Entanglement generation and Hamiltonian simulation in continuous-variable systems

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Several recent experiments have demonstrated the promise of atomic ensembles for quantum teleportation and quantum memory. In these cases, the collective internal state of the atoms is well described by continuous variables corresponding to the operators X_1, P_1 and the interaction with the optical field (X_2, P_2) by a quadratic Hamiltonian $X_1 X_2$. We show how this interaction can be used optimally to create entanglement and squeezing. We derive conditions for the efficient simulation of quadratic Hamiltonians and the engineering of all Gaussian operations and states.

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I. INTRODUCTION

After the first experiments [1] on quantum teleportation using two-mode squeezed states of light [2,3], as well as those [4–6] dealing with entanglement in atomic ensembles [7,8], a significant amount of work has been devoted to develop a quantum information theory of continuous-variable (cv) systems [9]. So far, most of the theoretical work has focused on the entanglement properties of the quantum states involved in all these experiments, the so-called *Gaussian states*. Some examples of the achievements in this field are the following. The problem of qualifying entanglement has been solved in the general bipartite setting [10–13] and in the three-mode case [14]. The distillation problem has also been answered in the general case [15], as well as in the case in which the class of allowed operations is restricted to those that conserve the Gaussian form [16–18]. In contrast to all this theoretical work on (the static) entanglement properties of Gaussian states, very few general results have been obtained on the dynamics of entanglement on these systems, i.e., on how to use the interactions provided by the physical setups in order to entangle the systems in the most efficient way, see, however, Refs. [19–22]. This paper provides a rather complete theory of the dynamics of entanglement in these experimental settings.

The dynamics of entanglement has been recently analyzed in systems of two or more qubits [23–28]. In that case one distinguishes between two scenarios. In the first one [23,25], the interaction between the qubits is described by a Hamiltonian H. The goal is to determine the sequence of local gates for which the increase of entanglement after some small (infinitesimal) time is maximal for a given initial state. In the second one [24,26,28], the interaction is given in terms of a nonlocal gate, which can be applied only once. Apart from its fundamental interest, these studies give some practical ways of creating entanglement in the most efficient way and may become relevant in several experimental situations. Another interesting and related problem is the one of Hamiltonian and gate simulation [29–34]. Here, one assumes that the two qubits interact via some given Hamiltonian H and the goal is to determine a sequence of local instantaneous gates in order to obtain in minimal time either a complete time evolution generated by some other Hamiltonian (Hamiltonian simulation) or some desired unitary gate (gate engineering).

In the present paper, we analyze all these problems for two-mode pure Gaussian states and interaction Hamiltonians which preserve the Gaussian character. We also study the generation of squeezing, since although it has no counterpart in the qubit case, it is a valuable resource in present experiments [35]. Given the fact that we touch on several different topics and therefore develop different mathematical tools, we have decided to write a section which explains in detail the different problems we consider and the corresponding results. In the following sections, we give detailed derivations of these results.

We stress the fact that the problems studied here are all motivated by the experimental situation in which light gets entangled with an atomic ensemble via a Kerr-like interaction [8,36–38]. We expect that the techniques developed in this paper can be easily extended to address other related problems, like the one of entangling two atomic ensembles using light.

The paper is organized as follows: Section II should be considered as a survey of the results presented in the paper. In Sec. III, we show which Hamiltonians can be simulated using a given interaction and how to do so optimally. We also show that, in fact, any general Gaussian operation can be generated in the considered setup. In Sec. IV B, we determine the optimal rate of entanglement generation as well as of squeezing generation for arbitrary input states. Finally, in Sec. IV C, we give an optimal entanglement generation scheme for finite times, starting out from a product (unsqueezed) state.

II. OVERVIEW

This section gives an overview of the content of this paper and it is further divided into sections. In the first one, we explain the physical setup that we are going to analyze. In the second one, we collect the main definitions used thereafter. In the third section, we give the main results of the paper without proving them. For the detailed derivations, we refer the reader to the following sections.

A. Setup

We consider a continuous-variable system composed of two one-mode systems coupled via some interaction Hamil-

tonian. The goal is to analyze which kind of evolutions we can achieve with such an interaction if certain instantaneous local operations can be applied at will. In particular, we study optimal methods of creating or increasing the entanglement shared by the two modes.

The interaction Hamiltonian has the general form

$$H = aX_1X_2 + bP_1P_2 + cP_1X_2 + dX_1P_2, \tag{1}$$

where $a,b,c,$ and d are real parameters, and $X_{1,2}$ and $P_{1,2}$ are canonical operators for the first and second mode, respectively [39]. We use dimensionless units throughout the paper. We assume that local operations, generated by the Hamiltonians

$$H_{\mathrm{loc},i} = g(X_i^2 + P_i^2), \tag{2}$$

can be applied instantaneously, where g is a real number that can be tuned at will [39]. These operations can neither change the entanglement nor the squeezing present in the state. Lastly, we assume that the initial state is pure and Gaussian.

Our choice of the Hamiltonian interaction as well as the instantaneous local operations is motivated by current experiments with atomic ensembles [4–6,40]. In particular, to those setups in which an atomic ensemble interacts with two modes of the electromagnetic field [41] with different polarizations [7,36,42,43]. If the atoms are sufficiently polarized along some given direction (say x), we can replace the total angular-momentum operators describing the internal state of the atoms by canonical operators. That is (if the involved levels have spin $\pm 1/2$), $S_y \to X_1/\sqrt{N/2}$, $S_z \to P_1/\sqrt{N/2}$, $S_x \to N/2$, with $[X_1,P_1] \simeq i$ ($\hbar = 1$), and where N is the number of atoms [44]. Similarly, if the electromagnetic field is sufficiently polarized along some direction, we can substitute the Stokes operators by canonical ones, X_2 and P_2 [45].

For some atomic structures and off-resonant interactions, the Hamiltonian describing the interaction between the atomic ensemble and the light can be written as [36]

$$H_0 = aX_1X_2, \tag{3}$$

which is a particular case of Eq. (1); in the following we will put the coupling constant $a=1$ when referring to H_0. In the same scenario, simple and fast local operations can be performed on the atoms and the electromagnetic field. For example, a magnetic field or a polarizer gives rise to the local Hamiltonians Eq. (2). Since the interaction between atoms and light is typically weak, with moderate magnetic fields the operations generated locally can be regarded as instantaneous. On the other hand, if the atoms and the light are completely polarized, the corresponding state in terms of our continuous-variable description is the tensor product of two vacuum states, in particular, it is a pure Gaussian state.

We emphasize that even though we have motivated our choices with some particular physical setup, our description is applicable to other physical situations and our results apply to the general interaction Hamiltonian equation (1).

Now we consider the following general *strategy* for state or gate engineering which can be realized using the tools described above. Starting with a pure initial state, described by the density operator $\rho(0)$, we perform fast local operations $V_0 \otimes W_0$ on the state and we then let H act on it for a time t_1. Then we perform again local rotations, $V_1 \otimes W_1$ followed by the nonlocal interaction generated by H for a time t_2 and so on until $\Sigma_k t_k = t$. This yields to the total time-evolution operator

$$\mathcal{U}(t) = [V_n \otimes W_n]U(t_n)\cdots U(t_2)[V_1 \otimes W_1]U(t_1)[V_0 \otimes W_0], \tag{4}$$

so that $\rho(t) = \mathcal{U}(t)\rho(0)\mathcal{U}(t)^\dagger$. Here $U(t) = e^{-iHt}$.

First, we want to analyze which $\mathcal{U}$ are achievable with this strategy. Second, for a given $\rho(0)$, we look for the best choice of n, $\{t_1,\ldots,t_n\}$, and the local operations $\{V_1 \otimes W_1,\ldots,V_n \otimes W_n\}$ in order to maximize either the created entanglement or the created squeezing. We consider two different regimes. First, we choose $\Sigma_k t_k = \delta t \ll \tau(H)$ (the characteristic time of the interaction) so that we can expand all the U as well as $\mathcal{U}(t)$ in lowest order in t_k. Second, we choose t_k finite. In the following, we refer to those two regimes as infinitesimal and finite, respectively.

B. Some definitions

Since all the Hamiltonians we are considering are at most quadratic in X and P, an initial Gaussian state will be Gaussian at all times. This means that we can fully describe it by the first and second moments of R_k, with $\vec{R} = (X_1,P_1,X_2,P_2)^T$, i.e., the expectation values $d_k = \mathrm{tr}(\rho R_k)$, (also called displacements) and $\mathrm{tr}[\rho(R_k - d_k)(R_l - d_l)]$. The latter are collected in the *correlation matrix* (CM) of the state ρ, the real, symmetric, positive matrix γ defined by

$$\gamma_{kl} = 2\,\mathrm{Re}\{\mathrm{tr}[\rho(R_k - d_k)(R_l - d_l)]\}. \tag{5}$$

In our description, the displacements are of no importance: they have no influence on the entanglement and squeezing properties of the states and can be brought to zero by local displacement operations, which can be easily implemented in our physical setup. Therefore, we take $d_k = 0$ in this paper.

We often write the correlation matrix in the block form

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \tag{6}$$

with 2×2 matrices $A,B,C,$ where A refers to the first system and B to second system. The matrix C describes the correlations between both systems and vanishes for product states.

