Density-matrix operatorial solution of the non-Markovian master equation for quantum Brownian motion

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An original method to exactly solve the non-Markovian master equation describing the interaction of a single harmonic oscillator with a quantum environment in the weak-coupling limit is reported. By using a superoperatorial approach, we succeed in deriving the operatorial solution for the density matrix of the system. Our method is independent of the physical properties of the environment. We show the usefulness of our solution deriving explicit expressions for the dissipative time evolution of some observables of physical interest for the system, such as, for example, its mean energy.

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I. INTRODUCTION

The problem of the quantum dynamics of a small system interacting with its environment has been extensively studied since the origin of quantum mechanics. However, in spite of the noticeable progresses in the theory of open quantum systems, many conceptual difficulties still remain. Indeed the understanding of the effects of the environment on the physical system of interest, in general, is not an easy task. The conventional way to afford the problem of the description of the time evolution of an open quantum system consists in eliminating the degrees of freedom of the environment in order to derive an equation of motion for the reduced density matrix of the small system. This procedure stems from the fact that we are usually only interested in the dynamics of this subsystem and do not really care about the state of the environment and its evolution. The procedure of tracing over the environmental variables thus amounts to declaring that everything other than the small system is indifferent to us. In this way one reduces the number of variables one needs to take into account to study the problem.

Only very few physical systems prove to be amenable to an analytic description of their open dynamics. In this paper, we deal with the simplest open quantum system worth being studied: the damped harmonic oscillator. Such a system is of great conceptual importance because it provides a simple, successful starting point in the theoretical description of many experimental situations in the quantum optics, solid-state physics, and quantum-field-theory contexts. For this reason, it is also one of the most extensively studied physical systems [1–7].

Understanding the dissipative behavior of open quantum systems is a crucial problem for both fundamental and applicative reasons. On one hand, indeed, during the past few years, the interpretation of Zurek, associating the quantum to classical transition with environment induced decoherence effects, has become increasingly popular. From this point of view, studying the open system dynamics of exemplary quantum systems, i.e., harmonic oscillator, would help in identifying the elusive border between quantum and classical descriptions of the world. On the other hand, the huge advances in experimental techniques for controlling the evolution of quantum systems have paved the way to the realization of the first quantum logic gates, the key elements of quantum computers. The biggest obstacle to building quantum computers is the decoherence process due to the unavoidable coupling with the external environment. Thus, studying a decohering quantum computer is an essential step for the identification of realistic quantum error correcting codes [8].

A standard method to describe the effects of the environment on a small quantum system is based on the Born-Markov master equation. Such an approach is valid whenever the environmental correlation time is much shorter than the typical time scales of the system dynamics. Under these conditions, it is possible to derive and, in some cases, to solve analytically the master equation ruling the dynamics. This is, for example, the case of a quantum harmonic oscillator coupled through bilinear interactions to a reservoir of harmonic oscillators. This model is a particular case of the quantum Brownian motion (QBM), where we specify the system to be just a quantum harmonic oscillator [9].

There exist situations, however, wherein the reservoir correlation time is longer than the system time scales of interest and thus the Born-Markov approximation does not hold anymore. This is, for instance, the case of atoms decaying in photonic band-gap materials or atom lasers [10]. Moreover, very recently, the potential interest of non-Markovian reservoirs for quantum information processing has been demonstrated [11] and a non-Markovian description of quantum computing, showing the limits of the Markovian approach, has been presented [8]. Several procedures have been developed to treat non-Markovian processes. Most of them are reviewed in Ref. [12]. Very often, in order to describe the dynamics of an open quantum system in terms of a master equation for its density matrix, a number of approximations involving the relevant system and reservoir parameters are required. One of the most common is the weak-coupling approximation based on the assumption of a weak system-environment coupling strength. As underlined by Paz and Zurek in Ref. [9], a perturbative approach in the coupling strength can always be shown to lead to master equations

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local in time. This fact could appear surprising at a first sight, since often non-Markovian quantum systems are described by generalized master equations involving a nonlocal memory kernel taking into account the past history of the reduced system. The advantage of dealing with a generalized master equation which is local in time is that the memory effects of the environment are incorporated into its time-dependent coefficients [13–15]. This circumstance makes it possible, in some cases, to find an analytic solution of the generalized master equation.

