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Anomalous diffusion and Tsallis statistics in an optical lattice

Eric Lutz
Sloane Physics Laboratory, Yale University, P.O. Box 208120, New Haven, Connecticut 06520-8120
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We point out a connection between anomalous transport in an optical lattice and Tsallis' generalized statistics. Specifically, we show that the momentum equation for the semiclassical Wigner function which describes atomic motion in the optical potential, belongs to a class of transport equations recently studied by Borland [Phys. Lett. A **245**, 67 (1998)]. The important property of these ordinary linear Fokker-Planck equations is that their stationary solutions are exactly given by Tsallis distributions. An analytical expression of the Tsallis index q in terms of the microscopic parameters of the quantum-optical problem is given and the spatial coherence of the atomic wave packets is discussed.

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Non-Gaussian distributions occur frequently in systems that do not follow the prescriptions of standard statistical mechanics. Prominent examples of non-Gaussian statistics are Lévy stable distributions [1]. The probability density of a one-dimensional symmetric Lévy stable distribution is defined by its Fourier transform as $\mathcal{L}_\alpha^C(x)=1/(2\pi)\int dk \exp[ikx-C|k|^\alpha]$, where $0<\alpha\leqslant 2$. A key feature of such a stable distribution is the presence of an asymptotic, non-Gaussian, power-law tail, $\mathcal{L}_\alpha^C(x)\sim 1/|x|^{\alpha+1}$ when $\alpha<2$. This leads to the important consequence that, except in the Gaussian case $\alpha=2$, a Lévy probability density has a divergent second moment [2]. Signatures of Lévy statistics have been experimentally observed in a variety of physical systems [3] ranging from micelle systems [4] to porous glasses [5] and subrecoil laser cooling [6]. Another non-Gaussian distribution, which naturally arises within the framework of nonextensive statistical mechanics [7,8], is the Tsallis distribution, $P_q(x)=Z_q^{-1}[1-\beta(1-q)x^2]^{1/(1-q)}$, with $1\leqslant q<3$. Similar to a Lévy stable distribution, the function $P_q(x)$ exhibits an asymptotic, non-Gaussian, algebraic tail, $P_q(x)\sim 1/x^{2/(q-1)}$, for a Tsallis index $q\neq 1$. Typical systems where Tsallis' generalized thermostatistics has been applied are those involving long-range correlations, such as self-gravitating systems [8] or long-range magnetic systems [9] and systems with fluctuating temperature [10]. In the last decade, the theory of nonextensive statistical physics witnessed a tremendous development [11] and there is now also growing experimental evidence of the relevance of Tsallis statistics in describing physical processes [8]. As an example, we mention fully developed turbulence, where it has recently been shown that velocity fluctuations can be described by a Tsallis-like distribution [12].

Our aim in this paper is to show that there is a connection between Tsallis statistics and anomalous transport in optical lattices. An optical lattice is a standing-wave potential that can be obtained by a superposition of counterpropagating laser beams with linear orthogonal polarizations (other configurations are also possible—for a recent review, see Ref. [13]). The optical potential so produced is spatially periodic and, as a consequence, shares many common properties with crystalline lattices in solid-state physics, such as Bragg scattering and Bloch oscillations. The main advantage of an optical crystal compared to its condensed-matter counterpart is that the optical periodic potential is exactly known and, furthermore, easily modified in a precise and controlled way. Originally designed for laser cooling (Sisyphus cooling—for an introduction, see, for example, Ref. [14]), optical lattices rapidly evolved to an active field of investigation on its own [13].

An important issue in this context is the understanding of atomic transport in the optical lattice. Depending on the depth of the optical potential, three different regimes can be identified [13,15–19]: (i) diffusive motion in deep potentials, (ii) ballistic motion in shallow potentials, and (iii) an intermediate regime in between (of main interest here), where anomalous (non-Gaussian) diffusion takes place. The existence of Lévy-like diffusion with long jumps below a given potential threshold has been predicted by Marksteiner *et al.* [18] and later experimentally verified by a group at the MPQ in Garching by studying the dynamics of a single ion in a one-dimensional optical lattice [19]. In the following, we show that the equation governing the evolution of the semiclassical momentum distribution of the atom in the optical potential belongs to a family of ordinary linear Fokker-Planck equations recently defined by Borland [20]. An interesting property of these equations is that their stationary solutions are exactly given by Tsallis statistics. This allows us not only to express the indices q and β of the Tsallis distribution in terms of the microscopic parameters of the quantum-optical problem, but also to give a physical explanation for the non-normalizability of the distribution, as well as for the divergence of its variance in some range of parameters to be specified. We finally evaluate the spatial correlation function of the atomic wave packets and conclude by discussing their spatial coherence properties.