All the states and operations we consider here are pure. Therefore, and since we look at two-mode states only, we can always write their CM in the form [46]

$$\gamma = (S_1 \oplus S_2)\begin{pmatrix} \cosh(r)\mathbb{1} & \sinh(r)\sigma_z \\ \sinh(r)\sigma_z & \cosh(r)\mathbb{1} \end{pmatrix}(S_1^T \oplus S_2^T), \tag{7}$$

which we refer to as the *pure state standard form* of γ. Here, $S_{1,2}$ are local symplectic matrices, $r \geqslant 0$, and σ_z is the Pauli matrix $\mathrm{diag}(1,-1)$. The parameter r contains all information

about the entanglement of the state, whereas S_1 and S_2 contain information about local squeezing. Given a CM γ, one can readily find its pure state standard form [47].

Concerning the bilinear interaction Hamiltonians, it is convenient to rewrite the Hamiltonian of Eq. (1) as follows:

$$H=(X_1,P_1)K\begin{pmatrix}X_2\\P_2\end{pmatrix},\quad \text{where}\quad K=\begin{pmatrix}a & d\\ c & b\end{pmatrix}. \tag{8}$$

We denote by $s_1=\sigma_1$, $s_2=\mathrm{sign}[\det(K)]\sigma_2$ [48] with $\sigma_1\geqslant\sigma_2\geqslant 0$ the singular values of K. We refer to the s_k as the *restricted singular values* of K. Note that, local rotations can always bring any H to the diagonal form $s_1X_1X_2+s_2P_1P_2$.

C. Results

We state here the main results of this paper. To give a clear picture of them we do not use more mathematical tools and definitions than necessary.

First, we characterize the interactions which we are able to generate within the setting described by Eq. (4). In the infinitesimal regime the problem is usually called Hamiltonian simulation, whereas for t finite it is usually called gate simulation. Then we use these results to find the optimal strategy to generate entanglement or squeezing both in the infinitesimal and finite regime.

1. Hamiltonian simulation

Given two Hamiltonians H and H' of the form (1), we want to see the conditions under which H can simulate H'. That is, for a given sufficiently small t' we want to find out if it is possible to have

$$\begin{aligned}e^{-iH't'}=&[V_n\otimes W_n]e^{-iHt_n}\cdots e^{-iHt_2}[V_1\otimes W_1]\\&\times e^{-iHt_1}[V_0\otimes W_0],\end{aligned} \tag{9}$$

with t_k small as well. If it is possible to choose $t\equiv\Sigma_k t_k=t'$, we say that H can simulate H' *efficiently*.

Defining the matrices K and K' as in Eq. (8), as well as their respective restricted singular values $s_{1,2}$ and $s'_{1,2}$, we find the following results: (i) The Hamiltonian H can efficiently simulate H' if and only if

$$s_1+s_2\geqslant s'_1+s'_2\quad\text{and}\quad s_1-s_2\geqslant s'_1-s'_2. \tag{10}$$

(ii) If it is not possible to simulate H' efficiently with H, then the minimal time needed to simulate the evolution corresponding to H' for the time t' is $t_{\min}:=\min_t\{t:(s_1+s_2)t\geqslant(s'_1+s'_2)t',(s_1-s_2)t\geqslant(s'_1-s'_2)t'\}$. Thus, except for the cases $s_1=\pm s_2$ every Hamiltonian of the form (1) can simulate all other Hamiltonians of that form (including the $s'_1=\pm s'_2$ case). In particular, with the Hamiltonian H_0 describing the atom-light interaction one can simulate every bilinear Hamiltonian (1) and can do so efficiently as long as $|s'_1|+|s'_2|\leqslant 1$. In this case, the interaction existing in the physical setup can be considered universal.

2. Gate simulation and state generation

We show that starting from the Hamiltonians H and $H_{\mathrm{loc},i}$ of Eqs. (1) and (2) it is possible to generate any desired unitary evolution of the form $\mathcal{U}=e^{-i\tilde{H}}$, where $\tilde{H}$ is an arbitrary self-adjoint operator quadratic in $\{X_1,P_1,X_2,P_2\}$, if and only if $|s_1|\neq|s_2|$. In particular, the Hamiltonian H_0, together with the local operations given in Eq. (2) and local displacements, allows to generate all unitary linear operations, and therefore to generate arbitrary Gaussian states out of any pure Gaussian state. This shows that $H_0,H_{\mathrm{loc},i}$ generate a set of universal linear gates for continuous variables smaller than the one given in Ref. [49].

Let us analyze some important applications of these results in the case of atomic ensembles interacting with light. They imply that with current experiments with atomic ensembles one can generate all unitary linear operations, as well as arbitrary Gaussian states. In particular, one can generate local squeezing operators for which $\tilde{H}=X_1^2-P_1^2$ [which are not included among the Hamiltonians of the form (1) and therefore cannot be simulated infinitesimally by any of them] and therefore one can generate squeezing in the atomic system, light system or both independently (without performing measurements). On the other hand, one can use H_0 to generate the swap operator, which (in the Heisenberg picture) transforms

$$X_1\leftrightarrow X_2,\quad P_1\leftrightarrow P_2. \tag{11}$$

This operation can be generated in a finite time. Thus, one can use the interaction H_0 to realize a perfect interface between light and atoms, which allows to use the atomic ensemble as a quantum memory for light, as opposed to the case in Ref. [40], where this result is obtained in the limit of very strong interaction.

3. Optimal entanglement generation: infinitesimal case

The problem that we consider now can be stated as follows. Let us assume that we have some initial pure Gaussian state and we have some interaction described by the general Hamiltonian (1) at our disposal for a short time δt. The initial state at time t_0 is described by some correlation matrix of the form $\gamma(t_0)$ and possesses an entanglement $E(t_0)$, where E is some measure of entanglement. We would like to increase the entanglement as much as possible. In order to simplify our notation we choose, without loss of generality, $t_0=0$. We omit the argument for all quantities referring to the initial state, e.g., $\gamma=\gamma(0)$.

Since for the case of two modes in a pure state there is a single parameter that describes the entanglement [cf. Eq. (35)], all entanglement measures are monotonically dependent on each other. One particular measure is the parameter r appearing in Eq. (7), $E_0(\gamma)=r$. In fact, E_0 is the log negativity [50] of the Gaussian state. Thus, every entanglement measure E can be expressed in terms of r. We use the obvious notation $E(t)\equiv E[\gamma(t)]$ when considering the time evolution of E. Mathematically, our goal is to maximize the *entanglement rate* [23],

$$\left.\frac{dE}{dt}\right|_{t_0} = \lim_{\delta t\to 0}\frac{E(t_0+\delta t)-E(t_0)}{\delta t}, \tag{12}$$

by using the fast local operations. We find the following result:

$$\left.\frac{dE}{dt}\right|_{t_0,\mathrm{opt}} = \left.\frac{dE}{dr}\right|_{r(t_0)} \Gamma_{E,\mathrm{opt}}(\gamma,H). \tag{13}$$

The function Γ_E, which genuinely contains the optimal entanglement increase, is given by

$$\Gamma_{E,\mathrm{opt}}(\gamma,H) = s_1 e^{l} - s_2 e^{-l}, \tag{14}$$

where s_1, s_2 characterize the given interaction Hamiltonian, while l is a parameter that only depends on the local squeezing of our state and can be determined through the following relation [using the notation of Eqs. (6) and (7)]:

$$\cosh(2l) = \frac{\det(A)}{-2\det(C)}\mathrm{tr}(A^{-2}CC^T) = \frac{1}{2}\mathrm{tr}[(S_1^T S_1)^{-1}\sigma_z S_2^T S_2 \sigma_z]. \tag{15}$$

Note that there is no divergence as $\det C\to 0$ as is seen by the second expression in Eq. (15) [51].

Thus, we see that the entanglement rate depends on the local symplectic matrices S_1, S_2, i.e., on both the amount of (local) squeezing in the two modes and the angle between the squeezed quadratures (which, e.g., is zero, if both X_1 and X_2 are squeezed). However, it does not depend on the entanglement of the state. Rewriting $\Gamma_{E,\mathrm{opt}}$ as $(s_1-s_2)\cosh l + (s_1+s_2)\sinh l$ we see that some Hamiltonians can produce entanglement even if there is no local squeezing present in the state (which implies that $l=0$), while others (notably the beam splitter with $s_1=s_2=1$) cannot.

Note that the rate goes to infinity as local squeezing is increased, in contrast to the case of qubits. Given a CM γ, there are typically local rotations that enhance the entanglement rate.

From these results, we conclude that if the goal is to create as much entanglement as possible it is more efficient to squeeze the state locally first (if possible) before using the interaction; in particular, the use of squeezed light [7] is advantageous compared to coherent light [42].

4. Optimal squeezing generation: infinitesimal case

Now we consider the problem of optimal squeezing generation in the same setup as in the preceding section. We take as a measure of squeezing of a correlation matrix γ, $\mathcal{S}=\mathcal{S}(Q)$, any monotonically increasing function of Q, where Q is minus the logarithm of the smallest eigenvalue of γ. We find

$$\left.\frac{d\mathcal{S}}{dt}\right|_{t_0,\mathrm{opt}} = \left.\frac{d\mathcal{S}}{dQ}\right|_{Q(t_0)} g_S[\gamma] C_S(H). \tag{16}$$

$C_S(H)$ is the *squeezing capability* of the Hamiltonian and it is given by s_1-s_2, where the s_i's are the restricted singular values of K, given in Eq. (8) and

$$g_S(\gamma) = 2\|\vec{x}_1\|\|\vec{x}_2\| \leqslant 1 \tag{17}$$

quantifies how "squeezable" the state γ is by interactions of the type (1). Here $\hat{x}^T=(\vec{x}_1,\vec{x}_2)$, with $\vec{x}_1,\vec{x}_2\in\mathbb{R}^2$ is the normalized eigenvector corresponding to the minimal eigenvalue of γ.

