

Tunneling time distribution by means of Nelson's quantum mechanics and wave-particle dualityKoh'ichiro Hara^{1,*} and Ichiro Ohba^{1,2,3,†}¹*Department of Physics, Waseda University, Tokyo 169-8555, Japan*²*Kagami Memorial Laboratory for Materials Science and Technology, Waseda University, Tokyo 169-0051, Japan*³*Advanced Research Center for Science and Technology, Waseda University, Tokyo 169-8555, Japan*

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We calculate a tunneling time distribution by means of Nelson's quantum mechanics and investigate its statistical properties. The relationship between the average and deviation of tunneling time suggests the existence of "wave-particle duality" in the tunneling phenomena.

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I. INTRODUCTION

It was suggested that there is a time associated with the passage of a particle under a tunneling barrier, so-called tunneling time [1]. Actually, several authors have tried to measure the time experimentally [2–4]. However, there is no clear consensus about any definition of tunneling time admissible for everyone. There are several approaches and methods to estimate the tunneling time. For example, Wigner time [5–7] is based on the time evolution of wave packet through the barrier, and the delay time of the peak or the centroid is expressed by an energy derivative of phase shift. Larmor time [8–12] is obtained from the Larmor precession angle caused by a magnetic field confined in the barrier region. The traversal time proposed by Büttiker and Landauer [13–15] is defined by the analysis of transmission coefficient through a static barrier augmented by a small oscillation in the barrier height. This time is obtained by measuring or analyzing the effect of "clock" added on the tunneling barrier. The dwell time [12,16,17] is defined as the total probability of the particle within the barrier divided by the incident probability current. This is only applicable to the stationary-state case. There are other types of methods based on the motion of "particle paths," for example, Bohmian mechanics [18–20], Feynman path integral [21–23], Nelson's quantum mechanics [27–29], and so on. In these methods, the tunneling time is defined as the time spent by particle paths under the tunneling barrier. See Refs. [24,25] and references therein for reviews of this problem.

We think that Nelson's quantum mechanics [26] have some characteristic properties to study the tunneling time as follows. Since this method is described by the real-time stochastic process, it enables us to describe the real-time evolution of individual events as the analogy of classical mechanics. We call such an event of evolution as a "sample path." Moreover, since a sample path has its own history, we can obtain information of time parameter, in particular, tunneling time.

In this paper, first, we calculate a tunneling time distribution by means of Nelson's quantum mechanics [26]. The tunneling phenomena should occur quantum mechanically

and accompany fluctuating properties. If a tunneling particle is described by a wave packet with distribution of wave numbers of finite width, then one would have a distribution of tunneling time. The width of the distribution should be reduced by using the spatially wider wave packet or the sharper distribution of wave-number wave packet. In such case, there should still remain the fluctuation of tunneling time coming from quantum effect. In spite of such an argument, many approaches centered discussion only on the averaged value, because it is difficult to consider the tunneling time distribution by the use of the conventional frameworks of quantum mechanics. However, Nelson's quantum mechanics can afford to predict such distribution, because this method enables us to obtain an ensemble of various sample paths in tunneling barrier. Next, from the tunneling time distribution calculated by this method, we investigate the statistical properties of the distribution, such as the average and the deviation of tunneling time. We found that the relationship between the deviation and average suggests the existence of the "wave-particle duality" in the tunneling phenomena. Last, we discuss the "quantum-classical correspondence" by analyzing the "Planck constant dependence" of tunneling time distribution.

In this paper, for simplicity, we analyze the tunneling phenomena with one-dimensional static rectangular potential barrier with height V_0 and width d as following:

$$V(x) = \begin{cases} 0, & x < -d/2 & \text{(region I)} \\ V_0, & -d/2 \leq x \leq d/2 & \text{(region II)} \\ 0, & x > d/2 & \text{(region III)} \end{cases} \quad (1)$$

The initial Gaussian wave packet with variance Δx^2 ,

$$\psi(x,0) = \left(\frac{1}{2\pi(\Delta x)^2} \right)^{1/4} \exp \left[-\frac{(x-\langle x \rangle)^2}{4(\Delta x)^2} + \frac{i}{\hbar} \langle p \rangle (x-\langle x \rangle) \right], \quad (2)$$

is injected into the potential barrier, where $\langle x \rangle$ and $\langle p \rangle$ are expectation value of position and momentum, respectively. We use the natural unit $m = \hbar = 1$, and perform numerical simulations of 100 000 sample paths. For the numerical

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simulation, we take $[-1000, 1000]$ as the total space for the unit of $1/k_0$, and adopt these parameters as follows, $\langle x \rangle = -500/k_0$, and $\langle p \rangle = k_0$.