Starting from the microscopic Hamiltonian that describes the atom-laser interaction in the optical lattice, an atomic quantum master equation can be derived [21]. After spatial averaging, the Rayleigh equation for the corresponding semiclassical Wigner function $W(p,t)$ can be written as [15,16,18]

$$\frac{\partial W}{\partial t}=-\frac{\partial}{\partial p}[K(p)W]+\frac{\partial}{\partial p}\left[D(p)\frac{\partial W}{\partial p}\right]. \tag{1}$$

The Rayleigh equation (1) is obtained under the following assumptions: (i) the laser intensity is low, that is, the saturation parameter is small, $s \ll 1$, (ii) the atoms move very fast, $mv^2/2 \gg U_0$, where U_0 is the potential depth, so that all positions along the lattice can be considered equiprobable (hence, allowing spatial averaging), and (iii) the semiclassical limit further imposes that $p \gg \hbar k$, where k is the wave number of the laser field. Equation (1) has the form of an ordinary linear Fokker-Planck equation with momentum-dependent drift and diffusion coefficients,

$$K(p) = -\frac{\alpha p}{1+(p/p_c)^2}, \quad D(p) = D_0 + \frac{D_1}{1+(p/p_c)^2}. \quad (2)$$

These two quantities have a simple physical interpretation: The drift $K(p)$ represents a cooling force (due to the Sisyphus effect) with damping coefficient α. This force acts only on slow particles with a momentum smaller than the capture momentum p_c. This is an important point as we shall discuss below. The diffusion factor $D(p)$, on the other hand, describes stochastic momentum fluctuations and accounts for heating processes. We note that $D(p)$ has two contributions [15]: A constant part D_0 that corresponds to fluctuations due to spontaneous photon emissions and fluctuations in the difference of photons absorbed in the two laser beams, plus a term proportional to D_1 which stems from fluctuations in the dipolar forces. This last term has the same limited momentum range p_c as the drift force. Interestingly, we remark that for vanishing D_1, Eq. (1) exactly reduces to the Fokker-Planck equation studied by Stariolo and gives rise to nonextensive statistics [22]. It should also be mentioned that in Ref. [18], an asymptotic approximation of Eq. (1) (constant diffusion and drift decaying for large p as $-1/p$) was considered to evaluate the long-time behavior of the momentum correlation function. Here, we shall be interested in the exact stationary solution of Eq. (1).

It is easily seen from Eq. (2) that K and D satisfy the following condition:

$$\frac{K(p)}{D(p)} = -\frac{\beta}{1-\beta(1-q)U(p)} \frac{\partial U(p)}{\partial p}, \quad (3)$$

with

$$\beta = \frac{\alpha}{2(D_0+D_1)}, \quad q = 1 + \frac{2D_0}{\alpha p_c^2}, \quad \text{and} \quad U(p) = p^2. \quad (4)$$

Equation (3) has been first obtained by Borland [20]. We mention that in her original work, Borland considered the Ito form of the Fokker-Planck equation, whereas here, Eq. (3) applies to the Stratonovich form (1). Condition (3) implies that the stationary solution $W_q(p)$ of the Rayleigh equation (1) is given by the Tsallis distribution:

$$W_q(p) = Z_q^{-1}[1-\beta(1-q)U(p)]^{1/(1-q)}. \quad (5)$$

Equation (5) is the exact general stationary solution of Eq. (1) with the requirement $W_q(p) \to 0$ when $p \to \pm\infty$, the constant Z_q being a normalizing factor. The fact that the steady-state solution of Eq. (1) is non-Gaussian is, of course, well known [15,16,18]. Surprisingly, however, it has not been realized that this *precisely* corresponds to a Tsallis distribution. Among infinitely many non-Gaussian distributions, Eq. (3) singles out the Tsallis distribution (5). It is worth noting that the Tsallis indices q and β can be simply expressed in terms of the microscopic parameters of the problem [see Eqs. (4)]. In particular, we see that q depends on the ratio of the diffusion constant D_0 to the product of the friction coefficient α with the square of the capture momentum p_c, and does not depend on D_1. Equations (4) thus provide a link between the macroscopic Tsallis distribution (5) and the underlying microscopic dynamics in the optical potential. This allows us to give a physical interpretation of the characteristics of distribution (5).

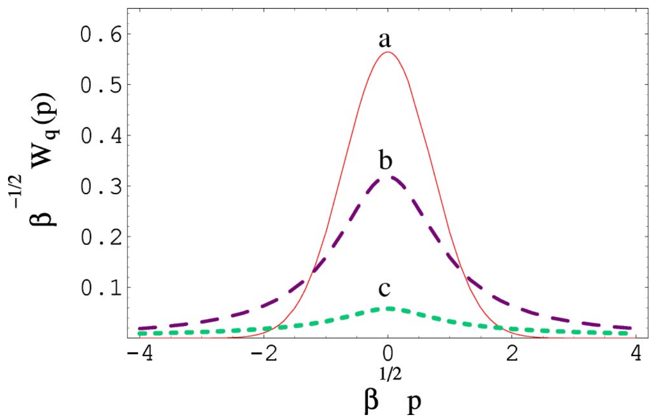

FIG. 1. Stationary momentum distribution $W_q(p)$ (5) for three values of the potential depth: (a) $U_0 \gg 44E_R$, (b) $U_0 = 44E_R$, and (c) $U_0 = 24E_R$.

