Nonpositive evolutions in open system dynamics

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The long-time evolution of a system in interaction with an external environment is usually described by a family of linear maps γ_t, generated by master equations of Block-Redfield type. These maps are, in general, nonpositive; a widely adopted cure for this physical inconsistency is to restrict the domain of definition of the dynamical maps to those states for which γ_t remains positive. We show that this prescription has to be modified when two systems are immersed in the same environment and evolve with the factorized dynamics $\gamma_t \otimes \gamma_t$ starting from an entangled initial state.

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I. INTRODUCTION

The dynamics of systems immersed in large, external environments can be described in terms of master equations: they generate the finite time-evolution for the reduced density matrix, obtained by tracing over the environmental degrees of freedom [1–9]. Their explicit form is, in general, rather complex, involving nonlinearities and memory effects. Nevertheless, when the coupling between subsystems and environment is sufficiently weak and for times much longer than the characteristic correlation time in the environment, suitable limiting master equations in Markovian form can be derived.

These derivations are often based on *ad hoc* approximations, lacking mathematical rigor, while the final result is justified on the basis of physical considerations. Despite these heuristic treatments, Markovian master equations have been applied to model various effects in open system dynamics, ranging from quantum optics to quantum chemistry.

It has been pointed out long ago that the heuristic derivations of the Markovian limit of master equations could lead, in general, to physical inconsistencies [10]. In particular, the resulting finite time evolution described by such equations would not, in general, preserve the positivity of the reduced density matrix, with some remarkable exceptions, based on rigorous mathematical treatments [1–4,11–19].

Although acknowledged in most subsequent literature on the subject, these inconsistencies were either dismissed as irrelevant for all practical purposes [20] or cured by adopting further *ad hoc* prescriptions [21–23]. In the latter case, the general attitude is to restrict the action of the dynamical maps generated by the nonpositive master equations to a subset of all possible initial reduced density matrices, those for which the time evolution remains positive. This is equivalent to a suitable selection of the initial conditions for the starting state of the subsystem, a procedure sometimes referred to as "slippage of the initial conditions." On physical grounds, this effect is viewed as the consequence of the short-time correlations in the environment, that have not been properly taken into account in the derivation of the Markovian limit of the original master equation.[1]

In the following, we shall reexamine this widely used prescription to cure possible inconsistencies produced by nonpositive, Markovian master equations, and point out further potential problems of this approach. We shall deal with two identical, noninteracting subsystems immersed in a same environment, both evolving, in the Markovian limit, with the same nonpositive master equation. We shall explicitly show that redefining the initial conditions to make positive the single-system time evolution is not enough to cure all possible inconsistencies of the two-system dynamics. These show up when the two-system state that emerges after the transient due to the short-time correlations in the environment is entangled; therefore, in order to have a physically acceptable time evolution for the two subsystems when entanglement is the most likely consequence of the initial transient phase, the above-mentioned procedure of restricting initial conditions should take into account also correlated states.

On the other hand, let us notice that maximal entanglement can be produced without any transient. A particularly interesting example is that of two neutral kaons that are pro-

[1]Let us point out that, instead of restricting the possible initial states, one can alternatively "smooth" the initial conditions on which the nonpositive dynamical map acts [24]; the resulting effective map turns out to be positive. Unless it results in being also completely positive, unphysical effects of the kind discussed below would affect this case as well.

duced, via the weak interaction, as decay products of a spin-one Φ resonance: due to the angular-momentum conservation, the two spin-zero kaons fly apart back to back, in a state that resembles that of the singlet for two spin-$\frac{1}{2}$ particles [25].

Using standard techniques, in the following section, we shall derive the Markovian limit of the master equation describing two two-level systems in interaction with a stochastic environment. After waiting for the correlations in the environment to die out, the resulting finite time evolution Γ_t turns out to be describable in terms of a factorized dynamics: $\Gamma_t = \gamma_t \otimes \gamma_t$, where γ_t represent a single open system dynamics, in general, nonpositive. In Sec. III, we shall then apply the derived time evolution Γ_t to a maximally entangled, pure initial state, and show that "negative probabilities" may arise even though the dynamics γ_t remains positive on the constituent single-system states. The case of partially entangled states and that of mixed entangled states is discussed in Sec. IV. The concluding Sec. V contains our final considerations.