In this paper, we present an original method for the derivation of the operatorial density matrix solution of the non-Markovian master equation for the QBM in the weak-coupling limit. We apply our procedure to the case in which the system is a harmonic oscillator interacting with the quantized reservoir through a bilinear coupling. Our method does not rely on any assumption other than the weak-coupling limit. In particular, it is independent of the type of environment considered and of the characteristic parameters of the system, provided a perturbative approach to the second order in the coupling constant is valid. We demonstrate the usefulness and simplicity of our analytic solution discussing, as an example, the dissipative dynamics of the mean energy of the harmonic oscillator.

The paper is organized as follows. In Sec. II, we introduce the mathematical formalism of superoperators and we present some key properties for solving the generalized master equation for our system. In Sec. III, we describe the non-Markovian master equation for the QBM in the weak-coupling limit and in Sec. IV we derive its operatorial solution. In Sec. V, we discuss some approximated forms of the density matrix and we present some applications. Finally, in Sec. VI we present conclusions.

II. SUPEROPERATOR FORMALISM

Let us begin introducing the mathematical formalism and properties which we use in the rest of the paper to derive the time dependence of density matrix of the system under scrutiny. Our method is based on the extension of the notion of eigensolution of an operator in the "*superoperator*" formalism.

Given two generic operators $\hat{A}$ and $\hat{\rho}$, let us define the *superoperators of the S type* $\mathbf{A}^S \equiv (A)^S$ *and of the* Σ *type* $\mathbf{A}^\Sigma \equiv (A)^\Sigma$ in the following way:

$$\mathbf{A}^S \rho = [\hat{A}, \hat{\rho}], \quad S \text{ type}, \tag{1a}$$

$$\mathbf{A}^\Sigma \hat{\rho} = \{\hat{A}, \hat{\rho}\}, \quad \Sigma \text{ type}, \tag{1b}$$

where the square and curly brackets indicate the commutator and anticommutator, respectively. Equations (1a) and (1b) define a particular class of superoperators hereafter also called commutator (S) and anticommutator (Σ) superoperators. It is important to note that a S superoperator $\mathbf{A}^S$ may be linearly combined with or multiplied by a Σ superoperator $\mathbf{B}^\Sigma$ giving rise to superoperators belonging neither to the S class nor to the Σ class. From Eqs. (1a) and (1b) linearity follows immediately,

$$(\alpha\hat{A} + \beta\hat{B})^{S(\Sigma)} = \alpha\mathbf{A}^{S(\Sigma)} + \beta\mathbf{B}^{S(\Sigma)}, \quad \alpha, \beta \in \mathbb{C}, \tag{2}$$

and the properties

$$[\mathbf{A}^S, \mathbf{B}^S] = [\hat{A}, \hat{B}]^S, \tag{3a}$$

$$[\mathbf{A}^\Sigma, \mathbf{B}^\Sigma] = [\hat{A}, \hat{B}]^\Sigma, \tag{3b}$$

$$[\mathbf{A}^S, \mathbf{B}^\Sigma] = [\mathbf{A}^\Sigma, \mathbf{B}^S] = [\hat{A}, \hat{B}]^\Sigma. \tag{3c}$$

The main consequence of such relations is that if the commutator of two operators is a c number, then the corresponding $S(\Sigma)$ superoperators do commute. For example, for the superoperators $\mathbf{X}^{S(\Sigma)}$ and $\mathbf{P}^{S(\Sigma)}$, corresponding, respectively, to position $\hat{X}$ and conjugate momentum $\hat{P}$ operators, the following relation holds:

$$[\mathbf{X}^{S(\Sigma)}, \mathbf{P}^{S(\Sigma)}] = 0. \tag{4}$$

Moreover, from Eq. (3c) one gets

$$[\mathbf{X}^S, \mathbf{P}^\Sigma] = [\mathbf{X}^\Sigma, \mathbf{P}^S] = 2\imath. \tag{5}$$

