Quantum limits to dynamical evolution

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We establish the minimum time it takes for an initial state of mean energy E and energy spread ΔE to move from its initial configuration by a predetermined amount. Distances in Hilbert space are estimated by the fidelity between the initial and final states. In this context, we study the role of entanglement among subsystems in speeding up the dynamics of a composite system.

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INTRODUCTION

How fast can a quantum system evolve in time, given a certain amount of energy? If the system is composed of a number of subsystems, is entanglement a useful resource in speeding up the dynamical evolution? To answer the first of these two questions, one typically defines some characteristic time of the dynamics and studies its connections with the energy resources of the initial state of the system. Most of the previous results in this field [1–3] trace back to the time-energy uncertainty relation in the form derived by Mandelstam and Tamm [4]: in this way, the various lifetimes are bounded by the energy spread ΔE of the system. More recently, however, Margolus and Levitin pointed out that one can relate the characteristic times of the system also to the average energy E of the initial state [5]. In particular, defining the lifetime of the system as the time it takes for it to evolve to an orthogonal state, the above results allow one to introduce a *quantum speed limit time* as the minimum possible lifetime for a system of average energy E and spread ΔE. In [6] we analyzed such a bound in the case of composite systems (i.e., systems composed of a collection of subsystems).

In this paper we extend these results by analyzing what happens when the quantum speed limit time is generalized by redefining it as the minimum time t it takes for the initial state ϱ to evolve through a unitary evolution to a state $\varrho(t)$ such that the fidelity $F(\varrho,\varrho(t))$ of [7] is equal to a given $\epsilon \in [0,1]$. Even though the scenario is more complex than the case $\epsilon=0$ of [6], in this case also it is possible to show that entanglement is useful to achieve speedup of the dynamics if one wants to share the energy resources among the subsystems.

In Sec. I we extend the definition of quantum speed limit time and derive its expression in terms of the energy characteristics of the initial state, first considering the case of pure states and then extending the analysis to the more complex case of nonpure states. In Sec. II we analyze the role that entanglement among subsystems plays in achieving the quantum speed limit. Most of the technical details of the derivations have been inserted in the Appendix.

I. QUANTUM SPEED LIMIT

Since the Hamiltonian H is the generator of the dynamical evolution and defines the energy of a system, one expects that the energy characteristics of a state are closely linked to the characteristic times of its dynamics. In particular, we are interested in how the mean energy E and the spread ΔE relate to the maximum ''speed'' the system can sustain in moving away from its initial state $|\Psi\rangle$.

Take the energy basis expansion of the initial state

$$|\Psi\rangle=\sum_n c_n|n\rangle, \tag{1}$$

which has average energy $E=\langle\Psi|H|\Psi\rangle$ and spread $\Delta E=\sqrt{\langle\Psi|(H-E)^2|\Psi\rangle}$. To characterize the departure of the system from $|\Psi\rangle$ we can use the fidelity

$$P(t)=|\langle\Psi|\Psi(t)\rangle|^2=\left|\sum_n |c_n|^2 e^{-iE_nt/\hbar}\right|^2, \tag{2}$$

where E_n is the energy eigenvalue of the Hamiltonian H relative to $|n\rangle$. The quantity $P(t)$ is the overlap between the time evolved state $|\Psi(t)\rangle$ and the initial state $|\Psi\rangle$. A measure of the ''speed of the dynamical evolution'' is obtained by analyzing how fast $P(t)$ changes in time: e.g., given a value $\epsilon\in[0,1]$, how long do we have to wait before the state ''rotates'' by an amount ϵ, i.e., before $P(t)=\epsilon$? Assuming (without loss of generality) a zero ground-state energy eigenvalue, it is possible to prove that the minimum time at which this happens is bounded by the quantity

$$\mathcal{T}_\epsilon(E,\Delta E)\equiv\max\left(\alpha(\epsilon)\frac{\pi\hbar}{2E},\beta(\epsilon)\frac{\pi\hbar}{2\Delta E}\right), \tag{3}$$

where $\alpha(\epsilon)$ and $\beta(\epsilon)$ are the functions plotted in Fig. 1. Of course, for $\epsilon=1$ this quantity is equal to zero: in fact, no time has to pass to obtain $P(t)=1$. On the other hand, since for $\epsilon=0$ we have $\alpha(\epsilon)=\beta(\epsilon)=1$, Eq. (3) reduces to the quantum speed limit time that was defined in [5,6]

$$\mathcal{T}_0(E,\Delta E)\equiv\max\left(\frac{\pi\hbar}{2E},\frac{\pi\hbar}{2\Delta E}\right), \tag{4}$$

which gives the minimum time it takes for a system to evolve to an orthogonal configuration. In the remainder of the paper, however, we use ''quantum speed limit time'' to refer to the generalized version $\mathcal{T}_\epsilon(E,\Delta E)$ of Eq. (3). The

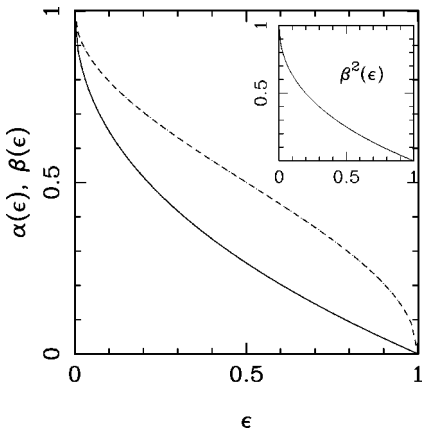

FIG. 1. Plot of $\alpha(\epsilon)$ (continuous line) and $\beta(\epsilon)$ (dashed line) introduced in Eq. (3). The inset shows the similarity between $\beta^2(\epsilon)$ and $\alpha(\epsilon)$.