II. MARKOVIAN MASTER EQUATION

The physical model we shall study is formed by two, noninteracting, two-level systems immersed in the same, external environment. The Markovian limit of their subdynamics will be derived using the same techniques and approximations widely adopted in analyzing single-system time evolutions [5–9].

For sake of definiteness, the action of the environment on the two subsystems will be assumed to be mediated by a weak time-dependent stochastic field, coupled to their spin-like degrees of freedom [26–28]. This choice is of sufficient generality for the considerations that follow. Let us point out that this model can describe real physical situations, like the ones occurring in interferometric setups, involving the propagation of neutrons in random magnetic fields [29–36] or photons in random optical media [5–8,37,38]. Moreover, it has been used to study dissipative effects in correlated neutral mesons under the action of weak, stochastic gravitational fields [39–42].

Without loss of generality, the total system Hamiltonian can be taken to be [36]

$$H = H_0^{(1)} + H_0^{(2)} + H_I^{(1)} + H_I^{(2)}, \tag{2.1}$$

$$H_0^{(A)} = \frac{\omega_0}{2}\sigma_1^{(A)}, \quad H_I^{(A)} = \sum_{i=1}^{3} V_i^{(A)}(t)\sigma_i^{(A)}, \quad A=1,2, \tag{2.2}$$

where $\sigma_i^{(1)} = \sigma_i \otimes \mathbf{1}$, $\sigma_i^{(2)} = \mathbf{1} \otimes \sigma_i$, are the two-system spin operators, represented by the Pauli matrices σ_i, $i=1,2,3$, while $\mathbf{V}^{(A)}(t) = (V_1^{(A)}(t), V_2^{(A)}(t), V_3^{(A)}(t))$, $A=1,2$, are the stochastic, Gaussian field variables, independently coupled to the spin degrees of freedom of the two systems. For simplicity, we assume $\mathbf{V}^{(A)}(t)$ to have zero mean, $\langle \mathbf{V}^{(A)}(t)\rangle = 0$, and stationary, real, positive-definite covariance matrix $[W_{ij}^{(AB)}(t)]$ with entries

$$W_{ij}^{(AB)}(t-s) = \langle V_i^{(A)}(t) V_j^{(B)}(s)\rangle = (W_{ij}^{(AB)})^*(t-s) = W_{ji}^{(BA)}(s-t). \tag{2.3}$$

For completeness, let us point out that the Hamiltonian (2.1) can be equivalently interpreted as describing two subsystems in interaction with two independent baths of identical physical characteristics; for definiteness, in the following we find more convenient to refer to the single bath picture.

Being coupled to a stochastic field, the complete 4×4 spin density matrix $R(t)$ is also stochastic; an effective, "reduced," spin density matrix $\rho(t)$ is obtained by averaging over the noise: $\rho(t) := \langle R(t)\rangle$. At the initial time $t=0$, we may suppose spin and noise to decouple, so that $\rho \equiv \langle R(0)\rangle = R(0)$.

The dynamical equation for $\rho(t)$ can be obtained in a standard way from the usual Liouville-von Neumann equation for $R(t)$, through the intermediate use of the interaction picture. The resulting master equation contains an infinite series of terms. As usually done in the case of a single-system subdynamics, a simplified, more manageable expression for it can be derived by means of physical considerations [5–9,26].

By hypothesis, the action of the external stochastic field on the two subsystems is weak; within this "weak-coupling limit" assumption, one can then focus on the dominant first term in the expansion of the general master equation, neglecting higher-order contributions. One explicitly finds

$$\partial_t \rho(t) = -i[H_0^{(1)} + H_0^{(2)}, \rho(t)] - \sum_{A,B=1}^{2} \sum_{i,j=1}^{3} C_{ij}^{(AB)}(t)[\sigma_i^{(A)}, [\sigma_j^{(B)}, \rho(t)]], \tag{2.4a}$$

$$C_{ij}^{(AB)}(t) = \sum_{k=1}^{3} \int_0^t ds\, W_{ik}^{(AB)}(s) U_{kj}(s), \tag{2.4b}$$

where

$$U_{ij}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_0 t & \sin\omega_0 t \\ 0 & -\sin\omega_0 t & \cos\omega_0 t \end{pmatrix} \tag{2.5}$$

is the orthogonal matrix $[U_{ij}(t)]$ that represents the rotations of the Pauli matrices due to the action of the free Hamiltonian: $e^{-itH_0^{(A)}}\sigma_i^{(A)} e^{itH_0^{(A)}} = \sum_{j=1}^{3} U_{ij}(t)\sigma_j^{(A)}$.