Another superoperatorial identity extensively used in the rest of the paper is

$$(\hat{A}\hat{B} + \hat{B}\hat{A})^S = (\mathbf{A}^S\mathbf{B}^\Sigma + \mathbf{B}^S\mathbf{A}^\Sigma), \tag{6}$$

and, in particular,

$$\mathbf{A}^S\mathbf{A}^\Sigma = (\hat{A}^2)^S = \mathbf{A}^{\mathbf{2}S}. \tag{7}$$

In a certain sense superoperators can be considered as a generalization of the concept of operator used in quantum mechanics. Superoperators act on a space whose elements are operators to give other operators. Let us consider a generalization of the concept of eigenstates. Looking at the equations

$$\mathbf{X}^S e^{-\imath(p\hat{X} - x\hat{P})} = -x e^{-\imath(p\hat{X} - x\hat{P})}, \tag{8a}$$

$$\mathbf{P}^S e^{-\imath(p\hat{X} - x\hat{P})} = -p e^{-\imath(p\hat{X} - x\hat{P})}, \quad x, p \in \mathbb{R}, \tag{8b}$$

we can identify $e^{\imath(p\hat{X} - x\hat{P})}$ as the eigenoperator of the superoperators $\mathbf{X}^S$ and $\mathbf{P}^S$ with eigenvalues $-x$ and $-p$, respectively. Another superoperatorial eigenvalue equation of interest in this paper is

$$\mathbf{N}(p\hat{X} - x\hat{P})^n = n(p\hat{X} - x\hat{P})^n, \quad n \in \mathbb{N}, \tag{9}$$

where $\mathbf{N} = -(\imath/2)(\mathbf{P}^\Sigma\mathbf{X}^S - \mathbf{X}^\Sigma\mathbf{P}^S)$. In particular, we have

$$\mathbf{N}\hat{X}^n = n\hat{X}^n, \quad \mathbf{N}\hat{P}^n = n\hat{P}^n. \tag{10}$$

In the following we will use, for the sake of simplicity, a matrix representation of the previous relations. Let us define

$$\hat{\vec{Z}} = \begin{pmatrix} \hat{X} \\ \hat{P} \end{pmatrix}, \quad \vec{\mathbf{Z}}^{S(\Sigma)} = \begin{pmatrix} \mathbf{X}^{S(\Sigma)} \\ \mathbf{P}^{S(\Sigma)} \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} \vec{x} \\ \vec{p} \end{pmatrix}. \tag{11}$$

With this notation, Eqs. (8) can be recast in a more compact form

$$\vec{\mathbf{Z}}^S e^{-\imath \vec{z}^t \mathsf{J} \hat{\vec{Z}}} = -\vec{z} e^{-\imath \vec{z}^t \mathsf{J} \hat{\vec{Z}}}. \tag{12}$$

In this equation, J is the following 2×2 matrix:

$$\mathsf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{13}$$

and thus satisfies the properties $\mathsf{J}^t = \mathsf{J}^{-1} = -\mathsf{J}$ and $\mathsf{J}^2 = -1$. Note that, using this notation, the superoperator $\mathbf{N}$ can be cast in the form $\mathbf{N} = -\frac{1}{2}\imath(\vec{\mathbf{Z}}^\Sigma)^t \mathsf{J} \vec{\mathbf{Z}}^S$.