detailed derivation of Eq. (3) is given in the Appendix. Here we give only a general idea of the procedure. The quantity $\mathcal{T}_\epsilon(E,\Delta E)$ is composed of two contributions. The first contribution relates the speed of the dynamical evolution to the average energy E through the function $\alpha(\epsilon)$ and it extends the Margolus-Levitin theorem [5]. It provides the value of $\mathcal{T}_\epsilon(E,\Delta E)$ in what we will refer to as the Margolus-Levitin regime, i.e., when $\Delta E/E \geqslant \beta(\epsilon)/\alpha(\epsilon)$. The function $\alpha(\epsilon)$ is derived by introducing two functions $\alpha_<(\epsilon)$ and $\alpha_>(\epsilon)$ such that $\alpha_<(\epsilon) \leqslant \alpha(\epsilon) \leqslant \alpha_>(\epsilon)$. The first one is obtained by directly analyzing the condition $P(t)=\epsilon$ and using a class of inequalities that maximizes sines and cosines with linear functions. The second one is obtained by studying the time evolution of a class of "fast" two level states. This procedure does not allow one to obtain an explicit analytical expression for $\alpha(\epsilon)$; however, the two bounding functions $\alpha_<$ and $\alpha_>$ can be shown numerically to coincide giving an estimate of $\alpha(\epsilon)$. The second contribution to $\mathcal{T}_\epsilon(E,\Delta E)$ relates the speed of the dynamical evolution to the spread ΔE by means of the function

$$\beta(\epsilon)=\frac{2}{\pi}\arccos(\sqrt{\epsilon}). \tag{5}$$

This term provides the value of $\mathcal{T}_\epsilon(E,\Delta E)$ in what we will refer to as the Heisenberg regime, i.e., when $\Delta E/E \leqslant \beta(\epsilon)/\alpha(\epsilon)$. Equation (5) was previously proved in [1,2] by employing the general form of the uncertainty relations; however, for the sake of completeness, in Appendix A we rederived the value of $\beta(\epsilon)$ starting directly from the expression (2) for $P(t)$.

Given E and ΔE, the quantum speed limit defines a forbidden evolution regime where the probability $P(t)$ is not allowed to enter. In fact, for $0 \leqslant t \leqslant \mathcal{T}_0(E,\Delta E)$, Eq. (3) implies

$$P(t) \geqslant \max\left\{ \alpha^{-1}\left(\frac{2Et}{\pi\hbar}\right), \beta^{-1}\left(\frac{2\Delta E t}{\pi\hbar}\right)\right\}, \tag{6}$$

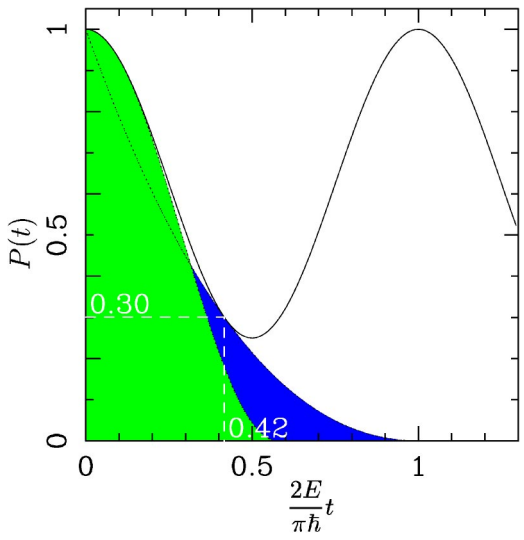

FIG. 2. Plot of the constraint given by Eq. (6), which shows the forbidden region where $P(t)$ is not allowed to enter (shaded areas). The time-energy uncertainty contribution to Eq. (6) defines the light gray region through β^{-1}. The Margolus-Levitin type contribution defines the dark gray region through α^{-1}. The condition plotted here is for $\Delta E/E=1.73$. The continuous line represents the trajectory $P(t)$ of a "fast" state $|\Omega_\xi\rangle$ of Eq. (A7) with $\xi=0.5$. This state touches the boundary of the forbidden region for $\epsilon=0.30$ and $t=\mathcal{T}_{\epsilon=0.30}(E,\Delta E)=0.42(\pi\hbar/2E)$.

where α^{-1} and β^{-1} are the inverse functions of $\alpha(\epsilon)$ and $\beta(\epsilon)$, respectively. This regime is shown on Fig. 2, where an example of $P(t)$ trajectory is plotted. By introducing the Margolus-Levitin type contribution (i.e., the term dependent on E) Eq. (6) generalizes the previous bounds for $P(t)$ [1,2]. Notice that, since the contribution dependent on ΔE to Eq. (6) exhibits a null derivative in $t=0$, it always provides a nontrivial bound to $P(t)$. On the other hand, since the contribution dependent on E exhibits a negative slope at $t=0$, it does not provide an achievable bound when $\Delta E \leqslant E$. For the same reason, the bound (A12) on the first derivative of $P(t)$ is not modified by the presence of the Margolus-Levitin type contribution of Eq. (6): $|dP(t)/dt|$ is limited only by the energy spread ΔE.

Quantum speed limit time for mixed states

Up to now we have focused on pure states of the system. What happens when the system is in a mixture ϱ? We will show that the notion of the quantum speed limit bound (3) can be extended to the density matrices in the sense that $\mathcal{T}_\epsilon(E,\Delta E)$ gives the lower bound to the time it takes for a state ϱ with energy E and spread ΔE to evolve to a configuration $\varrho(t)$ such that

$$F(\varrho,\varrho(t))=\epsilon, \tag{7}$$

where $F(\varrho,\varrho') \equiv \{\mathrm{Tr}[\sqrt{\sqrt{\varrho}\varrho'\sqrt{\varrho}}]\}^2$ is the fidelity introduced in [7].