Further, by the same physical arguments, the memory effects in Eq. (2.4) should not be physically relevant: within the above-mentioned hypothesis, the use of the Markovian limit is therefore justified; in practice, this can be implemented by extending to infinity the upper limit of integration in Eq. (2.4b). More precisely, for situations amenable to a rigorous mathematical treatment, one can show that a linear, local in time subdynamics is the general result of a limiting procedure in which the coupling constant ξ between system and external environment, and the ratio τ/T between the typical time scale of the system and the decay time of the correlations in the environment, become small [2–4]. The quantities ξ and τ/T regulate both the weak-coupling limit and the Markovian approximation.

In order to keep the discussion in the subsequent sections as simple as possible, we shall make some further simplifying assumptions on the environmental correlations (2.3). We first assume that the external stochastic field be oriented along the third direction, $\mathbf{V}^{(A)}(t)=(0,0,V_3^{(A)}(t))$, with exponentially suppressed correlation functions:

$$\langle V_3^{(1)}(t)V_3^{(1)}(s)\rangle=\langle V_3^{(2)}(t)V_3^{(2)}(s)\rangle=g^2e^{-\mu|t-s|}, \tag{2.6a}$$

$$\langle V_3^{(1)}(t)V_3^{(2)}(s)\rangle=f^2e^{-\nu|t-s|}. \tag{2.6b}$$

Furthermore, we make the physically sensible hypothesis that the nondiagonal, off-site correlations $W_{33}^{(12)}$ be subdominant with respect to the diagonal, on-site ones, $W_{33}^{(AA)}$; in practice, this can be achieved by assuming a hierarchy in the strength ($f^2 \ll g^2$) and decay constants ($\mu \ll \nu$) of the two types of correlations. In this way, the interaction between the two subsystems induced by the coupling with the environment becomes negligible. [The general case is briefly treated in the Appendix, where more details on the derivation of Eq. (2.7) below can also be found.] Then, to lowest order, the finite time evolution for the density matrix, $\rho(0)\to\rho(t)\equiv\Gamma_t[\rho(0)]$, assumes a factorized form, $\Gamma_t=\gamma_t\otimes\gamma_t$, being generated by the following Markovian master equation:

$$\partial_t\rho(t)=(L\otimes\mathbf{1}+\mathbf{1}\otimes L)[\rho(t)]. \tag{2.7}$$

The linear operator $L[\cdot]\equiv L_0[\cdot]+L_1[\cdot]$, the generator of γ_t, acts on 2×2 density matrices η, and its explicit form is as follows:

$$L_0[\eta]=-i[H_0,\eta],\quad H_0=\omega\sigma_1, \tag{2.8a}$$

$$L_1[\eta]=\alpha(\sigma_3\eta\sigma_3-\eta)-\beta(\sigma_2\eta\sigma_3+\sigma_3\eta\sigma_2), \tag{2.8b}$$

with

$$\alpha=\frac{2g^2\mu}{\omega_0^2+\mu^2},\quad \beta=\frac{g^2\omega_0}{\omega_0^2+\mu^2},\quad \omega=\frac{\omega_0}{2}+\beta. \tag{2.8c}$$

In this way, each system evolves independently, with the dynamics generated by[2]

$$\partial_t\eta(t)=L[\eta(t)]. \tag{2.9}$$

This equation is of the Bloch-Redfield type [5–9] and as such it is known not to be positive.[3] As already mentioned in the Introduction, to cure this pathology an *ad hoc* prescription has been proposed, widely adopted in the literature: restrict the possible initial states $\eta(0)$ to those for which $\eta(t)\equiv\gamma_t[\eta(0)]$, $t>0$, as generated by Eq. (2.9), is still a state. As we shall see in the following section, this requirement is, in general, not enough to guarantee the consistency of Eq. (2.7).