5. Optimal squeezing and entanglement: finite case

Finally, we consider the situation in which we start with both modes in the vacuum state and we have a Hamiltonian H for a finite time (as well as instantaneous local operations). We show that the optimal way to create entanglement is to apply local instantaneous operations flipping the X and P variables of both systems periodically after small times Δt. After a finite time t (and for $\Delta t\to 0$) this produces (up to local rotations) a two-mode squeezed state, which is both optimally squeezed and entangled. In particular, $Q(t)=(s_1-s_2)t$ and $E_0(t)=(s_1-s_2)t$.

We also show that it is not possible to increase the entanglement using Gaussian measurements during the evolution. We consider a system with CM γ and ancilla systems in vacuum state. We allow for linear-passive interactions (described by a symplectic and orthogonal matrix O) between one system and the ancillas and show that a Gaussian measurement does neither increase the squeezing nor the entanglement. This result implies that our method is optimal even if we allow for feedback, something which has been recently considered in the context of spin-squeezing generation [19,22].

For the case of atomic ensembles our result implies that there is a method to improve the entanglement generation in present experiments [4].

III. SIMULATION OF INTERACTIONS

In this section, we characterize all the unitary evolutions which we can generate within the given setup. That is, we define the set of unitary operators which can be written as Eq. (4). The first part of this section is devoted to the infinitesimal regime, where we will, in general, derive the necessary and sufficient conditions for Hamiltonian simulation. In the second part, we are concerned with the finite time regime. There we show that with (almost) any Hamiltonian H as in Eq. (1) and the local operations corresponding to the Hamiltonians given in Eq. (2) it is possible to generate any unitary gate.

A. Method of Hamiltonian simulation

A central result in the theory of Hamiltonian simulation [32] states that an alternating sequence of manipulations and interactions as given in Eq. (9) is equivalent to a fictitious-free evolution due to a certain effective Hamiltonian H_{eff}, i.e., produces a unitary transformation

$$\mathcal{U}=e^{-iH_{\rm eff}t'}$$

and

$$\kappa H_{\rm eff}=\sum_{k=1}^{n} p_k(\tilde{V}_k^\dagger\otimes\tilde{W}_k^\dagger)H(\tilde{V}_k\otimes\tilde{W}_k), \tag{18}$$

where $\kappa:=t'/t$, $t:=\Sigma_{i=1}^{n}t_i$, the $p_k:=t_k/t$ form a probability distribution and the $\tilde{V}_i\otimes\tilde{W}_i$ follow uniquely from the interspersed control operations $V_j\otimes W_j$ (and vice versa). Obviously one can in this way *simulate* an evolution due to a Hamiltonian $H_{\rm eff}$ by means of a given Hamiltonian H.

Equation (18) has a clear interpretation: A protocol proceeding in infinitesimal time steps yields a mean Hamiltonian which is a weighted sum of locally transformed variants of the original Hamiltonian H. The so-called *simulation factor* κ is the ratio of simulated time t' and time of simulation t and, therefore, is a measure for the efficiency of the simulation. The case $\kappa\geqslant 1$ corresponds to the *efficient simulation*.

B. Necessary and sufficient condition

We associate to the general nonlocal interaction Hamiltonian (1) the real 2×2 matrix K as in Eq. (8). The action of a local rotation $V(\varphi)=\exp[-i(X^2+P^2)\varphi/2]$ on the canonical operators X and P can be expressed by

$$V\begin{pmatrix}X\\P\end{pmatrix}V^\dagger=\overline{R}\begin{pmatrix}X\\P\end{pmatrix},$$

$$\text{where}\quad \overline{R}=R(\varphi)=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\in \mathrm{SO}(2,\mathbb{R}). \tag{19}$$

Thus, we can associate to all local rotations V_i, W_i (2) real orthogonal 2×2 matrices $\overline{R},\overline{S},\ldots$ with determinant $+1$. Consequently, we have

$$(V\otimes W)H(V^\dagger\otimes W^\dagger)=(X_1,P_1)\overline{R}^T K\overline{S}\begin{pmatrix}X_2\\P_2\end{pmatrix}. \tag{20}$$

Furthermore, we use that for any matrix K as given in Eq. (8) there exists a singular-value decomposition $K=OD\tilde{O}$, where $O,\tilde{O}\in O(2,\mathbb{R})$, $D=\mathrm{diag}(\sigma_1,\sigma_2)$ and the singular values $\sigma_1\geqslant\sigma_2\geqslant 0$ of K are unique. If we restrict ourselves on *special* orthogonal matrices, we can still find matrices $R,S\in \mathrm{SO}(2,\mathbb{R})$ such that

$$K=R\begin{pmatrix}s_1 & 0\\ 0 & s_2\end{pmatrix}S, \tag{21}$$

and $s_1=\sigma_1$, $s_2=\mathrm{sign}[\det(K)]\sigma_2$ [48]. Without loss of generality, we may always assume that

$$s_1\geqslant|s_2|. \tag{22}$$

Then these two values are uniquely defined and shall be called *restricted singular values* of K.

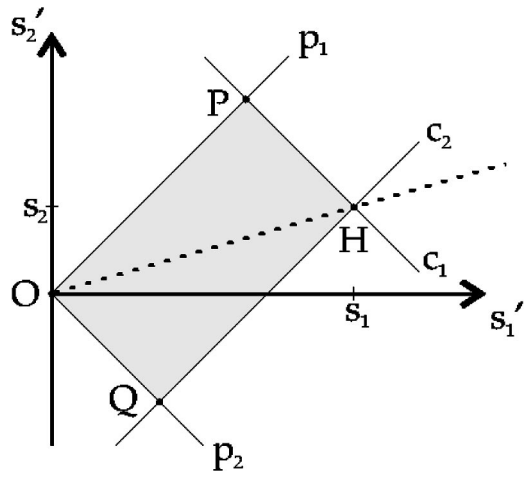

FIG. 1. Illustration of the accessible region in the (s_1',s_2') plane for the case $s_2>0$. Coordinates of relevant points: $H=(s_1,s_2)$, $P=[(s_1+s_2)/2](1,1)$, $Q=[(s_1-s_2)/2](1,-1)$. See the text for an explanation.

Assume now we want to simulate, in the above sense, some Hamiltonian H' by means of some other Hamiltonian H, both of the form (8). Let s_1,s_2 and s_1',s_2' denote their respective restricted singular values. Then we have the following result:

H can efficiently simulate H' iff

$$s_1+s_2\geqslant s_1'+s_2',$$

$$s_1-s_2\geqslant s_1'-s_2'. \tag{23}$$

The proof is elementary but requires some effort in notation such that we postpone it to Appendix A.

C. Discussion

Since the number of relevant parameters characterizing an interaction Hamiltonian is two, one can nicely illustrate the above result: The Fig. 1 illustrates the following geometrical relations: Point $H=(s_1,s_2)$ denotes the original general Hamiltonian. Lines p_1 and p_2 indicate the boundaries where $s_1'=\pm s_2'$, respectively, and are due to premise $s_1'\geqslant|s_2'|$. Lines c_1 and c_2 stem, respectively, from the first and second inequality constituting the necessary and sufficient condition. The region of accessible Hamiltonians, i.e., points $H'=(s_1',s_2')$ is thus contained in the rectangle $OPHQ$. One can even visualize how this set deepens with increasing time of simulation by parametrizing $H(t)=(s_1t,s_2t)$. Thus, H moves outward on the dashed line, while P and Q move on p_1 and p_2, respectively.

It is therefore just a matter of time to reach any point in the quadrant enclosed by p_1 and p_2. It is also quite instructive to consider certain special cases: (i) For $s_2=s_1$ ($s_2=-s_1$) the dashed line coincides with p_1 (p_2), respectively. This is a trivial case where we are confined to simulate locally equivalent variants of the original Hamiltonian (see Appendix A). Therefore, Hamiltonians whose restricted singular values are of equal modulus are nearly useless for the pur-

pose of Hamiltonian simulation. (ii) For $s_2=0$ or, equivalently, $\det(K)=0$ the picture gets symmetric with respect to the s_1' axis. This symmetrization can be interpreted in terms of time efficiencies, as we shall explain in the following.