To prove the above statement, first of all notice that in the case of pure states the fidelity reduces to the probability $P(t)$ of Eq. (2) and the definition (7) reduces to the quantum

speed limit bound given in the previous section. More generally, consider a generic decomposition of ϱ,

$$\varrho = \sum_n p_n |\phi_n\rangle\langle\phi_n|, \tag{8}$$

where $p_n > 0$, $\Sigma_n p_n = 1$, and $\{|\phi_n\rangle\}$ is a set of nonnecessarily orthogonal pure states. The fidelity F has been shown [7,9] to satisfy the property

$$F(\varrho, \varrho(t)) = \max_{|\chi\rangle, |\chi'\rangle} \{|\langle\chi|\chi'\rangle|^2\}, \tag{9}$$

where $|\chi\rangle$ and $|\chi'\rangle$ are purifications of ϱ and $\varrho(t)$, respectively, such as the states

$$|\chi\rangle = \sum_n \sqrt{p_n} |\phi_n\rangle |\xi_n\rangle, \tag{10}$$

$$|\chi'\rangle = \sum_n \sqrt{p_n} |\phi_n(t)\rangle |\xi'_n\rangle, \tag{11}$$

with $\{|\xi_n\rangle\}, \{|\xi'_n\rangle\}$ being two orthonormal sets of an ancillary system. Choosing $|\xi'_n\rangle = |\xi_n\rangle$ for all n and assuming that they are all eigenstates of the ground level of the ancillary system, $|\chi'\rangle$ becomes the time evolved of $|\chi\rangle$ [i.e., $|\chi'\rangle = |\chi(t)\rangle$] and Eq. (9) implies that the fidelity is bounded by

$$F(\varrho, \varrho(t)) \geqslant |\langle\chi|\chi(t)\rangle|^2. \tag{12}$$

Since the pure state $|\chi\rangle$ has the same energy E and energy spread ΔE of ϱ, it can rotate by a quantity ϵ in a time not smaller than $\mathcal{T}_\epsilon(E, \Delta E)$, as shown in the previous section. This, along with inequality (12), proves that the minimum time t for which $F(\varrho, \varrho(t)) = \epsilon$ is bounded by the quantity $\mathcal{T}_\epsilon(E, \Delta E)$, as stated in Eq. (7). Notice, finally, that for $\epsilon = 0$ we reobtain all the results that were given in [6], since in this case the condition (7) is equivalent to the condition $\mathrm{Tr}[\varrho(t)\varrho] = 0$ that was employed there [8].

Mixed states that reach the bound

Before concluding the section, let us analyze under what conditions a mixed state can reach the quantum speed limit. Assume that the state ϱ of energy E and spread ΔE achieves the bound for a value ϵ, i.e.,

$$F(\varrho, \varrho(\mathcal{T}_\epsilon)) = \epsilon, \tag{13}$$

where the dependence on E and ΔE has been dropped for ease of notation. Define the quantity $\epsilon_n \equiv |\langle\phi_n|\phi_n(\mathcal{T}_\epsilon)\rangle|^2$, which measures the rotation of the nth component of the mixture (8) at time $\mathcal{T}_\epsilon$. Applying quantum speed limit considerations to the state $|\phi_n\rangle$, one finds

$$\mathcal{T}_\epsilon(E, \Delta E) \geqslant \mathcal{T}_{\epsilon_n}(\mathcal{E}_n, \Delta\mathcal{E}_n), \tag{14}$$

where $\mathcal{E}_n$ and $\Delta\mathcal{E}_n$ are the energy and the spread of the state $|\phi_n\rangle$. Now, among all possible purifications of the form (10) and (11), choose one such that $\langle\xi_n|\xi'_m\rangle = \delta_{nm} e^{-i\varphi_{nm}}$, where $\varphi_{nm} = \arg[\langle\phi_n|\phi_m(\mathcal{T}_\epsilon)\rangle]$. From Eqs. (9) and (13), it follows that

$$\epsilon \geqslant \left| \sum_n p_n |\langle\phi_n|\phi_n(\mathcal{T}_\epsilon)\rangle| \right|^2 = \left(\sum_n p_n \sqrt{\epsilon_n} \right)^2 \equiv \overline{\epsilon}, \tag{15}$$

$$\mathcal{T}_{\overline{\epsilon}}(E, \Delta E) \geqslant \mathcal{T}_\epsilon(E, \Delta E), \tag{16}$$

where in Eq. (16) we employed the fact that $\alpha(\epsilon)$ and $\beta(\epsilon)$ are strictly decreasing functions. Combining Eqs. (14) and (16), we find that, for all n,

$$\mathcal{T}_{\overline{\epsilon}}(E, \Delta E) \geqslant \mathcal{T}_{\epsilon_n}(\mathcal{E}_n, \Delta\mathcal{E}_n). \tag{17}$$

Consider first the Margolus-Levitin regime, i.e., $\Delta E/E \geqslant \beta(\overline{\epsilon})/\alpha(\overline{\epsilon})$. From Eq. (17) it follows that

$$\alpha(\overline{\epsilon}) \frac{\pi\hbar}{2E} \geqslant \alpha(\epsilon_n) \frac{\pi\hbar}{2\mathcal{E}_n}. \tag{18}$$

Since the energy of the state ϱ is $E = \Sigma_n p_n \mathcal{E}_n$, Eq. (18) implies

$$\alpha(\overline{\epsilon}) \geqslant \sum_n p_n \alpha(\epsilon_n). \tag{19}$$

Analogously, in the Heisenberg regime, i.e., when $\Delta E/E \leqslant \beta(\overline{\epsilon})/\alpha(\overline{\epsilon})$, since $\Delta E^2 = \Sigma_n p_n [\Delta\mathcal{E}_n^2 + (E - \mathcal{E}_n)^2]$, one obtains

$$\beta^2(\overline{\epsilon}) \geqslant \sum_n p_n \beta^2(\epsilon_n). \tag{20}$$

The inequalities (19) and (20) must be satisfied if ϱ reaches the quantum speed limit. Since both $\alpha(\epsilon^2)$ and $\beta^2(\epsilon^2)$ are strictly convex functions [see Eqs. (A14) and (A15) of Appendix A 3], such conditions can be satisfied only when the equalities hold: this happens if $\epsilon_n = \epsilon$ for all n and if the equality holds also in (14). This shows that the fastest states ϱ are mixture composed of pure states $|\phi_n\rangle$ that all achieve the quantum speed limit bound for the same ϵ at the same time.