III. THE MASTER EQUATION IN THE WEAK-COUPLING LIMIT

In this section we specify the physical system we wish to study, namely, a single harmonic oscillator interacting with a quantized environment, and we introduce and discuss the generalized master equation governing its dynamics. Let us consider a harmonic oscillator of frequency ω_0 surrounded by a generic environment. We express the total Hamiltonian $\hat{H}$ as follows:

$$\hat{H} = \hat{H}_0 + \hat{H}_E + \alpha \hat{X}\hat{E}, \tag{14}$$

where $\hat{H}_0 = \frac{1}{2}(\hat{P}^2 + \hat{X}^2)$, $\hat{H}_E$ and $\alpha\hat{X}\hat{E}$ are the system, environment and interaction Hamiltonians, respectively, and α is the coupling constant. The interaction Hamiltonian here considered has a simple bilinear form with $\hat{X}$ as the position operator of the system and $\hat{E}$ as the generic environmental operator.

For the sake of simplicity, we have written the previous expressions in terms of adimensional position and momentum operators. Let us denote with $\hat{\tilde{\rho}} \equiv \tilde{\rho}$ the density-matrix operator for the oscillator-environment system.

Let us now assume the following.

(1) At $t=0$ system and environment are uncorrelated, that is, $\tilde{\rho}(0) = \hat{\rho}(0) \otimes \hat{\rho}_E(0)$, with $\hat{\rho}$ and $\hat{\rho}_E$ density matrices of the system and the environment, respectively.

(2) $[\hat{H}_E, \hat{\rho}_E(0)] = 0$ (stationarity of the environment).

(3) $\mathrm{tr}_E\{\hat{E}\hat{\rho}_E(0)\} = 0$ (as, for example, in the case of a thermal reservoir).

(4) A second-order perturbative approach in the coupling constant is possible.

Under these conditions one can show [9,21] that the non-Markovian generalized master equation describing the harmonic-oscillator dynamics, in the Schrödinger picture, is the following:

$$\frac{d\hat{\rho}(t)}{dt} = \left[\frac{1}{i\hbar}\mathbf{H}_0^S - \mathbf{D}_S(t) + \imath \mathbf{G}_S(t)\right]\hat{\rho}(t), \tag{15}$$

where the superoperator $\mathbf{D}_S(t)$ (not S or Σ type) is defined as

$$\mathbf{D}_S(t) = \int_0^t \kappa(\tau)\mathbf{X}^S[\cos(\omega_0\tau)\mathbf{X}^S - \sin(\omega_0\tau)\mathbf{P}^S]d\tau. \tag{16}$$

Also $\mathbf{G}_S(t)$ is neither an S-type nor a Σ-type superoperator being defined as follows:

$$\mathbf{G}_S(t) = \int_0^t \mu(\tau)\mathbf{X}^S[\cos(\omega_0\tau)\mathbf{X}^\Sigma - \sin(\omega_0\tau)\mathbf{P}^\Sigma]d\tau. \tag{17}$$

With reference to the assumption (1), we observe that the time-convolutionless master equation for quantum Brownian motion can be derived also for the more general event of correlated initial states of the total system. In such a case the generalized master equation for the system contains an inhomogeneity term $I(t)$ [14]. We have verified that a generalization of the solving procedure to be presented in the following section to the case of nonfactorizing initial conditions is possible provided that the quantum characteristic function associated to $I(t)$ is known.

In Eqs. (16) and (17) we have introduced the correlation $\kappa(\tau)$ and the susceptibility $\mu(\tau)$ functions [24]. Such quantities, characterizing the environmental temporal behavior, are defined as follows:

$$\kappa(\tau) = \frac{\alpha^2}{2\hbar^2}\langle\{\hat{E}(\tau), \hat{E}(0)\}\rangle, \tag{18}$$

$$\mu(\tau) = \frac{\imath\alpha^2}{2\hbar^2}\langle[\hat{E}(\tau), \hat{E}(0)]\rangle, \tag{19}$$

with $\langle \cdots \rangle = \mathrm{tr}_E\{\cdots \hat{\rho}_E(0)\}$.

The form of Eq. (15) has a clear physical meaning. The superoperator $\mathbf{D}_S(t)$, indeed, is strictly related to diffusion (decoherence) processes only [9]. The superoperator $\mathbf{G}_S(t)$, on the other hand, describes dissipation and frequency renormalization processes [9]. Such a superoperator arises from a quantum-mechanical treatment of the environment and, indeed, vanishes when the environment is treated as a classical quantity [see also Eq. (19)].