Based on the criterion above one can ask for time efficiencies and especially for *time optimal protocols*. Time optimal simulation is achieved if the simulation factor $\kappa=t'/t$ [see Eq. (18)] gets maximal. Without loss of generality, we set $t'=1$ such that $\kappa=1/t$. Given now H and H' with restricted singular values s_1,s_2 and s_1',s_2', we can determine the minimal time of simulation as $t_{\min}:=\min_t\{t:(s_1+s_2)t\geqslant(s_1'+s_2'),(s_1-s_2)t\geqslant(s_1'-s_2')\}$. We find

$$t_{\min}=\begin{cases}\dfrac{s_1'+s_2'}{s_1+s_2} & \text{if } \dfrac{s_2'}{s_1'}\geqslant\dfrac{s_2}{s_1}\\[2ex] \dfrac{s_1'-s_2'}{s_1-s_2} & \text{if } \dfrac{s_2'}{s_1'}<\dfrac{s_2}{s_1}.\end{cases}\tag{24}$$

Thus, the efficiency of simulation depends strongly on whether $\mathrm{sign}(s_2')=\mathrm{sign}(s_2)$ or not, the last case being more time consuming. Only when $s_2=0$ [case (ii) above] it is equally expensive (in terms of costs of interaction time) to simulate either kind of Hamiltonians H' [$\mathrm{sign}(s_2')\leqslant 0$], a fact which is reflected in the above-mentioned symmetrization. Correspondingly, the optimal time of simulation or, so to say, the *minimal interaction costs* [33] are in this case uniquely determined by

$$t_{\min}=(s_1'+|s_2'|)/s_1. \tag{25}$$

D. Application to X_1X_2 interaction

Let us outline some conclusions out of this result for the interaction $H=X_1X_2$. The restricted singular values of H are obviously $s_1=1$ and $s_2=0$. Therefore, we can efficiently ($\kappa=1$, i.e., $t'=t$) implement all Hamiltonians H' whose restricted singular values fulfill

$$s_1'+|s_2'|\leqslant 1. \tag{26}$$

As an example as well as to give a basis for further results we shall consider here two kinds of well known unitary transformations: the *beam-splitter* operator

$$U_{\mathrm{bs}}(t):=e^{-iH_{\mathrm{bs}}t} \quad \text{where} \quad H_{\mathrm{bs}}=X_1P_2-P_1X_2, \tag{27}$$

and the *two-mode squeezer*

$$U_{\mathrm{tms}}(t):=e^{-iH_{\mathrm{tms}}t}, \quad \text{where} \quad H_{\mathrm{tms}}=X_1X_2-P_1P_2. \tag{28}$$

As mentioned already, the action of $U_{\mathrm{bs}}(\pi/2)$ corresponds to swapping the states of the first and the second mode, i.e., it transforms $X_1,P_1\rightarrow -X_2,-P_2$ and $X_2,P_2\rightarrow X_1,P_1$. Note that the global phase thereby acquired by subsystem 1 can be corrected locally.

Application of $U_{\mathrm{tms}}(t)$ squeezes the quadratures (X_1+X_2) and (P_1-P_2) by a factor e^{-2t} and therefore also entangles the two systems, as we shall see.

In order to perform these operations by means of the X_1X_2 interaction, we have to determine the restricted singular values of H_{bs} and H_{tms}. One finds for H_{bs} $s_1=1,s_2=1$ and for H_{tms} $s_1=1,s_2=-1$. Since in both cases condition (26) is not met, we cannot *efficiently* simulate these Hamiltonians. But nevertheless, we can determine strategies for infinitesimal simulations being time optimal. The minimal time of simulation can be calculated using Eq. (25) and yields a maximal simulation factor $\kappa=1/t_{\min}=1/2$ for both, the beam splitter and the squeezer. Thus, in order to implement $U_{\mathrm{bs}}(t')$ we need at least a time $t=2t'$ and to create squeezing by a factor $e^{-2t'}$ it will take a time $2t'$, i.e., to implement $U_{\mathrm{tms}}(t')$ we need a time $t=2t'$. Explicit simulation protocols can be constructed following Appendix A.

E. Simulation of unitary operators and state engineering

Until now we have focused on the regime of infinitesimal times in order to clarify which *unitary evolutions* we can simulate by means of the given interaction. We found that we can do so—more or less efficiently—for all evolutions governed by Hamiltonians of the form (8), but no more. This leaves open the question which *unitary operations* can, in general, i.e., for finite times, be realized with a given interaction and local rotations.

As we show in the following, any interaction described by some Hamiltonian H, where $s_1\neq|s_2|$ together with local rotations is sufficient to realize *any unitary operation* of the form $\exp(iG)$ where G is a quadratic expression in the operators X_k,P_k. That is, any Gaussian unitary transformation of the two modes can be obtained. This implies, that any desired pure Gaussian state can be ''engineered'' starting from any given (pure Gaussian) input state.

As we show in Appendix B, any $U=\exp(-iG)$ can be decomposed as

$$\begin{aligned}U=&(V_5\otimes W_5)U_{\mathrm{bs}}(t_5)(V_4\otimes W_4)\\ &\times U_{\mathrm{tms}}(t_4)(V_3\otimes W_3)U_{\mathrm{bs}}(t_3)(V_2\otimes W_2)U_{\mathrm{tms}}(t_2)\\ &\times(V_1\otimes W_1)U_{\mathrm{bs}}(t_1)(V_0\otimes W_0),\end{aligned}\tag{29}$$

where all $(V_i\otimes W_i)$ are local rotations, $U_{\mathrm{bs}}(t_i)$ is a beam splitter and $U_{\mathrm{tms}}(t_i)$ a two-mode squeezing operation as defined in Eqs. (27) and (28). Since all Hamiltonians with $s_1\neq|s_2|$ can be used to simulate beam splitters and two-mode squeezers one can reach any desired unitary U and therefore also any desired Gaussian state.

IV. ENTANGLEMENT AND SQUEEZING

In the preceding section, we characterized the time evolutions on the joint system which can be realized using a given interaction Hamiltonian of the form (1) and the control operations provided by Eq. (2). In this section, we determine the optimal way to use these tools for the generation of en-

tanglement and squeezing between the two subsystems in both, the infinitesimal and the finite regime.

Our derivations make extensive use of the formalism of Gaussian states and operations. The necessary concepts and notation are introduced in Sec. IV A and then put to work in the cases of infinitesimal (Sec. IV B) and finite (Sec. IV C) times.

A. State transformations and measures of entanglement and squeezing

We show here how Gaussian states evolve under a general quadratic Hamiltonian and then introduce some entanglement and squeezing measures for Gaussian states.

1. State transformation

A quadratic interaction Hamiltonian (1) characterized by a matrix K as in Eq. (8) generates a linear time evolution of the X and P operators. Solving the Heisenberg equations for $\vec{R}=(X_1,P_1,X_2,P_2)^T$, we find

$$\vec{R}(t)=e^{Mt}\vec{R}(0)=S(t)\vec{R}(0), \tag{30}$$

where

$$M=\begin{pmatrix} 0 & L \\ \tilde{L} & 0 \end{pmatrix}, \tag{31}$$

with

$$L=\begin{pmatrix} c & b \\ -a & -d \end{pmatrix}=J^TK \quad \text{and} \quad \tilde{L}=-JL^TJ^T=J^TK^T, \tag{32}$$

where

$$J=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{33}$$

Note that for $0\neq -\det(L)=:\alpha$, we have $\tilde{L}=\alpha L^{-1}$. Using the fact that $M^2=\alpha\mathbb{1}$, we can easily reexpress Eq. (30) and find

$$S(t)=\cosh(\sqrt{\alpha}t)\mathbb{1}+\sinh(\sqrt{\alpha}t)/\sqrt{\alpha}M. \tag{34}$$

Thus, every evolution generated by a Hamiltonian (1) is uniquely characterized by a symplectic transformation $S(t)$ of the form (34). Note that any such transformation can be written in its *standard form*

$$S(t)=\cosh(\sqrt{\alpha}t)(O_1\oplus O_2)\begin{pmatrix} 1 & 0 & h_1 & 0 \\ 0 & 1 & 0 & -h_2 \\ h_2 & 0 & 1 & 0 \\ 0 & -h_1 & 0 & 1 \end{pmatrix} \times(O_1\oplus O_2)^T, \tag{35}$$

where $O_1,O_2\in \mathrm{SO}(2,\mathbb{R})$ perform the restricted singular-value decomposition of L, and $h_k=\tanh(\sqrt{\alpha}t)/\sqrt{\alpha}s_k$, where s_k are the restricted singular values of L, which clearly coincide with those of K. In particular, the Hamiltonian $H_0=X_1X_2$ of Eq. (3) generates an time evolution described by the symplectic matrix

$$S_0(t)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 1 & 0 \\ -t & 0 & 0 & 1 \end{pmatrix}, \tag{36}$$

i.e., $\alpha=0,(s_1,s_2)=(1,0)$, and $O_1=J$ [see Eq. (33)] and $O_2=-\mathbb{1}$.

In the Schrödinger picture a linear time evolution as in Eq. (30) transforms the CM γ as

$$\gamma(t)=S(t)\gamma S(t)^T. \tag{37}$$

In the following section, we address the case of very short interaction time, i.e., we consider $S(\delta t)$ for an infinitesimally short-time step δt. In this case, we obtain

$$S(\delta t)=\mathbb{1}+\delta tM, \tag{38}$$

and the correlation matrix $\gamma(t)$ transforms to first order as

$$\gamma(t+\delta t)=\gamma(t)+\delta t[M\gamma(t)+\gamma(t)M^T]. \tag{39}$$

Let us in the following write the 4×4 CM of the two-mode Gaussian state as a block matrix as in Eq. (6) with 2×2 matrices A,B,C. Then A refers to the first system and is the CM belonging to the reduced density operators of the system 1. Note that for all CMs $\det(\gamma)\geqslant1$, and equality holds if and only if (iff) the state is pure. Since our initial state is pure and we consider unitary transformations (and, later, complete Gaussian measurements) this implies that we are only concerned with pure states at all times.

2. Entanglement and squeezing of Gaussian states

As one can see in Eq. (7), the single parameter which characterizes the nonlocal properties of a pure state is the two-mode squeezing parameter r. This automatically implies that any monotonic function of this parameter can be used to quantify the entanglement of pure Gaussian two-mode states and we are free to choose [53] the most convenient measure.

One such quantity is $E_p(\gamma)=\det A=\cosh(r)^2$, the determinant of the CM corresponding to the reduced density. It is related to the *purity* of the reduced density matrix [55]. As mentioned before, the determinant of a CM is one, iff the state is pure, which implies that $E_p(\gamma)=1$ iff the state is not entangled, i.e., iff $r=0$.