II. ENTANGLED DYNAMICS

In a preceding paper [6], we analyzed the role of entanglement in achieving the quantum speed limit bound (4) for composite systems. We found that quantum correlations among the subsystems allow the state of the system to evolve to an orthogonal configuration faster if the energy resources are not devoted to a single subsystem and the initial state is pure. Here we analyze the generalized bound (3) and show that the same result holds even when we do not require the initial and final states to be orthogonal. Quantum correlations among subsystems allow the state of the system to rotate in Hilbert space faster if the energy resources are not devoted to a single subsystem.

In the following we consider the case of a noninteracting

composite system with M independent components. Its Hamiltonian is given by $H=\Sigma_k H_k$, where H_k is the free Hamiltonian of the kth subsystem. Since the Hamiltonian H is assumed to have zero ground state, we will redefine all the H_k to have zero ground states without loss of generality.

A. Pure states

Consider a composite system of M noninteracting parts in the initial pure separable state

$$|\Psi\rangle=|\psi_1\rangle_1\cdots|\psi_M\rangle_M, \tag{21}$$

which has energy and energy spread

$$E=\sum_k E_k, \tag{22}$$

$$\Delta E=\left(\sum_k \Delta E_k^2\right)^{1/2}, \tag{23}$$

where E_k and ΔE_k are the energy and the spread of the state $|\psi_k\rangle_k$ of the kth subsystem [10]. The state $|\Psi\rangle$ reaches the quantum speed limit if, for some value of ϵ, the following identity applies:

$$P(\mathcal{T}_\epsilon(E,\Delta E))=\epsilon, \tag{24}$$

where $P(t)$ is the probability (2) of the state $|\Psi\rangle$ and $\mathcal{T}_\epsilon(E,\Delta E)$ is the quantum speed limit time of Eq. (3). For a separable state, the quantity $P(t)$ is given by

$$P(t)=P_1(t)\cdots P_M(t), \tag{25}$$

where $P_k(t)=|_k\langle\psi_k|\psi_k(t)\rangle_k|^2$ is the overlap of the state of the kth subsystem at time t with its initial value. Defining $\epsilon_k=P_k(\mathcal{T}_\epsilon(E,\Delta E))$ and using Eq. (25), the condition (24) can be rewritten as

$$\epsilon=\epsilon_1\cdots\epsilon_M. \tag{26}$$

Moreover, applying the quantum speed limit relation (3) to the kth subsystem, one finds that

$$\mathcal{T}_\epsilon(E,\Delta E)\geqslant\mathcal{T}_{\epsilon_k}(E_k,\Delta E_k). \tag{27}$$

Consider first the Margolus-Levitin regime, i.e., $\Delta E/E\geqslant\beta(\epsilon)/\alpha(\epsilon)$. In this case, Eq. (27) and the definitions of $\mathcal{T}_\epsilon(E,\Delta E)$ and $\mathcal{T}_{\epsilon_k}(E_k,\Delta E_k)$ imply

$$\alpha(\epsilon)\frac{\pi\hbar}{2E}\geqslant\alpha(\epsilon_k)\frac{\pi\hbar}{2E_k}, \tag{28}$$

and hence, using the expression (22) for the total energy of the system,

$$\alpha(\epsilon)\geqslant\sum_{k=1}^{M}\alpha(\epsilon_k). \tag{29}$$

Analogously, in the Heisenberg regime Eq. (27) implies that

$$\beta(\epsilon)\frac{\pi\hbar}{2\Delta E}\geqslant\beta(\epsilon_k)\frac{\pi\hbar}{2\Delta E_k}, \tag{30}$$

and hence

$$\beta^2(\epsilon)\geqslant\sum_{k=1}^{M}\beta^2(\epsilon_k). \tag{31}$$

A necessary condition for the separable state $|\Psi\rangle$ to reach the quantum speed limit is that there exists a set of ϵ_k that satisfy at least one of the two inequalities (29) and (31) under the constraint (26). According to the strict subadditivity of $\alpha(\epsilon)$ and $\beta^2(\epsilon)$ [see Eqs. (A16) and (A17) of Appendix A 3], the relations (29) and (31) can be satisfied only when the equality holds: this happens if there exists a k (say k') such that $\epsilon_{k'}=\epsilon$ and $\epsilon_k=1$ for all $k\neq k'$. Such a solution corresponds to the case in which all the energy resources are devoted to the k'th subsystem. In fact, for $k=k'$, the relations (28) and (30) imply that

$$E_{k'}\geqslant\frac{\alpha(\epsilon_{k'})}{\alpha(\epsilon)}E=E, \tag{32}$$

$$\Delta E_{k'}\geqslant\frac{\beta(\epsilon_{k'})}{\beta(\epsilon)}\Delta E=\Delta E, \tag{33}$$

where Eq. (32) holds in the Margolus-Levitin regime, while Eq. (33) holds in the Heisenberg regime. Equation (32) and the form (22) of E require that $E_{k'}=E$ and $E_k=0$ for $k\neq k'$ [10]. Since H_k have all zero ground-state energy, this also implies that $\Delta E_{k'}=\Delta E$ and $\Delta E_k=0$ for $k\neq k'$. On the other hand, Eq. (33) and the form (23) of ΔE require that $\Delta E_{k'}=\Delta E$ and $\Delta E_k=0$ for $k\neq k'$.

In conclusion, the only states $|\Psi\rangle$ of the form (21) that can reach the quantum speed limit (3) for some value of ϵ are those in which all the energy spread ΔE is carried by the single subsystem k'. The other subsystems are in eigenstates of their Hamiltonians H_k. Moreover, if the system is in the Margolus-Levitin regime, then k' also carries all the mean energy E, the other subsystems being in their ground states. From the dynamical point of view, this means that k' is the only subsystem that rotates in the Hilbert space, while all the others do not evolve.