By using the properties of the superoperators introduced in the preceding section, one can show that $\mathbf{G}_S(t)$ can be recast in the following form:

$$\mathbf{G}_S(t) = \frac{1}{2}[r(t)\hat{X}^2 - \gamma(t)(\hat{X}\hat{P} + \hat{P}\hat{X})]^S - \imath\gamma(t)(\mathbf{N}+2), \tag{20}$$

where the superoperator $\mathbf{N}$ is defined by Eq. (9) and

$$r(t) = 2\int_0^t \mu(\tau)\cos(\omega_0\tau)d\tau, \quad \gamma(t) = \int_0^t \mu(\tau)\sin(\omega_0\tau)d\tau. \tag{21}$$

As for $\mathbf{D}_S(t)$, it is straightforward to see from Eq. (16) that it can be recast in the form

$$\mathbf{D}_S(t) = \overline{\Delta}(t)(\mathbf{X}^S)^2 - \Pi(t)\mathbf{X}^S\mathbf{P}^S, \tag{22}$$

where

$$\overline{\Delta}(t)=\int_0^t \kappa(\tau)\cos(\omega_0\tau)d\tau, \quad \Pi(t)=\int_0^t \kappa(\tau)\sin(\omega_0\tau)d\tau. \tag{23}$$

In view of Eq. (4), $\mathbf{D}_S(t)$ can be regarded as a quadratic form in the commutative operator variables $\mathbf{X}^S$ and $\mathbf{P}^S$:

$$\mathbf{D}_S(t)=(\vec{\mathbf{Z}}^S)^t \mathsf{M}(t)\vec{\mathbf{Z}}^S \quad \text{with} \quad \mathsf{M}(t)=\begin{pmatrix} \overline{\Delta}(t) & -\frac{\Pi(t)}{2} \\ -\frac{\Pi(t)}{2} & 0 \end{pmatrix}. \tag{24}$$

It is not difficult to check that $[\mathbf{D}_S(t),\mathbf{D}_S(t_1)]=0$, whatever t and t_1 are. Inserting Eq. (20) into Eq. (15) one can write the generalized non-Markovian master equation describing the dissipative dynamics of our system in the following final form [9,21]:

$$\frac{d\hat{\rho}(t)}{dt}=\left[\frac{1}{i\hbar}\overline{\mathbf{H}}_0^S(t)-\mathbf{D}_S(t)+\gamma(t)(\mathbf{N}+2)\right]\hat{\rho}(t), \tag{25}$$

with

$$\widehat{\overline{H}}_0(t)=\frac{\hbar\omega_0}{2}\left[\hat{P}^2+\hat{X}^2-\frac{r(t)}{\omega_0}\hat{X}^2+\frac{\gamma(t)}{\omega_0}(\hat{X}\hat{P}+\hat{P}\hat{X})\right]. \tag{26}$$

Let us note, first of all, that such a master equation is local in time, even if non-Markovian. This feature is typical of all the generalized master equations derived by using the time-convolutionless projection operator technique [13,14] or equivalent approaches such as the superoperatorial one sketched in this section.

Let us have a closer look at the form of our master equation. Equation (26) shows the appearance of two terms, produced by the interaction with the environment, modifying the free Hamiltonian of the system. The first one, proportional to $r(t)$, is a time-dependent renormalization of the frequency of the oscillator while the second one, proportional to $\gamma(t)$, describes a coupling between the the $\hat{X}$ and $\hat{P}$ operators. The perturbation of the free dynamics due to these terms can be easily visualized in phase space, as shown in Fig. 1. Indeed, the effect of the terms proportional to $r(t)$ and $\gamma(t)$ is equivalent to a compression and rotation in phase space of the circle describing the free oscillator dynamics. From a dynamical point of view one can show that the term proportional to $\gamma(t)$ gives rise to both a further frequency renormalization and a dynamical dephasing between position and momentum of the oscillator (see Fig. 1). Note that these features are analogous to the ones present in the dynamics of a classical dissipative oscillator [9].