[2]It is interesting to notice that essentially the same master equation (2.9) with (2.8) is also the result of the Markovian approximation of a stochastic dynamical evolution based on the "quantum state diffusion" approach [43,44].

[3]Indeed, for sufficiently small, but positive times, the evolution equation (2.9) will map the initial state $\eta(0)=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ into a nonpositive matrix $\eta(t)$.

III. MAXIMALLY ENTANGLED STATES

As shown in the preceding section, the dynamics of two noninteracting systems immersed in the same bath takes a factorized form, $\Gamma_t=\gamma_t\otimes\gamma_t$, at least for times much longer than the characteristic correlation times in the environment. However, the initial state $\rho(0)$ of the compound system, on which Γ_t acts, need not be in factorized form: due to the short-time interaction with the environment, the subsystems can emerge from the transient in an entangled state.

To avoid inconsistencies with the single-system dynamics γ_t, we shall adopt the previously mentioned prescription of restricting its action to those states for which positivity is guaranteed, for any $t\geqslant0$. For the compound system under study, this amounts to require that the partial traces $\eta^{(1)}=\mathrm{Tr}_2[\rho(0)]$ and $\eta^{(2)}=\mathrm{Tr}_1[\rho(0)]$ over the degrees of freedom of the second, respectively, the first, subsystem be admissible states for the map γ_t.

Let us decompose a generic 2×2 density matrix η along the Pauli matrices and the identity σ_0: $\eta=\sum_{\mu=0}^3\eta^\mu\sigma_\mu$ with $\eta^0=1/2$ and η^i real; its time evolution, generated by the Eqs. (2.8) and (2.9), is then given by $\eta(t)\equiv\gamma_t[\eta]=\sum_{\mu=0}^3\eta^\mu(t)\sigma_\mu$ with components

$$\eta^0(t)=\frac{1}{2}, \tag{3.1a}$$

$$\eta^1(t)=e^{-2\alpha t}\eta^1, \tag{3.1b}$$

$$\eta^2(t)=e^{-\alpha t}\left[\left(\cos2\Omega t-\frac{\alpha}{2\Omega}\sin2\Omega t\right)\eta^2-\frac{\omega+\beta}{\Omega}\sin2\Omega t\,\eta^3\right], \tag{3.1c}$$

$$\eta^3(t)=e^{-\alpha t}\left[\left(\cos2\Omega t+\frac{\alpha}{2\Omega}\sin2\Omega t\right)\eta^3+\frac{\omega-\beta}{\Omega}\sin2\Omega t\,\eta^2\right], \tag{3.1d}$$

with $\Omega=\sqrt{\omega^2-\beta^2-\alpha^2/4}$, and η^1, η^2, η^3 the initial density matrix components.

With the eigenstates $\eta_\pm\equiv|\pm\rangle\langle\pm|=(\sigma_0\pm\sigma_1)/2$ of the free systems Hamiltonian in Eq. (2.8a), $H_0|\pm\rangle=\pm\omega|\pm\rangle$, one can build the maximally entangled state

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|+\rangle\otimes|-\rangle-|-\rangle\otimes|+\rangle). \tag{3.2}$$

Let us assume that after the transient the compound system be in such a state, so that the initial density matrix, on which the total Markovian dynamics $\Gamma_t=\gamma_t\otimes\gamma_t$ acts, is given by

$$\rho(0)\equiv|\psi\rangle\langle\psi|=\frac{1}{2}(\eta_+\otimes\eta_-+\eta_-\otimes\eta_+-\eta_{+-}\otimes\eta_{-+}-\eta_{-+}\otimes\eta_{+-}), \tag{3.3}$$

with $\eta_{\pm\mp}=(\sigma_3\pm i\sigma_2)/2$. The two partial traces $\eta^{(1)}=\mathrm{Tr}_2[\rho(0)]$ and $\eta^{(2)}=\mathrm{Tr}_1[\rho(0)]$, being equal to $\sigma_0/2$, are left invariant by the dynamics (3.1), and therefore represent admissible states for the evolution γ_t.