A simple example: Separable vs entangled state

The gap between entangled states and nonentangled ones is particularly evident in the case in which the energy resources are homogeneously distributed among all subsystems, i.e., when $E_k=E/M$ and $\Delta E_k=\Delta E/\sqrt{M}$ for all k. For the sake of simplicity we analyze an example in which all the subsystems are in the same state $|\psi_k\rangle_k=|\psi\rangle_k$. In this case the minimum time t_ϵ for which the global state $|\Psi_s\rangle=|\psi\rangle_1\cdots|\psi\rangle_M$ rotates by a quantity ϵ is given by the minimum time it takes for each subsystem to rotate by a quantity $\epsilon^{1/M}$. Applying the quantum speed limit (3) to each subsystem, one obtains that $t_\epsilon\geqslant\mathcal{T}_{\epsilon^{1/M}}(E/M,\Delta E/\sqrt{M})$. The ratio $R(\epsilon)$ between t_ϵ and $\mathcal{T}_\epsilon(E,\Delta E)$, i.e.,

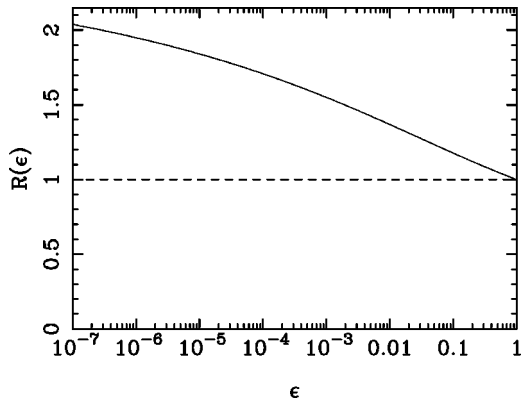

FIG. 3. Plot of the lower bound of $R(\epsilon)$ from Eq. (34). It shows that homogeneous separable states cannot achieve the quantum speed limit bound (given by the dashed line). Here $M=5$.

$$R(\epsilon)\geqslant \min\left(M\,\frac{\alpha(\epsilon^{1/M})}{\alpha(\epsilon)}, \sqrt{M}\,\frac{\beta(\epsilon^{1/M})}{\beta(\epsilon)}\right), \tag{34}$$

shows how much slower the separable state $|\Psi_s\rangle$ is in comparison with the maximum speed allowed for a system of the same energetic resources as $|\Psi_s\rangle$ (see Fig. 3). From the subadditivity properties (A16) and (A17), it follows that $R(\epsilon)\geqslant 1$. In particular, for $\epsilon=0$, $R(\epsilon)$ is always greater or equal to $\sqrt{M}$, as discussed in [6].

Consider now the following family of entangled states:

$$|\Psi_\xi\rangle=\sqrt{1-\xi^2}|0\rangle_1\cdots|0\rangle_M+\xi|E_0\rangle_1\cdots|E_0\rangle_M, \tag{35}$$

where $\xi\in[0,1]$ and $|0\rangle$ and $|E_0\rangle$ are eigenstates of the Hamiltonian of energy 0 and $E_0>0$, respectively. The state $|\Psi_\xi\rangle$ represents a homogeneous configuration where each subsystem has energy ξ^2E_0 and spread $\xi\sqrt{1-\xi^2}E_0$. However, unlike the separable state $|\Psi_s\rangle$ discussed before, for a suitable choice of the parameter ξ, $|\Psi_\xi\rangle$ achieves the quantum speed limit bound, as can be shown by comparison with the state $|\Omega_\xi\rangle$ of Eq. (A7) in Appendix A 1.

By proving that homogeneous separable states cannot exhibit speedup while at least one homogeneous entangled case that exhibits speedup exists, we have shown that entanglement is a fundamental resource in this context.

B. Entangled dynamics for mixed states

In this section we generalize the results of the previous section to mixed states.

The most general separable state of M subsystems can be written as

$$\varrho=\sum_n p_n|\Psi^{(n)}\rangle\langle\Psi^{(n)}|, \tag{36}$$

where $p_n>0$, $\Sigma_n p_n=1$, and

$$|\Psi^{(n)}\rangle=|\psi_1^{(n)}\rangle_1\cdots|\psi_M^{(n)}\rangle_M, \tag{37}$$

with $|\psi_k^{(n)}\rangle_k$ a state of the kth subsystem. As discussed in Sec. I A, $\mathcal{T}_\epsilon(E,\Delta E)$ is the minimum time it takes for the state ϱ to reach a configuration $\varrho(t)$ with fidelity ϵ. Moreover, we already know that the state ϱ will reach the quantum speed limit bound for mixed states of Eq. (7) only if all the states $|\Psi^{(n)}\rangle$ rotate by an amount ϵ in the time $\mathcal{T}_\epsilon(\mathcal{E}_n,\Delta\mathcal{E}_n)$, given $\mathcal{E}_n$ and $\Delta\mathcal{E}_n$, the energy and energy spread of $|\Psi^{(n)}\rangle$. Since $|\Psi^{(n)}\rangle$ is a separable pure state, from the previous section it follows that this is possible only if there exists a subsystem (say the k_nth) that possesses all the energy resources. This means that the only separable states ϱ that reach the bound (7) are those for which, for any statistical realization n of the mixture (36), a single subsystem evolves to an orthogonal configuration at its own maximum speed limit time [which coincides with $\mathcal{T}(E,\Delta E)$ of the whole system]. All the other subsystems do not evolve. Since the above derivation applies for any product state expansion p_n, one can say that in each experimental run only one of the subsystems evolves. Of course, other possible ensembles can be used in the decomposition of ϱ, which are not necessarily of the form (36). For instance, one can have unravelings in which $|\Psi^{(n)}\rangle$ is an entangled state of the M subsystems. This means that, as building blocks to produce a ''fast'' mixed state, we need either entanglement or to use product states in which only one of the subsystems evolves. What is definitely impossible is to produce such a state mixing product states where the energy is homogeneously distributed. This is essentially the same result that was obtained in [6], although here we considered the more general case of $\epsilon\neq 0$.