IV. OPERATORIAL SOLUTION

In quantum mechanics there exists a well established procedure to determine the dynamical evolution of a given closed physical system. It prescribes to write down the Schrödinger equation and calculate eigenstates and eigenvalues of the Hamiltonian of the system. Once we have determined these quantities, it is possible to describe the dynamics evaluating the action of the corresponding temporal evolution operator on the initial state. The dynamics of open systems, however, is much more complicated since the description of the state of the system in terms of a state vector is not sufficient anymore. In order to describe completely the physical system, one needs indeed to introduce the density-matrix operator whose time evolution is governed by a master equation definitely more difficult to handle than the Schrödinger equation.

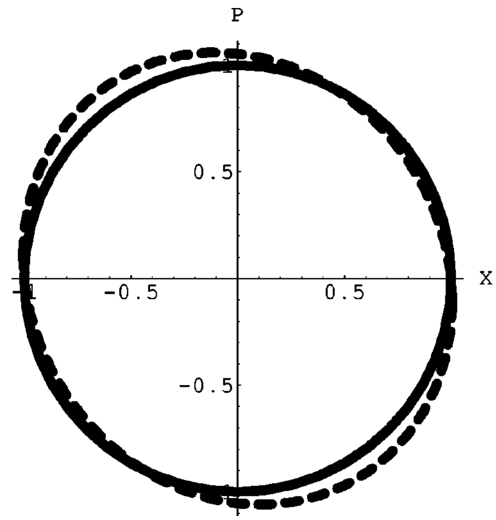

FIG. 1. The plain line represents the unperturbed oscillator. The dashed line represents, instead, an oscillator with a Hamiltonian of the form given in Eq. (26) [we have chosen $\gamma(t)/\omega_0=r(t)/\omega_0=0.1$].

In this section we present a method to solve the generalized master equation, given by Eq. (25), for the system of interest in the paper. In some sense, the method we describe can be seen as a generalization to open systems of the procedure for solving the Schrödinger equation of a closed system, since it is based on the solution of an appropriate eigenvalue equation. Indeed we will deal, in the formalism of superoperator, with generalized eigenvalue equations involving superoperators and operators, instead of operators and vectors, respectively. As we will see in this section, the existence of some useful algebraic properties of the superoperators will help us in treating the problem of the open system dynamics allowing, in particular, to find the operatorial solution for the density matrix of the system.

A remarkable virtue of the procedure we are going to describe is its independence on the expression of the time-dependent coefficients appearing in the master equation (25). This implies that it is applicable to all the master equations presenting the same structure.

Our method for solving the master equation (25) consists of two steps which can be summarized as follows: (1) singling out the temporal evolution superoperator corresponding to Eq. (25) and (2) understanding how this temporal evolution superoperator acts on $\hat{\rho}(0)$.

A. Temporal evolution superoperator

In the preceding section we have introduced the master equation related to the dissipative non-Markovian dynamics of our system:

$$\frac{d\hat{\rho}(t)}{dt}=\left[\frac{1}{i\hbar}\overline{\mathbf{H}}_0^S(t)-\mathbf{D}_S(t)+\gamma(t)(\mathbf{N}+2)\right]\hat{\rho}(t). \tag{27}$$

A formal solution of Eq. (27) can be written as

$$\hat{\rho}(t)=\mathbf{T}(t)\hat{\rho}(0), \tag{28}$$

where the temporal evolution superoperator $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t)=\exp_c\left(\int_0^t\left[\frac{1}{i\hbar}\overline{\mathbf{H}}_0^S(t_1)-\mathbf{D}_S(t_1)+\gamma(t_1)(\mathbf{N}+2)\right]dt_1\right). \tag{29}$$

