangle_2 e^{-\langle(\Delta\theta)^2\rangle_2/2}. \tag{F7}$$

This immediately leads to the identities (144) and (145).

[1] A. Görlitz, J. M. Vogels, A. E. Leanhardt, C. Raman, T. L. Gustavson, J. R. Abo-Shaeer, A. P. Chikkatur, S. Gupta, S. Inouye, T. P. Rosenband, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. **87**, 130402 (2001).

[2] F. Schreck, L. Khaykovich, K. L. Corwin, G. Ferrari, T. Bourdel, J. Cubizolles, and C. Salomon, Phys. Rev. Lett. **87**, 080403 (2001).

[3] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **22**, 1133 (1966).

[4] P. C. Hohenberg, Phys. Rev. **158**, 383 (1967).

[5] Y. Castin, R. Dum, E. Mandonnet, A. Minguzzi, and I. Carusotto, J. Mod. Opt. **47**, 2671 (2000).

[6] I. Carusotto and Y. Castin, J. Phys. B **34**, 4589 (2001).

[7] V. N. Popov, Theor. Math. Phys. **11**, 565 (1972); *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, 1983), Chap. 6.

[8] D. S. Petrov, G. V. Shlyapnikov, and J. T. M. Walraven, Phys. Rev. Lett. **85**, 3745 (2000).

[9] D. S. Petrov, M. Holzmann, and G. V. Shlyapnikov, Phys. Rev. Lett. **84**, 2551 (2000).

[10] T.-L. Ho and M. Ma, J. Low Temp. Phys. **115**, 61 (1999).

[11] S. Dettmer, D. Hellweg, P. Ryytty, J. J. Arlt, W. Ertmer, K. Sengstock, D. S. Petrov, and G. V. Shlyapnikov, Phys. Rev. Lett. **87**, 160406 (2001).

[12] F. Gerbier, S. Richard, J. H. Thywissen, M. Hugbart, P. Bouyer, and A. Aspect, e-print cond-mat/0210206.

[13] N. Bogoliubov, J. Phys. (Moscow) **11**, 23 (1947).

[14] C. Gardiner, Phys. Rev. A **56**, 1414 (1997).

[15] Y. Castin and R. Dum, Phys. Rev. A **57**, 3008 (1998).

[16] S. I. Shevchenko, Sov. J. Low Temp. Phys. **18**, 223 (1992).

[17] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1995), Chap. 28.

[18] M. Girardeau and R. Arnowitt, Phys. Rev. **113**, 755 (1959).

[19] M. Schwartz, Phys. Rev. A **10**, 1858 (1974).

[20] J. O. Andersen, U. Al Khawaja, and H. T. C. Stoof, Phys. Rev. Lett. **88**, 070407 (2002).

[21] U. Al Khawaja, J. O. Andersen, N. P. Proukakis, and H. T. C. Stoof, Phys. Rev. A **66**, 013615 (2002).

[22] U. Al Khawaja, J. O. Andersen, N. P. Proukakis, and H. T. C. Stoof, Phys. Rev. A **66**, 059902 (2002).

[23] W.-C. Wu and A. Griffin, Phys. Rev. A **54**, 4204 (1996).

[24] This can be seen by taking the derivative of Eq. (11) with respect to α.

[25] We note that the same problem of course arises in the continuous version of the theory; in the above reasoning one has to replace $\hat{\rho}(\mathbf{r})$ by the integral of $\hat{\rho}(\mathbf{r}')$ over a finite volume around the point $\mathbf{r}$.

[26] A. Sinatra, C. Lobo, and Y. Castin, J. Phys. B **35**, 3599 (2002).

[27] M. Lewenstein and Li You, Phys. Rev. Lett. **77**, 3489 (1996).

[28] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).

[29] M. Gaudin, *La Fonction d'Onde de Bethe* (Masson, Paris, 1983).

[30] V. Liu, Phys. Rev. Lett. **79**, 4056 (1997).

[31] L. P. Pitaevskii and S. Stringari, Phys. Lett. A **235**, 398 (1997).

[32] P. O. Fedichev and G. V. Shlyapnikov, Phys. Rev. A **58**, 3146 (1998).

[33] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1119 (1957).

[34] M. Schick, Phys. Rev. A **3**, 1067 (1971).

[35] D. M. Gangardt and G. V. Shlyapnikov, Phys. Rev. Lett. **90**, 010401 (2003).

[36] We have used the identity $\int_0^x du \, (1-\cos u)/u = \ln x + C - \mathrm{Ci}(x)$, where the cosine-integral function $\mathrm{Ci}(x)$ tends to zero in the limit $x \to +\infty$.

[37] V. N. Popov, JETP Lett. **31**, 526 (1980).

[38] The trick is to use the rewriting $-\rho A(k)/k^2 = (1+2\bar{v}_k^2)(n_k+1/2)-1/2$ where $2n_k+1 = \coth(\beta\epsilon_k/2)$ and $1+2\bar{v}_k^2 = (\mu+\hbar^2 k^2/2m)/\epsilon_k$. It is then clear that $A(k)$ can be extended to the domain $k<0$ to form an even C^∞ function of k since one can extend ϵ_k as an odd C^∞ function of k.

[39] M. Schwartz, Phys. Rev. B **15**, 1399 (1977).

[40] M. Olshanii and V. Dunjko, e-print cond-mat/0210629.