III. CONCLUSION

In this paper we have generalized the definition of quantum speed limit time [5,6] to take into account the case in which the system does not evolve to an orthogonal state. As a ''measure of the distance'' between the initial and the final states we have used the fidelity $F(\varrho,\varrho(t))$ [7]. However, the entire analysis can be performed also using other kinds of measures which are connected with the fidelity, such as the trace distance $\mathrm{Tr}[\varrho-\varrho(t)]/2$ or the relative entropy $S(\varrho||\varrho(t))\equiv\mathrm{Tr}[\varrho\log_2\varrho-\varrho\log_2\varrho(t)]$ [9,11]. In this context, we have analyzed the role of quantum correlations among noninteracting subsystems in a composite system. As a result, we have shown that entanglement plays an important role in the speedup of the dynamical evolution in a composite system. In fact, the only separable pure states that can achieve the quantum speed limit bound are those where only one subsystem at a time is evolving, while the others are stationary. Analogously, for mixed states, we have shown that one cannot create a fast separable configuration starting from product states in which the energy is shared among multiple subsystems.

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APPENDIX

In Secs. A 1 and A 2 we derive the forms of the functions $\alpha(\epsilon)$ and $\beta(\epsilon)$, respectively. In Sec. A 3 we study these two

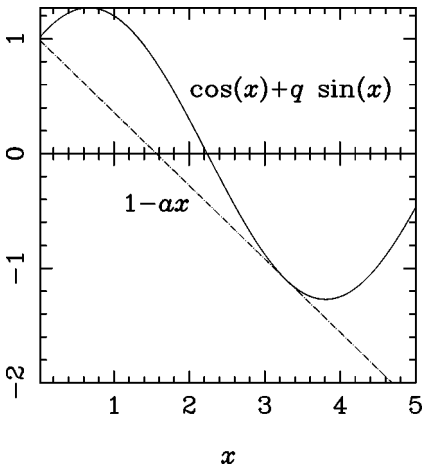

FIG. 4. Plot of the inequality (A2) for $q=\pi/4$. In this case $a \simeq 0.64$.

functions, giving some mathematical properties that are used in the paper.

1. Derivation of $\alpha(\epsilon)$

In order to determine $\alpha(\epsilon)$ we will: (i) give a lower bound for it; (ii) give an upper bound for it; (iii) show numerically that these two bounds coincide, thus providing an estimation of $\alpha(\epsilon)$.

(i) A lower bound for $\alpha(\epsilon)$ can be obtained by observing that if $P(t)=\epsilon$, then $\langle\Psi|\Psi(t)\rangle=\sqrt{\epsilon}e^{i\theta}$, i.e., from Eq. (1)

$$\sum_n |c_n|^2 \cos\frac{E_n t}{\hbar}=\sqrt{\epsilon}\cos\theta,$$

$$\sum_n |c_n|^2 \sin\frac{E_n t}{\hbar}=-\sqrt{\epsilon}\sin\theta, \tag{A1}$$

with $\theta\in[0,2\pi]$. Consider now the following class of inequalities for $q\geqslant 0$:

$$\cos x+q\sin x\geqslant 1-ax, \tag{A2}$$

which is valid for $x\geqslant 0$ and where a is a function of q defined implicitly by the set of equations

$$a=\frac{y+\sqrt{y^2(1+q^2)+q^2}}{1+y^2},$$

$$\sin y=\frac{a(1-qy)+q}{1+q^2}, \tag{A3}$$

for $y\in[\pi-\arctan(1/q),\pi+\arctan(q)]$. The inequality (A2) is obtained by bounding the term on the left with the linear function that is tangent to it and is equal to 1 for $x=0$, as shown in Fig. 4. Since we assumed zero ground-state energy, all the energy levels are positive and we can replace x with $E_n t/\hbar$ in Eq. (A2). Summing on n and employing Eq. (A1), we obtain the inequality

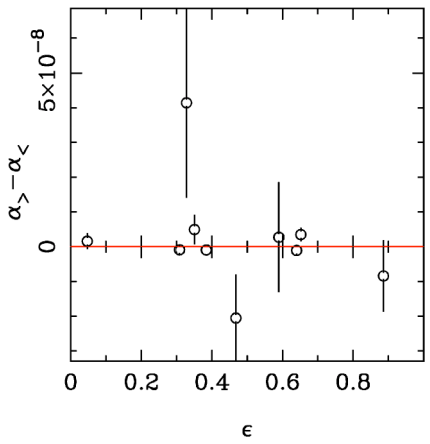

FIG. 5. Comparison of $\alpha_<(\epsilon)$ and $\alpha_>(\epsilon)$ for some random values of ϵ. The plot shows $\alpha_>(\epsilon)-\alpha_<(\epsilon)$ with the error bars denoting the $\delta\theta\to 0$ extrapolation error of $\alpha_<(\epsilon)$ (see text). Since all the values are compatible with zero, we can conclude that $\alpha_<(\epsilon)=\alpha_>(\epsilon)=\alpha(\epsilon)$.

$$\sqrt{\epsilon}(\cos\theta-q\sin\theta)\geqslant 1-a\frac{Et}{\hbar}. \tag{A4}$$

From the definition of $\alpha(\epsilon)$ introduced in Eq. (3), this implies

$$\alpha(\epsilon)\geqslant[1-\sqrt{\epsilon}(\cos\theta-q\sin\theta)]\frac{2}{\pi a}. \tag{A5}$$

Since, for a given value of θ, Eq. (A5) must be valid for all $q\geqslant 0$, then the following lower bound for $\alpha(\epsilon)$ can be obtained:

$$\alpha(\epsilon)\geqslant\alpha_<(\epsilon)\equiv\min_\theta\left[\max_q\left([1-\sqrt{\epsilon}(\cos\theta-q\sin\theta)]\frac{2}{\pi a}\right)\right]. \tag{A6}$$