For the last part of this section another measure of entanglement, namely, the *negativity* $\mathcal{N}$ introduced in Ref. [50] is most convenient to use. For a 1×1 Gaussian state with CM γ the negativity is given by the inverse of the smallest symplectic eigenvalue of the partially transposed CM $\tilde{\gamma}=\Lambda\gamma\Lambda$, which can easily be calculated [50] as

$$\mathcal{N}(\gamma)=[\min\{\text{sisngular values}(J_2^T\tilde{\gamma}J_2\tilde{\gamma})\}]^{-1/2}. \tag{40}$$

Here Λ is the 4×4 diagonal matrix diag(1,1,1,−1) (which implements partial transposition, see Ref. [11]) and $J_2 = J\oplus J$ is the symplectic matrix for two modes.

The other interesting quantity that characterizes Gaussian states besides the entanglement is the *squeezing* inherent in the state, i.e., by how much the variance of some (passive-linearly transformed) quadrature is reduced below the standard quantum limit. The reduced variance is given by the smallest eigenvalue $\lambda_{\min}(\gamma)$ of γ and we define the squeezing of a state with CM γ as the inverse of $\lambda_{\min}(\gamma)$,

$$\mathcal{S}(\gamma) = \min\{\mathrm{eig}(\gamma)\}^{-1} = [\lambda_{\min}(\gamma)]^{-1}. \tag{41}$$

In a situation like the one we consider here where only orthogonal operations are freely available, the squeezing of a state represents a valuable resource which can be used, e.g., for the creation of entanglement [35] and which should be created as efficiently as possible.

B. Optimal entanglement and squeezing rates

The goal of this section is to determine the optimal strategy for the generation of entanglement [squeezing] in an (infinitesimally) small time step δt. That is, given a pure Gaussian state ρ with CM γ and an interaction Hamiltonian H as in Eq. (1), we look for the best choice of the local rotations $V\otimes W$ such that $e^{-iH\delta t}(V\otimes W)\rho(V\otimes W)^\dagger e^{iH\delta t}$ is as entangled [squeezed] as possible. Stating this problem mathematically: We maximize the *entanglement* [*squeezing*] *rate*, that is the time derivative of the chosen entanglement [squeezing] measures E [$\mathcal{S}$] under the time evolutions obtainable in the given setting.

1. Maximizing the entanglement rate

As measure of entanglement we use E_0, where $E_0(\gamma)$ is the two-mode squeezing parameter r [53] defined in Eq. (7). The entanglement rate is then simply given by

$$\Gamma_E = \frac{dE_0}{dt}\Big|_{t=0} = \lim_{\delta t\to 0}\frac{r(\delta t)-r}{\delta t}, \tag{42}$$

where $r\equiv r(0)$ is the entanglement of the initial CM γ.

In order to determine Γ_E we use, following Eq. (13), the formula $\Gamma_{E_p} = \sinh(2r)\Gamma_E = 2\sqrt{-\det(A)\det(C)}\,\Gamma_E$, where Γ_{E_p} denotes the entanglement rate corresponding to the purity-related measure E_p.

Let H as in Eq. (8) be the given Hamiltonian. It generates an evolution given by the symplectic transformation $\overline{S}(\delta t)$, which we write in its standard form (35) as $\overline{S}(\delta t) := (\overline{O}_1\oplus\overline{O}_2)S(\delta t)(\overline{O}_1\oplus\overline{O}_2)^T$. Since local operations cannot increase the entanglement the only way in which the local control operations may help is to rotate the state by $\tilde{O}_1\oplus\tilde{O}_2$ before applying H. Thus, the best strategy yields a $\gamma(\delta t)$ that can be written as

$$\gamma(\delta t) = S(\delta t)(O_1\oplus O_2)\,\gamma\,(O_1\oplus O_2)^T S(\delta t)^T, \tag{43}$$

where we defined $O_i := \overline{O}_i^T\tilde{O}_i$ and omitted the irrelevant final local rotations coming from $\overline{S}(\delta t)$. Writing $\gamma(\delta t)$ in the form (6) and using Eq. (39) it is straightforward to determine the CM corresponding to the reduced state,

$$A(\delta t) = O_1 A O_1^T + \delta t(L_0 O_2 C^T O_1^T + \mathrm{H.c.}), \tag{44}$$

where $L_0 = \mathrm{diag}(s_2, -s_1)$ is determined by the Hamiltonian H, cf. Eqs. (35) and (31). One quickly sees that $\det[A(\delta t)] = \det(A)[1 + 2\,\delta t\,\mathrm{tr}(L_0 O_2 C^T A^{-1} O_1^T)]$, where we used the simple relation for 2×2 matrices: $\det(X+\delta t Y) = \det(X)[1 + \delta t\,\mathrm{tr}(X^{-1}Y)] + o(\delta t^2)$ and the fact that A is symmetric and invertible.

For the entanglement rate corresponding to E_p, we obtain $\Gamma_{E_p} = 2\det(A)\mathrm{tr}(L_0 O_2 C^T A^{-1} O_1^T)$. As mentioned before, we can from this easily determine the rate Γ_E corresponding to the two-mode squeezing parameter, namely, we have

$$\Gamma_E = \sqrt{\frac{\det(A)}{-\det(C)}}\,\mathrm{tr}(L_0 O_2 C^T A^{-1} O_1^T) = \mathrm{tr}(L_0 O_2 Y O_1^T), \tag{45}$$

where we have defined $Y := \sqrt{\det(A)/[-\det(C)]}\,C^T A^{-1}$.

Our aim is to maximize this expression with respect to the special orthogonal matrices O_1 and O_2. Note that $\det Y = -1$, which can be easily verified using Eq. (7). Therefore Y has the restricted singular values $e^l, -e^{-l}, l\geqslant 0$. Using that L_0 is diagonal it is straightforward to verify that the maximum of Eq. (45) is achieved when choosing O_1, O_2 such that they diagonalize Y such that $O_2 Y O_1^T = \mathrm{diag}(e^l, -e^{-l})$. Then the optimal choice for $\tilde{O}_i$ is

$$\tilde{O}_{i,\mathrm{opt}} = \overline{O}_i O_i, \tag{46}$$

with $\overline{O}_i$ given by $\overline{S}(\delta t)$. The best state to let H act on is thus $\gamma_{\mathrm{opt}} = (\tilde{O}_{1,\mathrm{opt}}\oplus\tilde{O}_{2,\mathrm{opt}})\,\gamma\,(\tilde{O}_{1,\mathrm{opt}}\oplus\tilde{O}_{2,\mathrm{opt}})^T$. Note that l which determines the singular values of Y can be easily determined by Eq. (15) [51].

In summary, given an interaction Hamiltonian H corresponding to a matrix K and an initial state with CM γ the optimal state preparation by local rotations (before letting H act) can be understood as a two-step procedure. First transform γ locally such that $C^T A^{-1}$ is diagonal [restricted singular-value decomposition, cf. Eq. (21)]. If K was already in its restricted singular-value decomposition, we are done. Otherwise, the second step of the state preparation can be viewed (in the Heisenberg picture) as the restricted singular-value decomposition of K. Then the optimal entanglement rate (entanglement is measured by E_0) is given by Eq. (14) in terms of the singular values s_k of the Hamiltonian matrix K and the local squeezing parameter l of the given state γ.

In the Fig. 2, we compare the entanglement rates and the entanglement obtained for different strategies using the "natural Hamiltonian" H_0. As initial state, we consider the product of the vacuum state in the first system and the squeezed vacuum in the second system, i.e.,

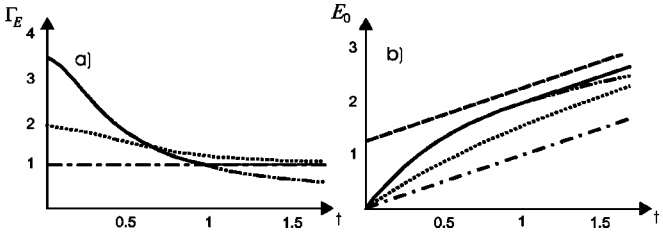

FIG. 2. (a) The entanglement rate obtained for the squeezed state $\gamma_{\rm in}$ Eq. (47) as initial state and various strategies. The solid line represents the optimal-rate strategy derived in this section; the dotted line represents the rate obtained by simulating the two-mode squeezing Hamiltonian $H_{\rm tms}$; the ''dot-dot-dashed'' line represents the rate obtained for the natural Hamiltonian $H_0=X_1X_2$. For the vacuum state as initial state we obtain the constant rate 1 (dashed line). (b) The entanglement created by the different strategies [same styles as in (a) for the different scenarios]. The dashed line represents the upper bound Eq. (56).