It is a matter of a simple computation to apply the evolution given in Eq. (3.1) to the four 2×2 matrices η_+, η_-, η_{+-}, η_{-+} and therefore obtain the explicit expression for the evolved 4×4 matrix $\rho(t)=\Gamma_t[\rho(0)]\equiv\gamma_t\otimes\gamma_t[\rho(0)]$. On the basis for which the Pauli matrices assume the standard form, $\sigma_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\sigma_2=\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)$, $\sigma_3=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, one explicitly gets

$$\rho(t)=\frac{1}{4}\begin{pmatrix}A_-(t)&C(t)&C(t)&B_+(t)\\-C(t)&A_+(t)&B_-(t)&-C(t)\\-C(t)&B_-(t)&A_+(t)&-C(t)\\B_+(t)&C(t)&C(t)&A_-(t)\end{pmatrix}, \tag{3.4}$$

where

$$A_\pm(t)=1\pm e^{-2\alpha t}\left[\left(\cos2\Omega t+\frac{\alpha}{2\Omega}\sin2\Omega t\right)^2+\left(\frac{\omega-\beta}{\Omega}\right)^2\sin^2 2\Omega t\right], \tag{3.5a}$$

$$B_\pm(t)=-e^{-4\alpha t}\pm e^{-2\alpha t}\left[\left(\cos2\Omega t-\frac{\alpha}{2\Omega}\sin2\Omega t\right)^2+\left(\frac{\omega+\beta}{\Omega}\right)^2\sin^2 2\Omega t\right], \tag{3.5b}$$

$$C(t)=ie^{-2\alpha t}\sin2\Omega t\left[\frac{2\beta}{\Omega}\cos2\Omega t-\frac{\alpha\omega}{\Omega^2}\sin2\Omega t\right]. \tag{3.5c}$$

The matrix $\rho(t)$ in Eq. (3.4) should represent the state of the compound system at time t, having been originally prepared in the initial entangled state (3.3). The matrix (3.4) should then be positive. However, one can easily check that one of its eigenvalues can become negative, precisely that corresponding to the eigenvector (1, 0, 0, -1).[4] Indeed, from its expression

$$\lambda(t)=\frac{1}{4}\left\{1+e^{-4\alpha t}-2e^{-2\alpha t}\left[\cos^2 2\Omega t+\frac{2\omega^2-\Omega^2}{\Omega^2}\sin^2 2\Omega t\right]\right\}, \tag{3.6}$$

one checks that $\lambda(0)=\dot\lambda(0)=0$, while $\ddot\lambda(0)=-8\beta^2$, so that $\lambda(t)$ starts assuming negative values as soon as t becomes nonzero.

In order for the map $\Gamma_t=\gamma_t\otimes\gamma_t$ to produce a physically acceptable dynamics, states like (3.3) must therefore be excluded from its domain of definition. Entanglement is crucial in revealing this physical inconsistency; indeed, on factorized states $\eta^{(1)}\otimes\eta^{(2)}$, with $\eta^{(1)}$, $\eta^{(2)}$ admissible starting density matrices for γ_t, the dynamics $\Gamma_t=\gamma_t\otimes\gamma_t$ remains positive.

The negativity of $\lambda(t)$ will not last forever: due to the damping factors, the expression in Eq. (3.6) becomes positive after a certain time, and actually asymptotically tends to 1/4, as the remaining three eigenvalues of Eq. (3.4). This is a consequence of the dynamics generated by Eq. (2.7) for which the von Neumann entropy, $S[\rho]=-\rho\ln\rho$, always increases (as already observed, the map γ_t, hence Γ_t, is unital, $\gamma_t[\sigma_0]=\sigma_0$); therefore, any initial state $\rho(0)$ of the compound system is asymptotically driven for long times to the maximally disordered state $\rho=\sigma_0\otimes\sigma_0/4$.

In general, master equations of the form (2.9) and (2.8b) may involve parameters α and β, not as in Eq. (2.8c), but totally independent. In such cases, contrary to Eq. (2.8c), α can become vanishingly small, without conflicting with the Markovian hypothesis. Consequently, the eigenvalue $\lambda(t)$ becomes a periodic function of time and assumes negative values even for arbitrary large times.

IV. PARTIALLY AND MIXED ENTANGLED STATES

As already observed, it is the entanglement of the initial state $\rho(0)$ of the otherwise independent two subsystems that allows revealing the unphysical effect of production of "negative probability" by the dynamics Γ_t. The magnitude of the phenomenon is directly connected to the amount of entanglement that the initial state $\rho(0)$ contains.