(ii) To provide an upper bound for $\alpha(\epsilon)$, consider the following family of two-level states:

$$|\Omega_\xi\rangle=\sqrt{1-\xi^2}|0\rangle+\xi|E_0\rangle, \tag{A7}$$

where $\xi\in[0,1]$, and $|0\rangle$ and $|E_0\rangle$ are Hamiltonian eigenstates of energy 0 and E_0, respectively. The state $|\Omega_\xi\rangle$ has average energy $E=\xi^2E_0$ and energy spread $\Delta E=\xi\sqrt{1-\xi^2}E_0$. Solving the dynamical evolution of the state $|\Omega_\xi\rangle$, one can show that the first time t for which $P(t)=\epsilon$ is given by

$$\frac{Et}{\hbar}=\xi^2\arccos\left[\frac{\epsilon-1+2\xi^2(1-\xi^2)}{2\xi^2(1-\xi^2)}\right]. \tag{A8}$$

Minimizing over ξ the right-hand term of Eq. (A8) gives the following upper bound for $\alpha(\epsilon)$:

$$\alpha(\epsilon)\leqslant\alpha_>(\epsilon)\equiv\frac{2}{\pi}z\arccos\left[\frac{\epsilon-1+2z(1-z)}{2z(1-z)}\right], \tag{A9}$$

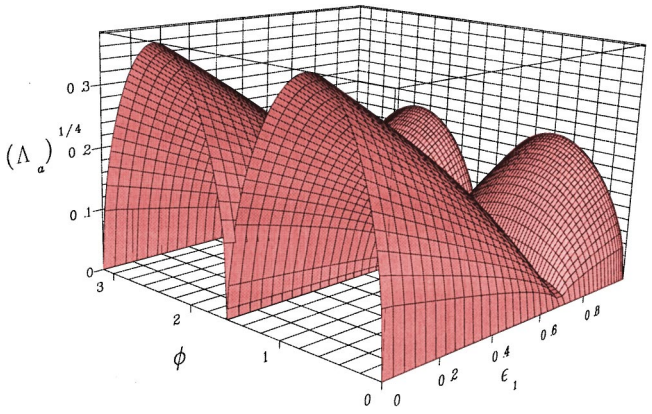

FIG. 6. Convexity condition (A14) for $\alpha(\epsilon^2)$ in the case $N=2$: $\Lambda_a \equiv \alpha(\epsilon_1^2)\cos^2\phi + \alpha(\epsilon_2^2)\sin^2\phi - \alpha((\epsilon_1\cos^2\phi + \epsilon_2\sin^2\phi)^2) \geqslant 0$. In this plot $\epsilon_2 = 0.7$. Notice that Λ_a is null only for $\epsilon_1 = \epsilon_2$ and for $\phi = 0, \pi/2, \pi$.

where the z is a function of ϵ defined implicitly by

$$\arccos\left[\frac{\epsilon - 1 + 2z(1-z)}{2z(1-z)}\right] = \frac{1-2z}{1-z}\sqrt{\frac{1-\epsilon}{\epsilon - 1 + 4z(1-z)}}. \tag{A10}$$

(iii) The obvious difficulty in deriving the explicit analytical form of the bounds $\alpha_<(\epsilon)$ and $\alpha_>(\epsilon)$ defined in Eqs. (A6) and (A9) can be overcome by performing a numerical study of these two conditions. We will show that $\alpha_<(\epsilon) = \alpha_>(\epsilon)$, thus giving an estimate of $\alpha(\epsilon)$.

In order to estimate $\alpha_>(\epsilon)$ numerically one has to solve Eq. (A10). Using a bisection algorithm, it is simple to get a machine-precision accurate solution very rapidly for all values of ϵ. On the other hand, the estimate of $\alpha_<(\epsilon)$ requires greater care, since two different parameters—q and θ of Eq. (A6)—are present in its definition. For each value of ϵ it is necessary to calculate the term on the right of Eq. (A5) on a bidimensional grid of values of q and θ and find for each θ the maximum on q. The value of $\alpha_<(\epsilon)$ is calculated by choosing the minimum among these maxima. Of course, this procedure is biased, since it depends on the grid spacings δq and $\delta\theta$. In order to remove the bias in the calculation result, one can repeat the whole procedure for different values of the grid spacing and then extrapolate the result for the spacings δq and $\delta\theta$ tending to zero. We have used a least squares linear interpolation, where χ^2 minimization allows us also to recover an ''error bar'' that measures how well the linear interpolation works for each value of ϵ. The error bar has no statistical meaning: it simply gives an idea of how well the linear extrapolation for $\delta\theta \to 0$ works for the value of ϵ under consideration. It can be used also to give a ''confidence interval'' for the result obtained. The $\delta q \to 0$ extrapolation error has been found negligible in all cases, meaning that a linear extrapolation is well suited. To reduce aliasing problems, instead of using an equispaced grid, it is preferable to adopt a random grid of values of q and θ uniformly distributed so that the average distance between distinct values is δq and $\delta\theta$, respectively.

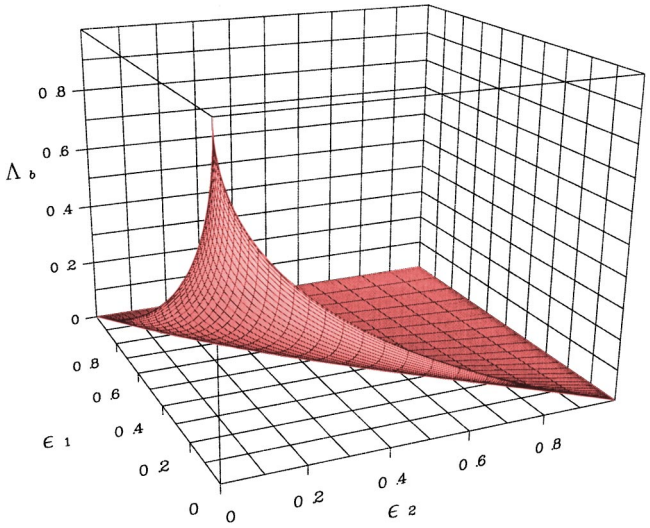

FIG. 7. Subadditivity condition (A16) for $\alpha(\epsilon)$ in the case $N=2$: $\Lambda_b \equiv \alpha(\epsilon_1) + \alpha(\epsilon_2) - \alpha(\epsilon_1\epsilon_2) \geqslant 0$, for $\epsilon_1, \epsilon_2 \geqslant 0$. Notice that Λ_b is null only for $\epsilon_1 = 1$ or $\epsilon_2 = 1$.