II. ESTIMATION OF THE TUNNELING TIME BASED ON THE NELSON'S QUANTUM MECHANICS

In this section, we give a brief review of the Nelson's approach of quantum mechanics [26] which plays a central role in this paper, and explain how to estimate the tunneling time. Nelson's quantum mechanics based on the real-time stochastic process, enables us to describe the quantum mechanics of a single particle in terminology of the "analog" of classical mechanics, i.e., the ensemble of sample paths. These sample paths are generated by the Ito-type Langevin equation

$$dx(t) = [u(x(t), t) + v(x(t), t)]dt + dw(t), \quad (3)$$

where $x(t)$ is a stochastic variable corresponding to the coordinate of the particle, and $u(x(t), t)$ and $v(x(t), t)$ are the osmotic velocity and the current velocity, respectively. The $dw(t)$ is the Gaussian white noise with the statistical properties of

$$\langle dw(t) \rangle = 0 \quad \text{and} \quad \langle dw(t)dw(t) \rangle = \frac{\hbar}{m} dt, \quad (4)$$

where $\langle \dots \rangle$ means the ensemble average with respect to the noise. In principle, the osmotic and the current velocities are given by solving coupled two equation, i.e., the kinetic equation and the "Nelson-Newton equation." Nelson showed that, for the expectation value of the dynamical variable, e.g., x, p , the whole ensemble of sample paths gives us the same results as quantum mechanics in the ordinary approach [26]. Once the equivalence between Nelson's framework and the ordinary quantum mechanics is proved, it is convenient to use the relation

$$u = \text{Re} \frac{\hbar}{m} \frac{\partial}{\partial x} \ln \psi(x, t) \quad \text{and} \quad v = \text{Im} \frac{\hbar}{m} \frac{\partial}{\partial x} \ln \psi(x, t), \quad (5)$$

where ψ is the solution of Schrödinger equation. Since an individual sample path has its own history, we obtain information on the time parameter, e.g., tunneling time definitely.

Now using the Nelson's quantum mechanics, we estimate the tunneling time of a particle crossing over a potential barrier. First, we prepare an incident wave packet given by Eq. (2) from the region I. Next, we solve the time-dependent Schrödinger equation. Last, using the relation Eq. (5), we obtain the drift term of the Langevin equation (3), and calculate sample paths. Assume a simulation of tunneling phenomena based on Eq. (3), starting for $t = -\infty$ and ending $t = \infty$. As we treat a wave packet satisfying the time-dependent Schrödinger equation, the wave packet is located in region I initially and turns finally into two spatially separated wave packets which are regions I and III, respectively. Figure 1 shows a typical transmitted sample path calculated by Eq. (3). Transmitted sample paths originate preferentially from the front of the initial wave packet as suggested by

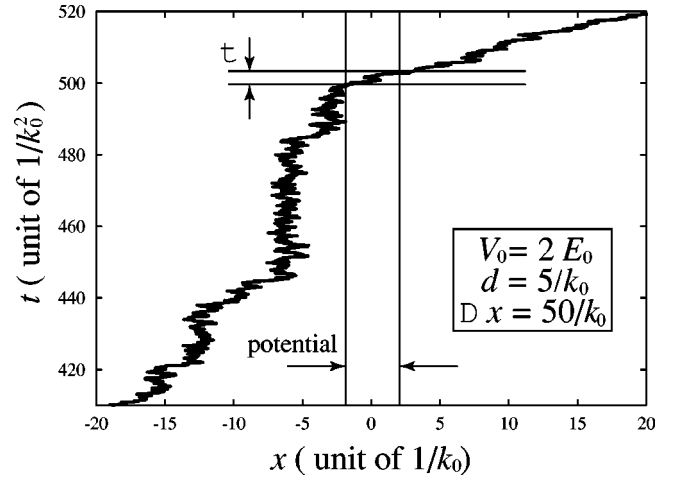


FIG. 1. Typical transmitted sample path calculated by Eq. (3). τ shows the traversal time of barrier, i.e., the tunneling time.