In Eq. (29) the subscript "c" stands for the Dyson chronological order. In the following, we shall prove that $\mathbf{T}(t)$ can be factorized as follows:

$$\mathbf{T}(t)=\mathbf{T}_{Os}(t)\mathbf{T}_{\Gamma}(t)\mathbf{T}_D(t), \tag{30}$$

where

$$\mathbf{T}_{Os}(t)=\exp_c\left[\frac{1}{i\hbar}\int_0^t\overline{\mathbf{H}}_0^S(t_1)dt_1\right], \tag{31a}$$

$$\mathbf{T}_{\Gamma}(t)=\exp\left[\frac{\Gamma(t)}{2}(\mathbf{N}+2)\right], \tag{31b}$$

$$\mathbf{T}_D(t)=\exp\left[-\int_0^t e^{\Gamma(t_1)}\overline{\mathbf{D}}(t_1)dt_1\right], \tag{31c}$$

with

$$\Gamma(t)=2\int_0^t\gamma(t_1)dt_1, \tag{32}$$

$$\overline{\mathbf{D}}(t)=\mathbf{T}_{Os}^{-1}(t)\mathbf{D}_S(t)\mathbf{T}_{Os}(t). \tag{33}$$

To this aim we exploit the Feynman's rule [25] stating that, whatever the operators or superoperators $A(t)$ and $B(t)$ are,

$$\exp_c\left[\int[A(t)+B(t)]dt\right]=\exp_c\left[\int A(t)dt\right]\exp_c\left[\int\overline{B}(t)dt\right], \tag{34}$$

with

$$\overline{B}(t)=\left(\exp_c\left[\int A(t)dt\right]\right)^{-1}B(t)\left(\exp_c\left[\int A(t)dt\right]\right). \tag{35}$$

Applying the Feynman's rule to the time-evolution superoperator defined in Eq. (29) and taking in consideration that, as shown in Appendix A, $[\mathbf{N},\overline{\mathbf{H}}_0^S(t)]=0$, one gets

$$\mathbf{T}(t)=\mathbf{T}_{Os}(t)\mathbf{T}'(t), \tag{36}$$

where

$$\mathbf{T}'(t)=\exp_c\left[\int_0^t[\gamma(t_1)(\mathbf{N}+2)-\overline{\mathbf{D}}(t_1)]dt_1\right]. \tag{37}$$

Applying again Feynman's rule in Eq. (37) and using the property (see Appendix A) $[\mathbf{N},\overline{\mathbf{D}}(t)]=-2\overline{\mathbf{D}}(t)$, leading to

$$\mathbf{T}_{\Gamma}^{-1}(t)\overline{\mathbf{D}}(t)\mathbf{T}_{\Gamma}(t)=e^{\Gamma(t)}\overline{\mathbf{D}}(t), \tag{38}$$

we have

$$\mathbf{T}'(t)=\mathbf{T}_{\Gamma}(t)\mathbf{T}_D(t). \tag{39}$$

Finally, inserting Eq. (39) into Eq. (36) one obtains the final factorized form of the time-evolution superoperator given by Eq. (30).

The key advantage of such a factorized form is that it makes easier estimating the action of the time-evolution superoperator on the initial density matrix of the system. As we shall see in the following, this allows not only to solve the master equation (25), but also to clarify the physical origin of each of its terms.

It is worth noting that the Dyson chronological order is present in only one of the three superoperators defined by Eqs. (30) and (31). Of course, this circumstance leads to a further simplification of the calculations. In fact, while for $\mathbf{T}_{\Gamma}(t)$ the reason why one can drop the subscript c is already clear from Eq. (31), in the case of $\mathbf{T}_D(t)$ some more comment is needed. As we will demonstrate in the following, the Dyson chronological order is not necessary in the expression of $\mathbf{T}_D(t)=\exp[-\int_0^t e^{\Gamma(t_1)}\overline{\mathbf{D}}(t_1)dt_1]$ because the superoperator $\overline{\mathbf{D}}(t)$, defined in Eq. (33), is a quadratic form in $\vec{\mathbf{Z}}^S$, that is,

$$\overline{\mathbf{D}}(t)=(\vec{\mathbf{Z}}^S)^t\mathsf{A}(t)\vec{\mathbf{Z}}^S, \tag{40}$$

with $\mathsf{A}(t)$ being the 2×2 matrix of time-dependent scalar quantities. Indeed it is straightforward to prove that, when Eq. (40) holds, $[\overline{\mathbf{D}}(t),\overline{\mathbf{D}}(t_1)]=0$ for all t,t_1.