Let us first remind that distribution (5) is not normalizable for a Tsallis index $3 \leqslant q$ or, equivalently, for $\alpha p_c^2 \leqslant D_0$. Physically, this means that the cooling force, as measured by αp_c^2, is too weak compared to the random momentum fluctuations, given by D_0, to maintain the particle in a steady state around $p=0$ (this is often referred to as *décrochage* [15,16]). On the other hand, in the limit where $q \to 1$ ($D_0 \ll \alpha p_c^2$), the stationary solution (5) reduces to the standard Maxwell-Boltzmann distribution, $W_1(p) = Z_1^{-1} \exp[-\beta U(p)]$, with an inverse temperature β (see Fig. 1). In this case, the cooling force is much stronger than the random momentum fluctuations. It thus appears that the Tsallis index q is intimately related to the interplay between stochastic heating processes (momentum fluctuations, as measured by D_0) and the cooling force with capture momentum p_c. It is important to remark that the finiteness of the latter is directly responsible for the occurrence of the non-Gaussian Tsallis distribution in this problem. This is confirmed by the observation that for infinite p_c, Eq. (1) reduces to the Ornstein-Uhlenbeck equation with well-known Gaussian dynamics. Using the parametrization of Ref. [15], the index q can be further written as $q = 1 + 44E_R/U_0$, where E_R is the recoil energy. We thus see that the Tsallis index can be related to the ratio of the recoil energy to the potential depth. This means that the nature of the atomic dynamics can be simply tuned by varying the depth of the optical lattice. We also notice that the inverse temperature β is written as the ratio of the friction coefficient to the sum of the diffusion coefficients, in analogy with the fluctuation-dissipation relation. We hasten to add that Eq. (5) corresponds to a steady state and not to an equilibrium state, and as such, temperature is not well defined in this problem.

We now turn to the intermediate regime with a Tsallis index $5/3<q<3(D_0<\alpha p_c^2<3D_0)$. Here, the second moment, $\langle p^2\rangle=\int p^2 W_q(p)dp$, of the Tsallis distribution is infinite. As a consequence, the mean kinetic energy of the particle, $E_K=\langle p^2\rangle/2m$, diverges. In this regime, rare but large momentum fluctuations occur which shove the particle outside the range of the cooling force before it is recaptured again. This leads to an anomalous momentum diffusion. The transition from Gaussian to anomalous diffusion as the depth of the optical lattice is decreased has recently been investigated experimentally and the divergence of the mean kinetic energy has been observed [19]. This is a clear signature of the underlying non-Gaussian statistics. A dissipative optical lattice hence appears as a unique system that allows an investigation of the Tsallis distribution in a whole range of q by simply varying a single parameter, the depth of the optical potential.

We emphasize that the non-Gaussian Tsallis statistics is here generated by an *ordinary linear* Fokker-Planck equation (3), which is often associated with the usual Boltzmann-Gibbs statistics. To our knowledge, atomic transport in an optical lattice constitutes the only physical system, known so far, where this occurs. Again, this results from the subtle interplay between the deterministic (drift) and stochastic (diffusion) forces (2) that act on the particle [20]. This is, for instance, at variance with the fully developed turbulence problem discussed in Ref. [12]. In the latter case, Tsallis statistics are obtained from a generalized Langevin equation with *fluctuating* friction and diffusion coefficients (the probability distribution of the corresponding temperature fluctuations being arbitrary). For comparison, the Langevin equation that corresponds to the Rayleigh equation (1) reads

$$\dot{p}=K(p)+\frac{\partial D(p)}{\partial p}+\sqrt{2D(p)}\,\eta(t), \tag{6}$$

where $\eta(t)$ is a centered Gaussian random force with variance $\langle \eta(t)\eta(t')\rangle=\delta(t-t')$. Equation (6) is a Langevin equation with multiplicative white noise and *deterministic* coefficients.

Broad momentum distributions also occur in velocity selective coherent population trapping (VSCPT), a subrecoil laser cooling method (for a recent review, see Ref. [6]). The physical mechanism that leads here to power-law distributions is based on a succession of trapping and recycling processes: during their random walk in momentum space, the particles can remain trapped for a very long time in a region $|p|<p_{trap}$, around the origin $p=0$, before leaving again. The broad momentum distribution of the cold atoms can then be shown to result from the competition between the rates of entry and departure in the trap. Interestingly, VSCPT can be successfully modeled using Lévy statistics—Lévy statistics being even used as a tool to optimize the cooling process. A recent measurement of the momentum distribution nicely confirmed the prediction of this statistical model [6].