Imafuku *et al.* [27]. Every transmitted sample path has its traversal time of barrier, i.e., the tunneling time which is described as

$$\tau_i \equiv \int_0^{t_f} \Theta(x_i(t)) dt \quad (i=1, 2, \dots, N), \quad (6)$$

where $x_i(t)$ is the i th sample path, t_f is the final time, the function $\Theta(x)$ is unity for $-d/2 \leq x \leq d/2$ and zero otherwise. Collecting these events, we can construct a statistical distribution of tunneling time (Fig. 2).

III. TUNNELING TIME DISTRIBUTION AND WAVE-PARTICLE DUALITY

From the ensemble of sample paths, we define a distribution of tunneling time as follows:

$$P(\tau) \delta\tau \equiv \frac{\delta n(\tau)}{N}, \quad (7)$$

$\delta n(\tau)$ is the number of sample paths with the tunneling time from τ to $\tau + \delta\tau$, and N is the total number of sample paths.

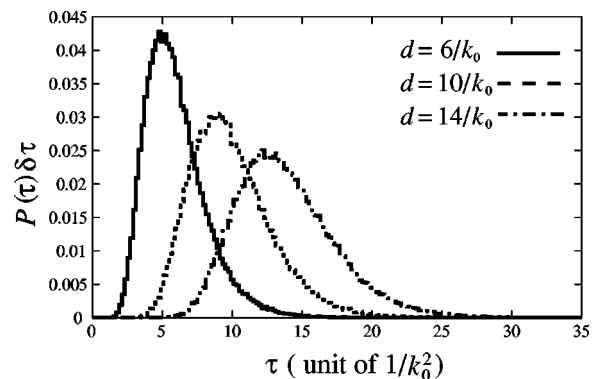


FIG. 2. Numerical results of tunneling time distribution as a function of tunneling time with potential height $V_0/E_0 = 2.0$.

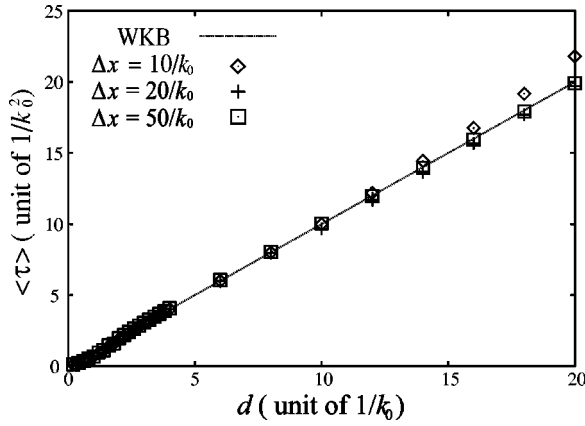


FIG. 3. Numerical results of tunneling time average versus potential width with potential height of $V_0/E_0=2.0$.

Figures 3 and 4 show the average of tunneling time $\langle \tau \rangle$ and its deviation $\Delta \tau$ versus potential barrier width for various width of wave packet Δx . We can see that in the case of Δx greater than $20/k_0$, behaviors of average and deviation are independent on Δx . However, in the case of $\Delta x = 10/k_0$, the data deviate remarkably from the others in the region $d \geq 10/k_0$. It is suggested that this effect originates from the wave packet spreading during the propagation. Indeed the spatial deviation of free wave packet at time t is

$$\sqrt{\Delta x^2 + \frac{\hbar^2}{4m^2 \Delta x^2} t^2}.$$

Therefore, the wave packet spreads twofold at $t = 2m(\Delta x)^2/\hbar$. By taking account of the fixed time $t = 500/k_0^2$ in which the peak of initial wave packet arrives at the left edge of the harder barrier, this occurs in the case of width $\Delta x \leq \sqrt{[250/(mk_0^2)]\hbar} \sim 16/k_0$.