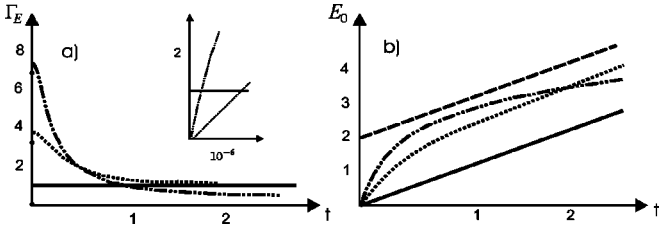

FIG. 3. (a) The entanglement rate obtained for the initial state $\gamma_{\rm in,2}=S_{r_1,r_2}\gamma_{\rm tms}(t_0/2)S_{r_1,r_2}^T$, where $S_{r_1,r_2}=\mathrm{diag}(e^{r_1/2},e^{-r_1/2},e^{r_2/2},e^{-r_2/2})$ and $r_1=r_2=2, t_0=10^{-3}$. The solid line $\Gamma_E=1$ is obtained with the strategy that optimizes the entanglement rate at each time; the dotted line represents the rate obtained for optimal simulation of $H_{\rm tms}$; the ''dot-dot-dashed'' line represents the rate obtained for the natural Hamiltonian $H_0=X_1X_2$. The inset shows that one has to ''pay'' with initial entanglement rates smaller than the optimal value of 1 to reach a state that allows for the large rates later on. (b) The entanglement created by the different strategies [same styles for different scenarios as in (a)] and the upper bound Eq. (56).

$$\gamma_{\rm in}=\mathbb{1}_2\oplus\begin{pmatrix} e^{-r} & 0 \\ 0 & e^{r}\end{pmatrix}, \tag{47}$$

with squeezing parameter $r=2.5$. We compare the strategy in which the rate of entanglement creation is optimized at each time to two simpler ones, namely, to just apply the natural Hamiltonian H_0 or to simulate the two-mode squeezing Hamiltonian $H_{\rm tms}=X_1X_2-P_1P_2$ using the optimal scheme of Sec. III. The rate-optimization strategy leads in fact to combination of the other two: one applies first the natural Hamiltonian for a finite time and then (when the ''local squeezing'' l has all been converted to two-mode squeezing) one simulates $H_{\rm tms}$. Having initially local squeezing available clearly helps with entanglement generation: for an initial unsqueezed state the optimal rate is constant $\Gamma_E=1$.

Figure 2(b) shows that the optimization strategy can lead to noticeably more entanglement in the resulting state after finite time: when the entanglement rate is optimized at each point, more entanglement is produced than, e.g., with the interactions H_0 or $H_{\rm tms}$. However, optimizing the rate is, in general, not the best strategy for the creation of entanglement, see Fig. 3.

2. Maximizing the squeezing rate

As in the preceding section we are given an interaction Hamiltonian of the form (1), an initial Gaussian state with CM γ, and we consider the case of infinitesimal interactions. Our goal is here to determine for each H and γ the strategy which maximizes the squeezing rate. We measure squeezing by $Q(\gamma)=\ln[\mathcal{S}(\gamma)]$, where $\mathcal{S}$ was defined in Eq. (41) as the inverse of the smallest eigenvalue of γ. The rate we are interested in is

$$\Gamma_S=\frac{d}{dt}\ln\mathcal{S}[\gamma(t)]|_{t=0}=\frac{-1}{\lambda_{\min}(\gamma)}\lim_{\delta t\to 0}\frac{\lambda_{\min}[\gamma(\delta t)]-\lambda_{\min}(\gamma)}{\delta t}. \tag{48}$$

Note that we use the logarithm of $\mathcal{S}$ instead of $\mathcal{S}$ for convenience. It simplifies the formulas but since ln is a monotonic function maximizing the rate of $\ln\mathcal{S}$ implies a maximal rate for $\mathcal{S}$ as well [53].

After applying the general strategy to the input state with CM γ, we obtain $\gamma(\delta t)$ as in Eq. (39). Doing first-order perturbation theory, we find that $\lambda_{\min}[\gamma(\delta t)]=\lambda_{\min}(\gamma)+\delta t\hat{x}^T(M^T\gamma+\gamma M)\hat{x}=\lambda_{\min}[1+\delta t\hat{x}^T(M^T+M)\hat{x}]$, where $\hat{x}$ is the normalized eigenvector corresponding to the smallest eigenvalue $\lambda_{\min}(\gamma)$ of γ. We obtain for the squeezing rate

$$\Gamma_S=\frac{-1}{\lambda_{\min}(\gamma)}[\hat{x}^T(M^T+M)\hat{x}], \tag{49}$$

which is maximized when $-\hat{x}^T(M^T+M)\hat{x}$ is as large as possible. Note that

$$M^T+M\equiv\begin{pmatrix} 0 & N \\ N^T & 0\end{pmatrix}, \tag{50}$$

where $N=\tilde{L}+L^T=J^TK^T+K^TJ$, where J is the SO(2) matrix of Eq. (33) and we have used the definitions (32) and (8). One quickly sees that $N=N^T$. Writing K in its restricted singular-value decomposition $K=SK_0R$, where $S,R\in\mathrm{SO}(2,\mathbb{R})$ and $K_0=\mathrm{diag}(s_1,s_2)$ as in Eq. (21), and using that R,S commute with J we see that $N=R^T(J^TK_0+K_0J)S^T=C_S(H)R^TJ^T\sigma_zS^T$, where

$$C_S(H)=s_1-s_2 \tag{51}$$

is the *squeezing capability* of the Hamiltonian H. Note that the matrix $\tilde{O}:=R^TJ^T\sigma_zS^T$ is orthogonal with $\det(\tilde{O})=-1$ and that we can obtain any such $\tilde{O}$ choosing $R,S\in\mathrm{SO}(2,\mathbb{R})$, i.e., by the local operations applied to the initial state. Using the notation $\hat{x}^T=(\vec{x}_1^T,\vec{x}_2^T)$, where $\vec{x}_1,\vec{x}_2\in\mathbb{R}^2$, we find $\Gamma_S=2C_S(H)\vec{x}_1^T\tilde{O}\vec{x}_2\leq 2C_S(H)\max_{\tilde{O}}|\vec{x}_1^T\tilde{O}\vec{x}_2|=2C_S(H)\|\vec{x}_1\|\,\|\vec{x}_2\|$, which gives an upper bound

$$\Gamma_S \leqslant 2C_S(H)\|\vec{x}_1\|\|\vec{x}_2\|,$$

for Γ_S. This maximum can be reached for $\tilde{O}_{\rm opt}$ such that $(-\tilde{O}_{\rm opt}\vec{x}_2)||\vec{x}_1$. Given γ (i.e., $\vec{x}_1,\vec{x}_2$) we can calculate $\tilde{O}_{\rm opt}$ with $\det\tilde{O}_{\rm opt}=-1$ which satisfies this condition. This then determines the optimal choice of $R,S\in{\rm SO}(2,\mathbb{R})$, i.e., how to transform the initial state with CM γ before letting H act in order to maximize the squeezing rate. One simple choice yielding $\tilde{O}=\tilde{O}_{\rm opt}$ is $S=\mathbb{1}$, i.e., nothing has to be done on the second system and $R_{\rm opt}=J^T\sigma_z\tilde{O}_{\rm opt}^T\in{\rm SO}(\mathbb{R},2)$. Thus, the optimal input state is given by $\gamma_{\rm opt}=(R_{\rm opt}^T\oplus\mathbb{1})\,\gamma(R_{\rm opt}\oplus\mathbb{1})$.

In summary, we have shown that the maximal squeezing rate is given by Eq. (16) as a product of the squeezing capability $C_S(H)$ of the given Hamiltonian and the squeezability $g_S(\gamma)$ of the given state. The optimal CM to let H act on is $\gamma_{\rm opt}=(R_{\rm opt}^T\oplus\mathbb{1})\,\gamma(R_{\rm opt}\oplus\mathbb{1})$, where

$$R_{\rm opt}=J^T\sigma_z\tilde{O}^T, \tag{52}$$

and $-\tilde{O}_{\rm opt}$ parallelizes $\vec{x}_1$ and $\vec{x}_2$. Note that the fact that $\hat{x}$ is normalized implies that $\Gamma_S\leqslant C_S(H)$ for any input state. Since we look at the logarithm of the squeezing this implies that $(d\mathcal{S}(\gamma)/dt)\leqslant\mathcal{S}(\gamma)C_S(H)$.

C. Optimal entanglement generation from the vacuum state

In practice, we are interested in creating the largest amount of entanglement when H acts for a *finite* total time t. Optimizing the rate of entanglement creation at each time does lead to a local but not necessarily, as we saw, the global maximum of the entanglement at time t [24].

We now show how to employ the interaction H to create the most entanglement in a given time t. To this end, we make use of the *squeezing* of γ which was introduced in Eq. (41) as the smallest eigenvalue of γ. The squeezing of γ is known [35] to give an upper bound for the amount of entanglement of γ, with $\mathcal{N}(\gamma)\leqslant\mathcal{S}(\gamma)$. We proceed as follows: First, we calculate the strongest squeezing that can be achieved after time t. This also gives an upper bound for the entanglement that can be obtained during this time. Then we point out a strategy that achieves the optimal squeezing and at the same time the strongest entanglement compatible with the given squeezing, thus being optimal on both counts.

The squeezing capability of a symplectic map S, i.e., the factor by which the squeezing in a CM can be increased by the application of S, is given by the inverse square of the smallest singular value of S, since $\mathcal{S}(S\gamma S^T)\leqslant[\sigma_{\rm min}(S)]^{-2}\mathcal{S}(\gamma)$. Here, and in the following we use that for the smallest singular value of a product AB we have $\sigma_{\rm min}(AB)\geqslant\sigma_{\rm min}(A)\sigma_{\rm min}(B)$. Now consider the symplectic map $S(t)$ corresponding to the unitary evolution generated by an interaction Hamiltonian H after time t, cf. Eq. (34). The singular values of $S(t)$ can easily be calculated analytically. We need them only for small times to first order in t, in which case we find

$$\sigma_\pm[S(t)]=\sqrt{1\pm\frac{1}{2}(s_1-s_2)t+o(t)^2}, \tag{53}$$

where s_1,s_2 are the restricted singular values of the matrix K [cf. Eq. (8)] corresponding to H.