An interesting quantity to look at is the spatial correlation function $G(a)=\int dx\,\psi(x,t)\,\psi^*(x+a,t)$, where $\psi(x,t)$ is the wave function of an atomic wave packet at time t [23]. In the case of a free evolution, $G(a)$ is simply the Fourier transform of the initial momentum distribution, $G(a)=\int dp\,\exp[-ipa/\hbar]|\psi(p,0)|^2$. The spatial correlation function is an important quantity both from a theoretical and an experimental point of view. Theoretically, the function $G(a)$ is the overlap integral between two identical wave packets separated by a distance a and, hence, $G(a)$ gives a measure of the spatial coherence of a state between two different points. On the other hand, experimentally it is often easier to directly measure $G(a)$ rather than $|\psi(p,0)|^2$, especially when the momentum distribution is very narrow [24]. This method has been recently used to measure the temperature of ultracold atoms obtained by VSCPT [25]. For atoms in an optical lattice, the measurement of the function $G(a)$ could be achieved by first switching off the optical potential, splitting the atomic wave packet, and then, after a time interval t, projecting one of the wave packets onto the other. If the atoms are not in a pure state but in a mixture of states, the spatial correlation function can be generalized to $G(a)=\int dx\,\rho(x,x+a,t)=\int dp\,\exp[-ipa/\hbar]\rho(p,p,0)$, where $\rho(t)$ is the density operator at time t. Now making use of a well-known property of the Wigner transform, we readily infer that $\rho(p,p,0)=W_q(p)$, where $\rho(p,p,0)$ is the initial momentum distribution of the atoms just after the lattice has been switched off and $W_q(p)$ is the stationary solution of the Fokker-Planck equation as given by Eq. (5). After Fourier transformation, we find

$$G_q(a)=\frac{\sqrt{\pi}2^{(3/2)-(1/r)}}{Z_q\Gamma\left(\frac{1}{r}\right)(\beta r)^{(1/4)+(1/2r)}}\left|\frac{a}{\hbar}\right|^{(1/r)-(1/2)} \times K_{(1/2)-(1/r)}\left(\frac{1}{\sqrt{\beta r}}\left|\frac{a}{\hbar}\right|\right), \tag{7}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind of order ν and $r=q-1$. In the Gaussian limit $q\to 1$, the spatial correlation function is given by a Gaussian,

$$G_1(a)=\frac{1}{Z_1}\sqrt{\frac{\pi}{\beta}}\exp\left[-\frac{a^2}{4\beta\hbar^2}\right], \tag{8}$$

with a correlation length $\lambda_1=2\hbar\sqrt{\beta}$. For $q>1$, we can use the asymptotic representation of the modified Bessel function for large arguments, $K_\nu(x)\sim\sqrt{\pi/2x}\exp[-x]$, to write for $a\gg 1$,

$$G_q(a)\sim a^{(2-q)/(q-1)}\exp\left[-\frac{a}{\hbar\sqrt{\beta(q-1)}}\right]. \tag{9}$$

We see that due to the power-law tails of the momentum distribution, the function $G_q(a)$ now asymptotically decays according to an exponential—hence much slower than in the Gaussian case—with a spatial correlation length $\lambda_q = \hbar\sqrt{\beta(q-1)}$, which is explicitly q dependent. We also note that λ_q increases with increasing q. We can therefore conclude that for $q>1$, the atomic wave packets show more spatial coherence than in the Gaussian regime. Nonetheless, in the limit of very large separation, the non-Gaussian wave packets do become orthogonal since the function $G_q(a)$ vanishes. As a final remark, we also mention that $\rho(p,p,0)$, as given by Eq. (5), has exactly the same form as the density operator corresponding to the power-law quantum wave packets recently introduced by Lillo and Mantegna [26].

In conclusion, we have shown that Tsallis statistics naturally appear in anomalous transport in a one-dimensional optical lattice. Remarkably, the Tsallis distribution is here generated by an ordinary linear Fokker-Planck equation and not by some generalized (nonlinear) diffusion equation. Furthermore, the Tsallis index q can be simply expressed in terms of the microscopic parameters of the quantum-optical problem, in particular, the potential depth U_0. This shows that the shape of the distribution can be straightforwardly modified—from a Gaussian to a uniform distribution—by solely varying U_0. We have also discussed the spatial coherence of the atomic wave packets with the help of the spatial correlation function $G_q(a)$ and have found a higher degree of spatial coherence in the non-Gaussian regime. Can these results be transposed to higher-dimensional optical lattices? For isotropic potentials, one might expect that this is indeed the case, however, for more general potentials the question remains open.

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