It seems that the numerical results of $\langle \tau \rangle$ are roughly similar to the WKB time $\tau_{\text{WKB}} = md/\hbar \kappa$, where $\kappa = \sqrt{2m(V_0 - E_0)}/\hbar$, expect for the case of $\Delta x = 10/k_0$. However, let us examine much in detail those values in the thin region ($d \leq 4/k_0$). Figure 5 is an enlarged copy of this

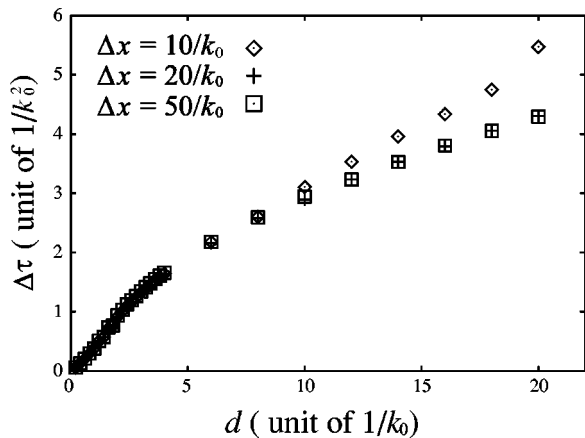


FIG. 4. Numerical results of tunneling time deviation versus potential width for potential height of $V_0/E_0=2.0$.

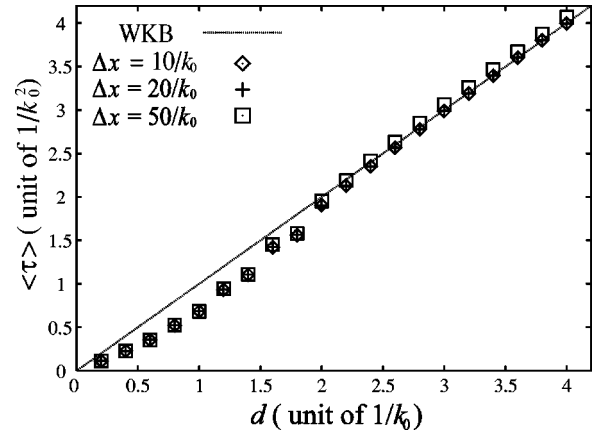


FIG. 5. An enlarged copy of thin barrier region ($d \leq 4/k_0$) in Fig. 3.

part in Fig. 3. It has been shown that, in the opaque case, the numerical simulation gives almost same values of the WKB times [27,28]. While we can see these features of opaque case in Figs. 3 and 5, the numerical values deviate from the WKB time in the translucent case characterized by small κd which is approximately less than 2. The WKB approximation is not proper in the latter case. This suggests that the tunneling phenomena make a “phase transition” in a sense around $\kappa d \sim 2$.

Next we show in Fig. 6 the deviation $\Delta \tau$ as a function of the average $\langle \tau \rangle$, which is calculated by changing the width d with fixed potential height V_0 for several cases of incident energy E_0 . For each case we can see a common feature characterized by the fact that $\Delta \tau$ is proportional to $\langle \tau \rangle$ for $\kappa d \leq 2$ and to $\sqrt{\langle \tau \rangle}$ for $\kappa d \geq 2$. In order to check this feature quantitatively, we fit the tunneling time distribution using the gamma distribution,

$$P(\tau) = \frac{1}{\beta^{\alpha+1} \Gamma(\alpha+1)} \tau^\alpha e^{-\tau/\beta} \quad (\alpha, \beta, \tau > 0). \quad (8)$$

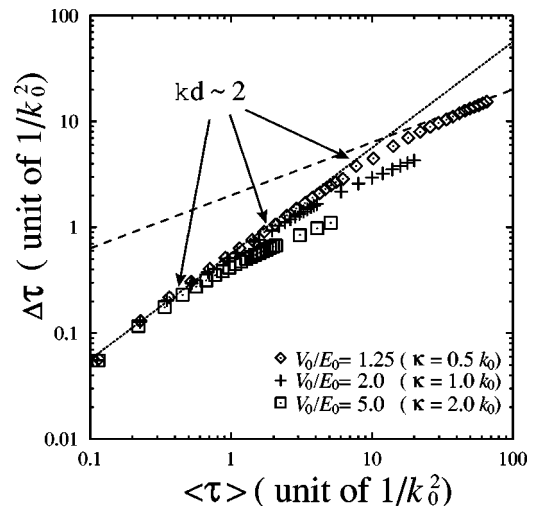


FIG. 6. Relationship between the average and the deviation of tunneling time with the width of wave packet $\Delta x = 50/k_0$. The dotted line shows the relation $\Delta \tau \propto \langle \tau \rangle$. The dashed line shows the relation $\Delta \tau \propto \sqrt{\langle \tau \rangle}$.