This can be easily shown by taking the partially entangled state

$$|\psi_\theta\rangle=\cos\theta|+\rangle\otimes|-\rangle-\sin\theta|-\rangle\otimes|+\rangle, \tag{4.1}$$

as starting state, instead of the maximally entangled one in Eq. (3.2). The evolution in time of the corresponding density matrix $\rho_\theta(0)=|\psi_\theta\rangle\langle\psi_\theta|$ can be easily obtained as before using the explicit expressions in Eq. (3.1).[5] One finds that also in this case the eigenvalue $\lambda_\theta(t)$ of $\rho_\theta(t)=\Gamma_t[\rho_\theta(0)]$ corresponding to the eigenvector (1,0,0,-1) can assume negative

[4] Using the definitions in Eq. (3.5), the four eigenvalues can be explicitly written as $(A_+-B_-)/4$, $(A_--B_+)/4$, $\{(A_++A_-+B_++B_-)\pm[(A_+-A_-+B_--B_+)^2-16C^2]^{1/2}\}/8$. The corresponding eigenvectors turn out to be time dependent; their explicit expressions are involved and not particularly inspiring.

[5] Notice that as before the partial traces $\eta^{(1)}_\theta=\mathrm{Tr}_2[\rho_\theta(0)]=\sigma_0/2+\cos2\theta\sigma_1/4$ and $\eta^{(2)}_\theta=\mathrm{Tr}_1[\rho_\theta(0)]=\sigma_0/2-\cos2\theta\sigma_1/4$ are perfectly admissible states of the dynamics γ_t.

values; in fact, $\lambda_\theta(0)=\dot{\lambda}_\theta(0)=0$, while $\ddot{\lambda}_\theta(0)=2(\alpha^2\cos^2 2\theta-4\beta^2\sin^2 2\theta)$, which is negative provided $\tan^2 2\theta \geqslant \alpha^2/4\beta^2$.

In other terms, once the evolution γ_t is given, and therefore the parameters α and β of the corresponding master equation are fixed, the time evolution $\Gamma_t=\gamma_t\otimes\gamma_t$ of the compound system becomes physically inconsistent on initial states that possess a sufficiently high degree of entanglement. Therefore, in order for Γ_t to be an acceptable Markovian evolution, one has to further restrict its domain of definition, in order to exclude also those partially entangled states.

The discussion can be extended to entangled mixed initial states, like the Werner states [45]:

$$\rho_W=p\rho+\frac{1-p}{4}\sigma_0\otimes\sigma_0, \quad -\frac{1}{3}\leqslant p\leqslant 1, \tag{4.2}$$

where ρ is again the maximally entangled state in Eq. (3.3). Also in this case one can show that the eigenvalue $\lambda_W(t)$ of $\rho_W(t)=\Gamma_t[\rho_W]$ corresponding to the eigenvector $(1,0,0,-1)$ can take negative values, provided the parameter p, that measures the degree of entanglement, is sufficiently close to one.

The discussion becomes particularly transparent when $\alpha=0$. In this case, the eigenvalues of $\rho_W(t)$ become a periodic function of time; for $\lambda_W(t)$ one then explicitly obtains

$$\lambda_W(t)=\frac{1}{4}\left\{1+p\left[1-2\left(\cos^2 2\Omega t+\frac{2\omega^2-\Omega^2}{\Omega^2}\sin^2 2\Omega t\right)\right]\right\}. \tag{4.3}$$

From this expression, one sees that the minimum value of $\lambda_W(t)$ becomes periodically negative provided $p>(\omega^2-\beta^2)/(\omega^2+3\beta^2)$. This possibility is excluded in absence of noise: $\beta=0$; in this case, the evolution γ_t (and hence $\Gamma_t=\gamma_t\otimes\gamma_t$), being unitary, results automatically positive.

V. DISCUSSION

The dynamics of a subsystem in weak interaction with an external environment can be described in terms of linear maps obeying a Markovian master equation. This result has been rigorously proven in very special cases; it is nevertheless believed to hold, in general, on the basis of simple physical considerations: when all correlations in the environment have died out, nonlinearities and memory effects should disappear from the reduced subsystem dynamics.