The extrapolated value of $\alpha_<(\epsilon)$ with its error bars is compared with the calculated value of $\alpha_>(\epsilon)$ in Fig. 5. Machine-precision accuracy is rapidly attainable in the calculation of $\alpha_>(\epsilon)$ and we have considered it as unaffected by error. Since the values of $\alpha_>(\epsilon)$ and $\alpha_<(\epsilon)$ are compatible for arbitrary values of ϵ we can conclude that the two functions coincide and are thus both equal to $\alpha(\epsilon)$. This allows us to give the numerical estimations of this function that have been used throughout the paper. Notice, however, that $\alpha(\epsilon)$ is roughly approximated (up to a few percent error) by the function $\beta^2(\epsilon)$, as can be seen from Fig. 1.

2. Derivation of $\beta(\epsilon)$

The function $\beta(\epsilon)$ can be derived starting from Eq. (2) by the following chain of relations:

$$\begin{aligned}\left|\frac{d}{dt}P(t)\right| &= \frac{2}{\hbar}\left|\sum_{n,m}|c_n|^2|c_m|^2(E_n - E)\sin\left(\frac{E_n - E_m}{\hbar}t\right)\right| \\ &\leqslant \frac{2}{\hbar}\left|\sum_{n,m}|c_n|^2|c_m|^2(E_n - E)e^{-i(E_n - E_m)t/\hbar}\right| \\ &= \frac{2}{\hbar}\left|\sum_n |c_n|^2(E_n - E)\right. \\ &\quad \left.\times\left(\sum_m |c_m|^2 e^{-i(E_n - E_m)t/\hbar} - P(t)\right)\right|, \end{aligned} \tag{A11}$$

where the last identity has been obtained by adding a zero term to the sum on n. Applying the Cauchy-Schwarz inequality to Eq. (A11), we find

$$\left|\frac{d}{dt}P(t)\right| \leqslant \frac{2\Delta E}{\hbar}\sqrt{P(t)[1 - P(t)]}, \tag{A12}$$

which for $0 \leqslant t \leqslant \pi\hbar/(2\Delta E)$ implies [1,2]

$$P(t) \geqslant \cos^2\left(\frac{\Delta E}{\hbar} t\right). \tag{A13}$$

This means that the smallest time t for which $P(t)=\epsilon$ is bounded by the quantity $\beta(\epsilon)\pi\hbar/(2\Delta E)$ with $\beta(\epsilon)$ defined in Eq. (5). Notice that the bound (A13) is achievable since, for example, the state $|\Omega_{\xi=1/\sqrt{2}}\rangle$ of Eq. (A7) reaches it.

3. Mathematical properties of $\alpha(\epsilon)$ and $\beta(\epsilon)$

Both $\alpha(\epsilon)$ and $\beta(\epsilon)$ are strictly decreasing functions (see Fig. 1). Moreover they satisfy the following constraints.

(a) The functions $\alpha(\epsilon^2)$ and $\beta^2(\epsilon^2)$ are strictly convex, i.e., for $\epsilon_n \in [0,1]$,

$$\alpha\left(\left(\sum_{n=1}^{N} p_n \epsilon_n\right)^2\right) \leqslant \sum_{n=1}^{N} p_n \alpha(\epsilon_n^2), \tag{A14}$$

$$\beta^2\left(\left(\sum_{n=1}^{N} p_n \epsilon_n\right)^2\right) \leqslant \sum_{n=1}^{N} p_n \beta^2(\epsilon_n^2), \tag{A15}$$

where $p_n > 0$ and $\Sigma_n p_n = 1$. The identity in (A14) and (A15) holds only if $\epsilon_n = \epsilon_{n'}$ for all n and n'.

(b) The functions $\alpha(\epsilon)$ and $\beta^2(\epsilon)$ are strictly subadditive, i.e., for $\epsilon_k \in [0,1]$,

$$\alpha\left(\prod_{k=1}^{N} \epsilon_k\right) \leqslant \sum_{k=1}^{N} \alpha(\epsilon_k), \tag{A16}$$

$$\beta^2\left(\prod_{k=1}^{N} \epsilon_k\right) \leqslant \sum_{k=1}^{N} \beta^2(\epsilon_k), \tag{A17}$$

with the identity holding only when there exists a k (say k') such that $\epsilon_k = 1$ for all $k \neq k'$.

To prove these properties, one can discuss the case of $N=2$ and then extend it by induction to arbitrary N. When referred to $\beta(\epsilon)$, both properties can be analytically proved using its definition (5). For $\alpha(\epsilon)$ we must instead resort to numerical verification (e.g., see Figs. 6 and 7).

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$$\varrho = \cos^2\phi |\chi_1\rangle\langle\chi_1| + \sin^2\phi |\chi_2\rangle\langle\chi_2|,$$

where $|\chi_{1,2}\rangle = (|0\rangle + |E_{1,2}\rangle)|\zeta_{1,2}\rangle/\sqrt{2}$, with $|0\rangle$ and $|E_{1,2}\rangle$ energy eigenstates and $|\zeta_{1,2}\rangle$ ancillary orthogonal states with null energy. This state will reach the condition $\mathrm{Tr}[\varrho(t)\varrho] = \epsilon\,\mathrm{Tr}[\varrho^2]$ in a time $t = 0.400\hbar/E_1$ smaller than $\mathcal{T}_\epsilon(E,\Delta E) = 0.410\hbar/E_1$ if the following parameters are employed: $\epsilon = 0.2$, $\phi = 1.070$, $E_2/E_1 = 1.425$.

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