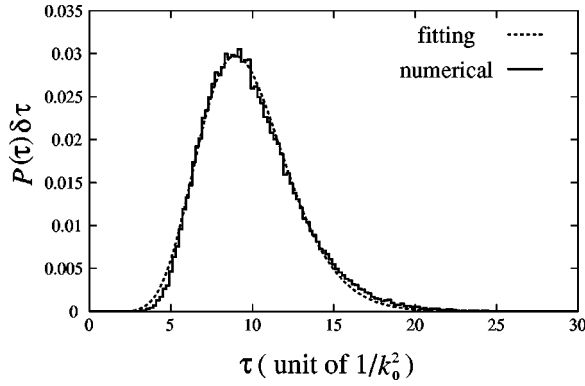


FIG. 7. Example of tunneling time distribution fitted by the gamma distribution in the case of $V_0/E_0=2.0, d=10/k_0, \Delta x=50/k_0$. In this case, we choose the fitting parameters $\alpha=11.3$ and $\beta=0.79$, using least-squares method.

This distribution has the following statistical properties:

$$\langle \tau \rangle = \beta(\alpha + 1) \quad \text{and} \quad \Delta \tau^2 = \beta^2(\alpha + 1). \quad (9)$$

If α is constant, the relation $\Delta \tau \propto \langle \tau \rangle$ holds good, and this is a typical feature of such coherent phenomena as deviation versus average value of photon number in coherent photon state. On the other hand, if β is constant, the relation $\Delta \tau \propto \sqrt{\langle \tau \rangle}$ holds good, and this is a typical feature of such random phenomena as Poisson process and Brownian motion. Figure 7 is an example of fitting distribution.

Figures 8 and 9 show the fitting parameters α and $\kappa^2\beta$ as a function of κd , respectively. We see that the κd dependence of fitting parameters is universal even for different potential heights. Note that κ is kept a constant value for each case. Furthermore, we found that α is constant in the translucent region ($\kappa d \leq 2$), and $\kappa^2\beta$ is constant in the opaque region ($\kappa d \geq 2$). Therefore, this assures the $\Delta \tau - \langle \tau \rangle$ relation as mentioned above.

There is no doubt that the tunneling time is a pure quantum process. However, the tunneling time is not an observable in the quantum mechanics. Actually, it can be closely connected to the time evolution of some specific observable

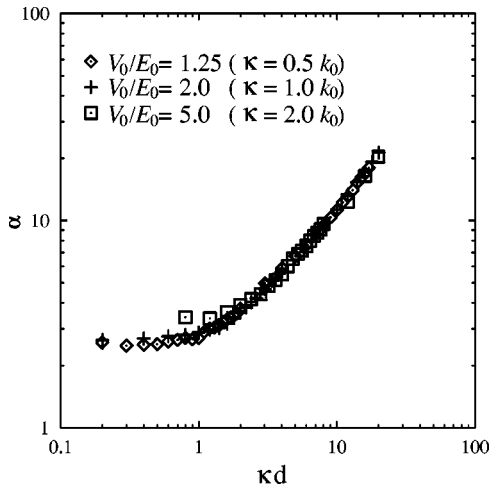


FIG. 8. Fitting parameter α versus κd for fixed κ .

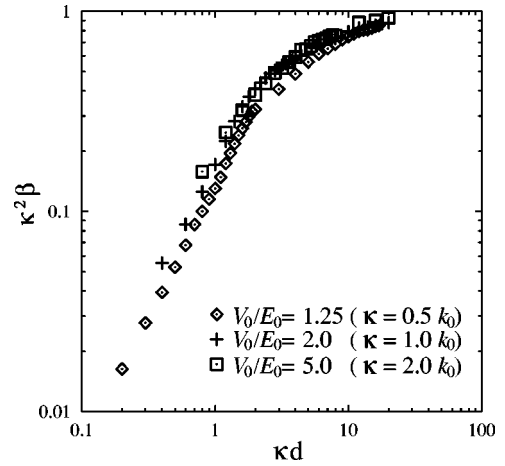


FIG. 9. Fitting parameter β multiplied by κ^2 versus κd for fixed κ .

of the tunneling system and it should be measured experimentally through the time dependence of the observable. Thus the statistical property of the tunneling time distribution should reflect the characteristic features of underlying quantum process. From the above discussion, we think that the tunneling may occur coherently, or in mode of wave picture dominantly in the translucent case and randomly, or in mode of particle picture dominantly in the opaque case. The tunneling time distribution reveals the wave-particle duality in the tunneling phenomena.