Since $S(t)=S(t/2)S(t/2)=\Pi_{k=1}^N S(t/N)$ we see immediately that $(\sigma_{\rm min}[S(t)])^2\geqslant e^{-(s_1-s_2)t}$, which implies that the squeezing capability of $S(t)$ is bounded by $e^{(s_1-s_2)t}$. Now consider a strategy as in Eq. (4), alternating the use of H for time t_k with local rotations $V_k\otimes W_k$. Note that the $t_k,k=1,\ldots,N$, which sum to t, are not assumed to be infinitesimal. The time-evolution effected by this strategy is described by a symplectic map

$$S(t)=\Pi_k\tilde{S}_k, \tag{54}$$

where $\tilde{S}_k=O_kS(t_k)O_k'$ and O_k,O_k' are the local rotations corresponding to $V_k\otimes W_k$. Clearly, $\sigma_{\rm min}[S(t)]\geqslant\Pi_k e^{-(s_1-s_2)t_k/2}=e^{-(s_1-s_2)t/2}$. Hence, $\mathcal{S}[S(t)S(t)^T]\leqslant e^{(s_1-s_2)t}$, i.e., we have an upper bound to the amount of squeezing that can be produced from an initially unsqueezed pure state by applying H for a total time t.

A strategy to achieve this optimum is the following: we choose the local rotations V_k,W_k as $\pi/2$ rotation in system 1 and $3\pi/2$ in system 2, the times t_k all equal, and consider the limit $t_k\to 0$. This corresponds to the situation considered in Sec. III and simulates the Hamiltonian related to $K'=(K+JKJ)/2$. Let $K=O_1\,{\rm diag}(s_1,s_2)O_2$, then we have that $K'=\frac{1}{2}\,O_1[{\rm diag}(s_1,s_2)+{\rm diag}(-s_2,-s_1)]O_2$, since rotations commute with J. That is, apart from local rotations the strategy, which simulates the two-mode squeezing Hamiltonian with an efficiency $(s_1-s_2)/2$, which is the optimal factor according to Eq. (24). Letting $H_{\rm tms}$ act for a time $t'=t(s_1-s_2)/2$ (using up an interaction time t) transforms the vacuum state into the two-mode squeezed state with CM

$$\gamma_{\rm tms}(t')=\begin{pmatrix}\cosh 2t'\mathbb{1} & \sinh 2t'\sigma_z\\ \sinh 2t'\sigma_z & \cosh 2t'\mathbb{1}\end{pmatrix} \tag{55}$$

which saturates the bounds derived above, since $\mathcal{S}[\gamma_{\rm tms}(t')]=e^{(s_1-s_2)t}$.

Now we show that $\gamma_{\rm tms}$ in Eq. (55) is also the most entangled state that can be obtained after letting H act for a total time t. Using Eq. (40) for the negativity of a Gaussian state with CM $\gamma=S(t)S(t)^T$ (i.e., an arbitrary strategy applied to the vacuum state), we get

$$\mathcal{N}(\gamma)=[\mathcal{S}(J^T\tilde{\gamma}J\tilde{\gamma})]^{-1/2}\leqslant\mathcal{S}(\tilde{\gamma})=\mathcal{S}(\gamma)=e^{(s_1-s_2)t}.$$

Since $\mathcal{N}[\gamma_{\rm tms}(t')]=e^{(s_1-s_2)t}$ the simulation of two-mode squeezing is the optimal strategy for both squeezing and entanglement generation. Note that even a rough approximation of the optimal strategy, i.e., a strategy consisting of just two or three steps already yields a marked improvement in generated squeezing and entanglement.

Up to now, we have only considered the unitary evolution of the initial state. There are, however, further tools available in current experiments. There might be additional light modes (ancillas) in the vacuum state on which passive linear optical operations (described by orthogonal and symplectic transformations) as well as complete or partial homodyne measurements can be performed. In principle, these

might help to increase the entanglement in γ, but in the following we show that this is not the case. We consider the following general setup: consider system with CM γ, ancilla systems in vacuum state, i.e., $\gamma_{\mathrm{anc}}=\mathbb{1}$, linear passive interactions (described by a symplectic and orthogonal matrix O) between the system light mode and the ancillas (e.g., beam splitter between light and ancillary modes), such that the whole system is described by the CM $\gamma'=O^T(\gamma\oplus\gamma_{\mathrm{anc}})O$; clearly, $\mathcal{S}(\gamma')=\mathcal{S}(\gamma)$ and now we show that a Gaussian measurement does not increase $\mathcal{S}(\gamma)$. We write γ' as

$$\gamma'=\begin{pmatrix} A' & C' \\ C'^T & B' \end{pmatrix},$$

where the block matrix B' refers to the ancillary modes to be measured. Then the resulting state is described by the CM $\gamma_{\mathrm{out}}=A'-C'B'^{-1}C'^T$ [16]. Using the following characterization of the smallest eigenvalues [57] it is straightforward to see that measurement has reduced the squeezing of the state:

$$\begin{aligned}\mathcal{S}(\gamma_{out}) &= \min_{x\in\mathbb{C}^n}\left\{\frac{x^\dagger(A'-C'B'^{-1}C'^T)x}{x^\dagger x}\right\}^{-1} \\ &\leqslant \min_x\left\{\frac{x^\dagger(A'-C'B'^{-1}C'^T)x}{x^\dagger(\mathbb{1}+C'B'^{-2}C'^T)x}\right\}^{-1} \\ &= \min_x\left\{\frac{y^\dagger\gamma' y}{y^\dagger y}:y=\begin{pmatrix} x \\ -B'^{-1}C'^T x\end{pmatrix}\right\}^{-1} \\ &\leqslant \min_{y\in\mathbb{C}^{2n}}\left\{\frac{y^\dagger\gamma' y}{y^\dagger y}\right\}=\mathcal{S}(\gamma').\end{aligned}$$

Consequently, unsqueezed ancilla systems and Gaussian measurements are of no help in increasing the squeezing or entanglement in a Gaussian state.

The preceding discussion does not completely solve the problem of optimal entanglement generation with a Hamiltonian H, since only one particular initial state (the vacuum) has been considered. If, e.g., the initial state of the light field is squeezed, we have seen in Sec. IV B that better rates can be achieved (see Fig. 2), which will translate into larger entanglement after finite times. The methods used above easily yield an upper bound for the entanglement that can be obtained from initially squeezed states: Consider an initial product state with squeezing e^{r_1} and e^{r_2} in systems 1 and 2 and let $r_1\geqslant r_2$. By the same arguments as above, after H has acted for a time t the squeezing in the final state and the negativity are bounded by $e^{(s_1-s_2)t+r_1}$. We can find a better bound on the achievable entanglement drawing on results from Ref. [35], where it was shown that the negativity of a two-mode CM γ is bounded by $1/\sqrt{\lambda_1\lambda_2}$, where λ_1,λ_2 are the two smallest eigenvalues of the γ. This implies that

$$\mathcal{N}(\gamma_{\mathrm{out}})\leqslant e^{(s_1-s_2)t+(r_1+r_2)/2}, \tag{56}$$

which yields the dashed curve in Fig. 2(b). This bound is most probably not tight for $r_k\neq 0$, not even as $t\to\infty$.

One might think that in order to optimize the entanglement after some finite time t it always suffices to optimize the rate at each time as for the case of a vacuum input. For qubit systems this was indeed shown to be true [23]. In contrast, it does not hold for cv systems as the counterexample depicted in Fig. 3 shows: We start with a slightly entangled state with CM $\gamma_{\mathrm{in},2}$ which can be obtained from the two-mode squeezed state $\gamma_{\mathrm{tms}}(t_0/2)$ squeezing both X_1 and X_2 by $r_1=r_2$. Then the ''local squeezing parameter'' l is zero and the optimal rate, therefore $\Gamma_E=1$. If t_0 is small and r_1,r_2 large it is possible to sacrifice some entanglement in order to ''activate'' the local squeezing thus enhancing the rate later on and obtaining significantly more entanglement at time $t\gg t_0$. The difference to the qubit case is related to the fact that in the cv context not all local transformations are available and hence not all equally entangled states are locally equivalent.

V. DISCUSSION AND CONCLUSION

We have investigated how a quadratic interaction between two continuous-variable systems (as it occurs naturally in certain quantum optical systems) can be optimally used to perform several quantum information tasks when certain simple local control operations (phase-space rotations) can be implemented as well. First, we have given necessary and sufficient conditions for the simulation of a Hamiltonian evolution given a fixed interaction and fast local rotations. In particular, we have shown that the naturally occurring Hamiltonian equation (3) allows to simulate all bilinear Hamiltonians and is in fact of the most versatile kind for this purpose. Moreover, we have seen that almost all the Hamiltonians of the form (1) (and, in particular, H_0) allow to generate all symplectic transformations on two modes, i.e., the complete group $\mathrm{SP}(2,\mathbb{R})$ can be generated starting from no more than the three Hamiltonians $H_0,H_{\mathrm{loc},1},H_{\mathrm{loc},2}$.

With these results, we have addressed the questions of optimal creation of entanglement and squeezing for a two-mode Gaussian state using a given interaction of the form (1) and local rotations of the form $H_{\mathrm{loc},i}=g(X_i^2+P_i^2)$, both of which are available in current experiments. For the case of small (infinitesimal) interaction times, we have determined the optimal strategy to increase entanglement or squeezing for any input state, i.e., we have derived the maximal entanglement and squeezing rates and determined the strategies which lead to these maxima. For the general case (finite interaction time), we have derived the optimal strategy for the creation of entanglement and squeezing starting with the vacuum state. We have also shown that (in contrast to qubit systems) for continuous variables optimizing the entanglement rate is not necessarily the best way to generate a finite amount of entanglement.