As shown in Sec. II, the same type of arguments allow deriving a Markovian limit for the master equation describing the time evolution of two noninteracting subsystems in contact with the same reservoir; the corresponding dynamical map Γ_t for the compound system turns out to assume a factorized form $\Gamma_t=\gamma_t\otimes\gamma_t$.

In the case of two-level systems, γ_t usually takes a Bloch-Redfield-type form, and therefore it is not, in general, positive. To avoid inconsistencies, one usually restricts the possible initial states to those for which γ_t remains positive (the so-called "slippage of initial conditions"). This prescription works also in the case of the evolution $\Gamma_t=\gamma_t\otimes\gamma_t$ for two identical subsystems, provided the initial state is in separable form: $\rho(0)=\Sigma_i p_i\,\eta_i^{(1)}\otimes\eta_i^{(2)}$, $p_i\geqslant 0$, $\Sigma_i p_i=1$, where $\eta_i^{(1)}$ and $\eta_i^{(2)}$ are admissible states for the first and second subsystems, respectively.

On the contrary, as shown in the previous sections, when the initial state $\rho(0)$ is not in factorized form and the degree of entanglement is sufficiently high, the evolved matrix $\rho(t)=\Gamma_t[\rho(0)]$ fails to be positive at all times. In keeping with the same attitude adopted for a single-subsystem dynamics γ_t, to cure this additional inconsistency one can further restrict the domain of applicability of $\gamma_t\otimes\gamma_t$. However, this is again a temporary solution: indeed, the whole discussion needs be repeated when three or more subsystems in contact with the same bath are considered; clearly, further restrictions on γ_t need to be imposed.

These considerations cannot be dismissed as being purely academic; on the contrary, they seem to have a direct experimental relevance: as mentioned in the Introduction, couples of systems in an entangled state are in fact actively studied, and the ongoing experiments on correlated neutral kaons constitute a significative example. From this perspective, the widely used cure of redefining the initial conditions in case of nonpositive Markovian dynamics does not appear to be completely satisfactory.

In closing, let us mention that in the few cases for which the Markovian limit of the subdynamics can be obtained in a rigorous way, the resulting evolution map γ_t turns out to be not only positive, but also completely positive [1–4,11–16]. In these cases, the compound map $\Gamma_t=\gamma_t\otimes\gamma_t$ is also completely positive and therefore no inconsistencies can arise, even when Γ_t acts on entangled states [46,47].

APPENDIX

In Sec. II, we have seen that the master equation that describes the time evolution of two, noninteracting subsystems in contact with the same stochastic bath can be written in the following closed form:

$$\partial_t\rho(t)=-i[H_0^{(1)}+H_0^{(2)},\rho(t)] - \sum_{A,B=1}^{2}\sum_{i,j=1}^{3}C_{ij}^{(AB)}[\sigma_i^{(A)},[\sigma_j^{(B)},\rho(t)]], \tag{A1}$$

$$C_{ij}^{(AB)}=\sum_{k=1}^{3}\int_0^\infty ds\,W_{ik}^{(AB)}(s)U_{kj}(s), \tag{A2}$$

after the weak-coupling limit and the Markovian approximation have been taken into account; here, $W_{ij}^{(AB)}$ represent the correlation functions in the environment, while U_{ij} is the orthogonal matrix in Eq. (2.5) that takes into account the rotation of the Pauli matrices generated by the free Hamiltonian.

From the properties (2.3) of the correlation functions, it follows that the diagonal, on-site coefficients $C_{ij}^{(AA)}$, $A=1,2$, are real matrices, that can thus be decomposed into symmetric, $\mathcal{S}_{ij}^{(A)}\equiv(C_{ij}^{(AA)}+C_{ji}^{(AA)})/2$, and antisymmetric, $\mathcal{A}_{ij}^{(A)}\equiv(C_{ij}^{(AA)}-C_{ji}^{(AA)})/2$, components. Correspondingly, the second term in Eq. (A1), can be decomposed into Hamiltonian and purely dissipative pieces, so that the total master equation can be rewritten as

$$\partial_t\rho(t)=-i[H,\rho(t)]+L_D[\rho(t)], \tag{A3}$$

where the total Hamiltonian is now given by

$$H=H_0^{(1)}+H_0^{(2)}+H_D^{(1)}+H_D^{(2)}, \quad H_D^{(A)}=\sum_{i,j,k=1}^{3}\epsilon_{ijk}\mathcal{A}_{ij}^{(A)}\sigma_k^{(A)}, \tag{A4}$$