IV. QUANTUM-CLASSICAL CORRESPONDENCE

We discuss the Planck constant dependence of tunneling time distribution. First, we introduce the parameter ϵ ($0 < \epsilon \leq 1$) and the Planck constant $\tilde{\hbar} = \epsilon \hbar$, and consider ‘‘Schrödinger equation’’ of a wave function $\tilde{\psi}(x, t)$,

$$i\tilde{\hbar} \frac{\partial}{\partial t} \tilde{\psi}(x, t) = \left[-\frac{\tilde{\hbar}^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \tilde{\psi}(x, t), \quad (10)$$

under the conditions with fixed values of m , V_0 , and p_0 ($= \hbar k_0$). This wave function describes a virtual quantum system with scaled Planck constant $\tilde{\hbar}$. This equation is formally transformed into the ordinary Schrödinger equation

$$i\hbar \frac{\partial}{\partial T} \Phi(X, T) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial X^2} + W(X) \right] \Phi(X, T), \quad (11)$$

by a scale transformation as following:

$$X \equiv x/\epsilon, \quad T \equiv t/\epsilon, \quad W(X) \equiv V(\epsilon X), \quad \text{and}$$

$$\Phi(X, T) \equiv \tilde{\psi}(\epsilon X, \epsilon T). \quad (12)$$

Here let us clear up the procedure of the simulation. We fix the parameters $m, V_0, E_0 = p^2/2m$, and d . We scale the space-time variables as $X = x/\epsilon$, $T = t/\epsilon$ and the potential width as $D = d/\epsilon$. In this $X-T$ reference frame de Broglie plane wave is scaled as $e^{i(kx - \omega t)} \rightarrow e^{i(\tilde{k}X - \tilde{\omega}T)}$, where $\tilde{k} = \epsilon k$ and $\tilde{\omega}$

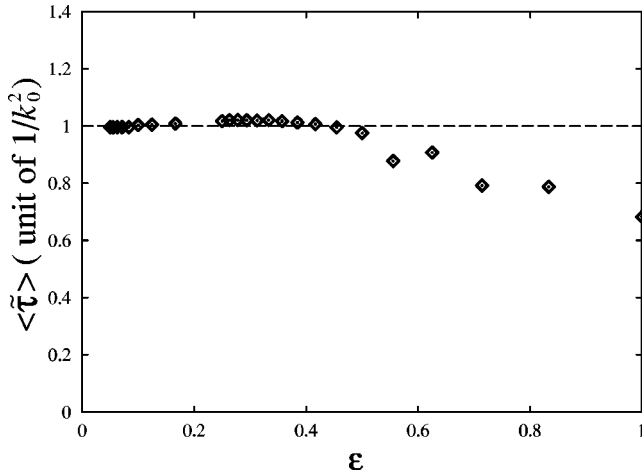


FIG. 10. ϵ dependence of tunneling time average with potential height $V_0/E_0=2.0$ and width $d=1/k_0$ ($\kappa d=1$). The dashed line shows the WKB time $\tilde{\tau}_{\text{WKB}} = \sqrt{m/[2(V_0-E_0)]}d$.

$=\epsilon\omega$. Then we perform the numerical simulation based on Eq. (11) and obtain the distribution of τ in the $X-T$ reference frame. Eventually we obtain the distribution of $\tilde{\tau}$, the tunneling time corresponding to a Planck constant $\tilde{\hbar}$, by the scaling of $\tilde{\tau} = \epsilon\tau$.