There are several interesting applications of our results for quantum information processing. In particular, we have seen that the beam-splitter Hamiltonian $H_{\mathrm{bs}}=X_1P_2-P_1X_2$ can be simulated with an efficiency factor 1/2 by H_0. When acting for a time $t=\pi$ the Hamiltonian H_{bs} generates the swap operation between the systems 1 and 2, thus performing the ''write-in'' and ''read-out'' operations needed when the

atomic ensemble is to be used as a *quantum memory* for the state of the light mode [58].

Another interesting application for atomic ensembles is enabled by the so-called spin-squeezed states [59] which have been prepared experimentally in settings similar to the one described in this paper [4,6]. It has been shown that these states allow for a significant increase in the precision of atomic clocks [60]. While the methods presented above show efficient ways to create squeezed atomic states (e.g., by using the interaction to create squeezing or entanglement optimally and then project the atoms into a pure squeezed state by measuring the light), it would also be interesting to find the *optimal* such procedure.

Note that the argument in Sec. IV C is easily adapted to similar circumstances. For example, it was shown in Ref. [60] that the interaction between the atoms of a suitably prepared Bose-Einstein condensate (BEC) can be described by the quadratic Hamiltonian $J_z^2 \approx P^2$, which can be used to drive the BEC into a spin-squeezed state. By the same reasoning as in Sec. IV C, we see that after an interaction time t a squeezing of e^t is the maximum achievable. This shows optimality of the procedure suggested in Ref. [60] (which employs effectively the so-called ''two-axes countertwisting'' Hamiltonian).

In summary, we have investigated the capabilities of cv interaction Hamiltonians H. We have shown which other Hamiltonians can be simulated with such an H and the available control operations and how to do so efficiently. Then we have derived the optimal entanglement generation rates achievable with this Hamiltonian and given an optimal protocol for the generation of entanglement between the two modes for finite times.

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APPENDIX A: PROOF OF THE NECESSARY AND SUFFICIENT CONDITION FOR HAMILTONIAN SIMULATION

First, we prove necessity. If H can simulate H' efficiently Eq. (18) has to hold for $\kappa=1$ and $H_{\text{eff}}=H'$. Therefore, and because of Eqs. (8) and (20) there must exist a probability distribution $\{p_i\}_{i=1}^n$ and special orthogonal matrices $\{R_i,S_i\}_{i=1}^n$ such that

$$\begin{pmatrix} s_1' & 0 \\ 0 & s_2' \end{pmatrix} = \sum_{i=1}^{n} p_i R_i \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} S_i . \tag{A1}$$

Rotation matrices which should, in principle, appear on the left-hand side can be removed by left and right multiplication with corresponding transposed matrices. In Eq. (A1), we assume these ones to be already included in the R_i, S_i on the right-hand side.

By using the fact that the vector of the diagonal elements of a product $R\ \mathrm{diag}(s_1,s_2)\ S$ can be written as $(R\circ S^T)\times(s_1,s_2)^T$, where $R\circ S^T$ denotes the component-wise (so-called Hadamard) product of matrices we can express the last equation in compact form as

$$\begin{pmatrix} s_1' \\ s_2' \end{pmatrix} = \sum_{i=1}^{n} p_i (R_i \circ S_i^T) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} =: N \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}. \tag{A2}$$

The definition of the matrix N in Eq. (A2) is obvious. Using that all matrices R_i, S_i are elements of $\mathrm{SO}(2,\mathbb{R})$ it can be seen easily that

$$N_{11}=N_{22}, \quad N_{12}=N_{21}$$

$$\text{and} \quad |N_{11} \pm N_{21}| \leqslant 1.$$

Conditions (23) follow now directly from (A2) and these properties of N,

$$s_1'+s_2' = (N_{11}+N_{21})(s_1+s_2) \leqslant s_1+s_2 .$$

The same holds identically for all plus signs replaced by minus signs proving necessity.

To demonstrate sufficiency, we show that conditions (23) guarantee the existence of a matrix N as in Eq. (A2) which in turn admits to connect the primed and unprimed restricted singular values as in Eq. (A1). This provides an efficient simulation protocol of the form (9).

Given s_1, s_2 and s_1', s_2' fulfilling (23), we can for the time being assume that $s_1 \neq |s_2|$ and define

$$N := \begin{pmatrix} e & f \\ f & e \end{pmatrix},$$

$$\text{where} \quad e = \frac{s_1 s_1' - s_2 s_2'}{s_1^2 - s_2^2}, \quad f = \frac{s_1 s_2' - s_2 s_1'}{s_1^2 - s_2^2}.$$

With this definition we have $(s_1', s_2')^T = N(s_1, s_2)^T$. Next we have to show that N can be written as a convex sum of Hadamard products of rotation matrices which is in fact exactly what inequalities (23) ensure.

It is again easy to check that if $|e|+|f| \leqslant 1$, we can find probabilities $\{p_i : p_i \geqslant 0, \Sigma_{i=1}^4 p_i\}_{i=1}^4$ such that $e=p_1-p_2$ and $f=p_3-p_4$ and therefore

$$\begin{aligned} N = {} & p_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + p_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ & + p_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + p_4 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{A3}$$

This decomposition of N allows to pass from (A2) to (A1) conserving the diagonal structure as can be checked easily.

Thus, it suffices to show how (23) implies $|e|+|f|\leq 1$. Multiplying the first [second] line of Eq. (23) by (s_1-s_2) $[(s_1+s_2)]$ yields, respectively,

$$s_1^2-s_2^2 \geq (s_1 s_1'-s_2 s_2')+(s_1 s_2'-s_2 s_1'),$$

$$s_1^2-s_2^2 \geq (s_1 s_1'-s_2 s_2')-(s_1 s_2'-s_2 s_1').$$

The first term on the right-hand sides is non-negative due to premise (22) such that these inequalities are equivalent to

$$s_1^2-s_2^2 \geq |s_1 s_1'-s_2 s_2'|+|s_1 s_2'-s_2 s_1'|,$$

which is, regarding the definition of e and f, exactly what we had to show and proves sufficiency for the case $s_1 \neq |s_2|$.

The complementary cases $s_1=|s_2|$ turn out to be trivial, since conditions (23) then require $s_1'=s_2'=s_1$ or $s_1'=-s_2'=s_1$, respectively, and this means that we can exclusively simulate Hamiltonians, where $H'=(U\otimes V)H(U^\dagger\otimes V^\dagger)$ for some local rotations $U\otimes V$, i.e., H' has to be—in this sense—locally equivalent to H. Hence, nothing has to be shown in this case.

We point out that this proof provides the possibility to construct simulation protocols explicitly. Given H and H' one has to calculate the decomposition in Eq. (A3). Then the probabilities and rotations appearing there will fix the time steps t_i and control operations $U_i\otimes V_i$ in Eq. (9). As can be seen such a protocol will contain at most three intervals of interaction and control operations being rotations about $\pm\pi/2$ and π.

APPENDIX B: GATE SIMULATION

To show that any unitary $U=\exp(-iG)$ where G is quadratic expression in the operators X_k, P_k can be decomposed as given in Eq. (29) we will proceed in three steps.

(i) As shown in Refs. [46,61] any such U can be decomposed into a sequence of one passive transformation, single mode squeezing and another passive transformation. That is to say the symplectic matrix S corresponding to the unitary transformation U can be decomposed as $S=OD\tilde{O}$, where $O,\tilde{O}$ are orthogonal, symplectic and, therefore, passive transformations and the diagonal matrix $D=\mathrm{diag}(e^{\alpha+\beta},e^{-(\alpha+\beta)},e^{\alpha-\beta},e^{-(\alpha-\beta)})$ amounts to local squeezing. Note that this is basically a singular-value decomposition of S.

(ii) Passive transformations contain essentially beam-splitter transformations and local rotations and it is well known from quantum optics that any such transformation on two modes can be decomposed into a sequence of a pair of local rotations, one beam-splitter operation and another pair of local rotations. Thus, a unitary U_O corresponding to a orthogonal symplectic transformation O can be decomposed as $U_O=(V\otimes W)U_{\mathrm{bs}}(t_0)(\tilde{V}\otimes\tilde{W})$ where $U_{\mathrm{bs}}(t)$ is defined in Eq. (27).

(iii) What is left to show is how to attain single-mode squeezing. For this we split the matrix D into two components, $D=\mathrm{diag}(e^{\alpha},e^{-\alpha},e^{\alpha},e^{-\alpha})\mathrm{diag}(e^{\beta},e^{-\beta},e^{-\beta},e^{\beta})$ and show how each of them can be attained by means of beam splitters and two-mode squeezing. Let us denote by $\overline{U}_{\mathrm{bs}}(t)$ and $\overline{U}_{\mathrm{tms}}(t)$ the variants of beam splitter and two-mode squeezing operators which are attained from Eqs. (27) and (28), respectively, by locally rotating $X_2\rightarrow P_2, P_2\rightarrow -X_2$. Then it can be easily shown that the sequence $\overline{U}_{\mathrm{bs}}(-\pi/4)U_{\mathrm{tms}}(\alpha)\overline{U}_{\mathrm{bs}}(\pi/4)$ generates a symplectic transformation $\mathrm{diag}(e^{\alpha},e^{-\alpha},e^{\alpha},e^{-\alpha})$ and $U_{\mathrm{bs}}(-\pi/4)\overline{U}_{\mathrm{tms}}(\beta)U_{\mathrm{bs}}(\pi/4)$ correspondingly $\mathrm{diag}(e^{\beta},e^{-\beta},e^{-\beta},e^{\beta})$.

Collecting things together and ordering all passive components as in (ii), i.e., such that it contains only one application of a beam-splitter operation, decomposition (29) follows immediately.

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