while the dissipative contribution $L_D\equiv L_D^{(1)}+L_D^{(2)}+L_D^{(12)}$ has diagonal and off-diagonal pieces

$$L_D^{(A)}[\rho]=\sum_{i,j=1}^{3}\mathcal{S}_{ij}^{(A)}(2\sigma_i^{(A)}\rho\sigma_j^{(A)}-\{\sigma_i^{(A)}\sigma_j^{(A)},\rho\}), \quad A=1,2, \tag{A5a}$$

$$L_D^{(12)}=\sum_{i,j=1}^{3}(C_{ij}^{(12)}+C_{ji}^{(21)})([\sigma_i^{(1)},\rho(t)\sigma_j^{(2)}] +[\sigma_j^{(2)}\rho(t),\sigma_i^{(1)}]). \tag{A5b}$$

Without additional knowledge on the behavior of the correlation functions $W_{ij}^{(AB)}$ of the environmental variables, the form of the evolution equations (A3)–(A5) cannot be further simplified.

On the other hand, with the assumptions (2.6),

$$W_{33}^{(11)}(t-s)=W_{33}^{(22)}(t-s)=g^2e^{-\mu|t-s|},$$
$$W_{33}^{(12)}(t-s)=f^2e^{-\nu|t-s|}, \tag{A6}$$

and all remaining entries zero, the form of the coefficients in Eq. (A2) can be explicitly computed, and a more manageable expression for Eq. (A3) can be obtained. Indeed, after some simple manipulations, one finds

$$\partial_t\rho(t)=(L_0\otimes\mathbf{1}+\mathbf{1}\otimes L_0)[\rho(t)]+(L_1\otimes\mathbf{1}+\mathbf{1}\otimes L_1)[\rho(t)] -L_2[\rho(t)], \tag{A7}$$

where L_0 ands L_1 are linear operators acting on 2×2 density matrices η

$$L_0[\eta]=-i[H,\eta], \quad H=\left(\frac{\omega_0}{2}+\beta\right)\sigma_1, \tag{A8a}$$

$$L_1[\eta]=\alpha(\sigma_3\eta\sigma_3-\eta)-\beta(\sigma_2\eta\sigma_3+\sigma_3\eta\sigma_2), \tag{A8b}$$

while L_2 takes the form

$$\begin{aligned}L_2[\rho]=\gamma[\{\sigma_3\otimes\sigma_3,\rho\}-\sigma_3\otimes\sigma_0\rho\sigma_0\otimes\sigma_3\\ -\sigma_0\otimes\sigma_3\rho\sigma_3\otimes\sigma_0]-\delta[\{\sigma_3\otimes\sigma_2,\rho\}\\ +\{\sigma_2\otimes\sigma_3,\rho\}-\sigma_3\otimes\sigma_0\rho\sigma_0\otimes\sigma_2\\ -\sigma_0\otimes\sigma_2\rho\sigma_3\otimes\sigma_0-\sigma_2\otimes\sigma_0\rho\sigma_0\otimes\sigma_3\\ -\sigma_0\otimes\sigma_3\rho\sigma_2\otimes\sigma_0].\end{aligned} \tag{A8c}$$

The four constants α, β, γ, δ, that measure the relative strength of the various dissipative contributions, are determined by the parameters appearing in the correlation functions (A6):

$$\alpha=\frac{2g^2\mu}{\omega_0^2+\mu^2}, \quad \beta=\frac{g^2\omega_0}{\omega_0^2+\mu^2}, \quad \gamma=\frac{2f^2\nu}{\omega_0^2+\nu^2},$$
$$\delta=\frac{f^2\omega_0}{\omega_0^2+\mu^2}. \tag{A9}$$

When the strength and decay constants of the off-diagonal correlations $W_{33}^{(12)}$ are much smaller than the corresponding ones in $W_{33}^{(AA)}$, $f^2\ll g^2$, $1/\nu\ll1/\mu$, the dissipative constants γ and δ can be neglected with respect to α and β, and the resulting evolution equation (A7) reduces to that presented in Eqs. (2.7) and (2.8).

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