Now we will examine how the average $\langle \tilde{\tau} \rangle = \epsilon \langle \tau \rangle$ and the deviation $\Delta \tilde{\tau} = \epsilon \Delta \tau$ depend on $\tilde{\hbar}$. Let us start the simulation at a translucent case with $\epsilon=1$, where the tunneling process proceeds in the wave mode. Then, we decrease the value of ϵ gradually, the effective width of tunneling barrier becomes wider and the tunneling process should proceed in the particle mode, or in other words, quasiclassically. This procedure gives us the ϵ dependence of $\langle \tilde{\tau} \rangle$ shown in Fig. 10. We can see that $\langle \tilde{\tau} \rangle$ increases with a tendency to approach the WKB time as ϵ is decreased in the region near $\epsilon=1$, whereas it is almost independent of ϵ in the region with smaller ϵ . These tendencies may be understood as follows. As it is seen in Fig. 5, the simulated values of $\langle \tau \rangle$ fit almost to the WKB time,

$$\tau_{\text{WKB}} = \sqrt{\frac{m}{2(V_0-E_0)}}D, \quad (13)$$

in the latter case. Thus, the tunneling time corresponding to $\tilde{\hbar}$ is

$$\langle \tilde{\tau} \rangle = \epsilon \tau_{\text{WKB}} = \sqrt{\frac{m}{2(V_0-E_0)}}d.$$

Actually, note that the WKB time can be expressed only by classical quantities. On the other hand, in the translucent case, the simulated values are smaller than the WKB time and approach gradually it as the effective width of tunneling barrier becomes wider, that is, ϵ is decreased. Moreover, these two tendencies cross each other at $\epsilon \sim 1/2$, which suggests that the tunneling phenomena change their phase around $\hbar \kappa d \sim 2\tilde{\hbar}$.

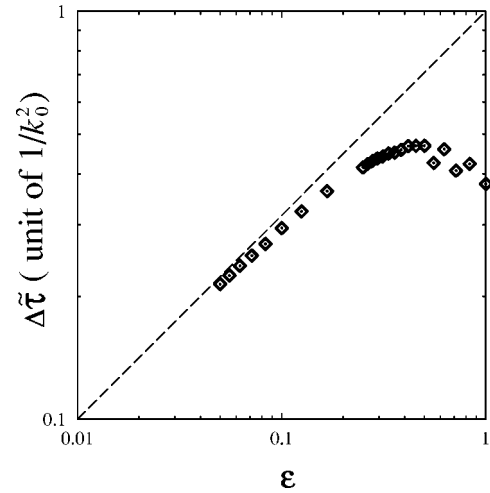


FIG. 11. ϵ dependence of tunneling time deviation with potential height $V_0/E_0=2.0$ and width $d=1/k_0$ ($\kappa d=1$). The dashed line shows the relation $\Delta \tilde{\tau} \propto \sqrt{\epsilon} \sqrt{\langle \tilde{\tau} \rangle}$ in the case of particle mode.

Next, let us consider the ϵ dependence of $\Delta \tilde{\tau}$. In this simulation, we may guess that in the region near $\epsilon=1$ (wave mode and $\Delta \tau \propto \langle \tau \rangle$), the deviation should be proportional to the average,

$$\Delta \tilde{\tau} \propto \langle \tilde{\tau} \rangle, \quad (14)$$

whereas in the region with smaller ϵ (particle mode and $\Delta \tau \propto \sqrt{\langle \tau \rangle}$), it should behave as

$$\Delta \tilde{\tau} \propto \sqrt{\epsilon} \sqrt{\langle \tilde{\tau} \rangle}. \quad (15)$$

Moreover, the change from the former tendency to the latter one may occur around $\hbar \kappa d \sim 2\tilde{\hbar}$.

We can see these tendencies in Fig. 11, which give another support of the idea of “wave-particle duality” in the tunneling phenomena.

V. SUMMARY

We calculate the tunneling time distribution by means of Nelson’s quantum mechanics. From the resulting distribution, we derived the statistical properties of it, the average and deviation of tunneling time. First, we found that if an incident wave packet is so large as to look plane-wave-like, the Δx dependence of them is negligible. Next, fitting the data by gamma distribution, we found that the shape of distribution is universally determined only by κd . Furthermore, by investigating the statistical properties of the distribution in two characteristic “translucent” and “opaque” regions roughly divided by $\kappa d \sim 2$, we found that the wave-particle duality may be seen in the tunneling phenomena. Last, we consider the Planck constant dependence of tunneling time, introducing the parameter ϵ as $\tilde{\hbar} = \epsilon \hbar$. Consequently, we found that the dependences of the average and the deviation suggest another support of the idea of wave-particle duality in tunneling phenomena.

We are interested in the comparison between the tunneling time distribution based on the Nelson's quantum mechanics and that based on the other method, e.g., Bohmian mechanics. The detailed study of such a comparison is a subject in the near future.

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