In order to prove Eq. (40), let us note that, as demonstrated in Appendix B, the following chain of equalities holds:

$$\mathbf{T}_{Os}^{-1}(t)\vec{\mathbf{Z}}^S\mathbf{T}_{Os}(t)=(\hat{T}_{Os}^{-1}(t)\hat{\vec{Z}}\hat{T}_{Os}(t))^S=(\mathbf{T}_{Os}^{-1}(t)\hat{\vec{Z}})^S. \tag{41}$$

According to these equations, the transformation operated by $\mathbf{T}_{Os}(t)$ on $\vec{\mathbf{Z}}^S$ is a linear transformation corresponding to the one operated by $\hat{T}_{Os}(t)$ on $\hat{\vec{Z}}$. One can show (see Appendix B) that

$$\hat{T}_{Os}^{-1}(t)\hat{\vec{Z}}\hat{T}_{Os}(t)=\mathsf{R}(t)\hat{\vec{Z}},\quad\text{where } \mathsf{R}(t)=\begin{pmatrix}c(t) & s(t)\\ -s_r(t) & c_r(t)\end{pmatrix}. \tag{42}$$

The functions $c(t)$and $s(t)$ are the solutions of the Cauchy problems

$$\ddot{y}+\omega_0^2\left[1-\frac{r(t)}{\omega_0}-\frac{\gamma^2(t)}{\omega_0^2}-\frac{\dot{\gamma}(t)}{\omega_0^2}\right]y=0,$$

$$\text{with initial conditions}\quad \begin{cases} c(0)=1, & \dot{c}(0)=0,\\ s(0)=0, & \dot{s}(0)=\omega_0. \end{cases} \tag{43}$$

The functions $\dot{s}_r(t)$ and $\dot{c}_r(t)$ are defined as

$$c_r(t)=\frac{\dot{s}(t)}{\omega_0}-\frac{\gamma(t)}{\omega_0}s(t), \quad -s_r(t)=\frac{\dot{c}(t)}{\omega_0}-\frac{\gamma(t)}{\omega_0}c(t). \tag{44}$$

Inserting Eq. (24) into Eq. (33) and using Eqs. (41) one gets

$$\overline{\mathbf{D}}(t)=(\vec{\mathbf{Z}}^S)^t\bar{\mathsf{M}}(t)\vec{\mathbf{Z}}^S,$$

$$\text{where we have put}\quad \bar{\mathsf{M}}(t)=\mathsf{R}^t(t)\mathsf{M}(t)\mathsf{R}(t). \tag{45}$$

These considerations allow us to recast $\mathbf{T}_D(t)$ in the form

$$\mathbf{T}_D(t)=\exp[-(\vec{\mathbf{Z}}^S)^t\mathsf{W}(t)\vec{\mathbf{Z}}^S], \tag{46}$$

where

$$\mathsf{W}(t)=\int_0^t e^{\Gamma(t_1)}\bar{\mathsf{M}}(t_1)dt_1. \tag{47}$$

Summarizing, in this section we have manipulated the expression of the time evolution superoperator in order to put it in a form more convenient for the calculations. In fact we have proved that such a superoperator can be factorized in the form given by Eq. (30). Moreover we have shown that two of the superoperators appearing in Eq. (30) do not need the Dyson chronological order. As we will see in the following section, this circumstance allows us to find an analytic expression for the density-matrix solution.

B. Evolution of the density matrix

In this section, we calculate the action of the factorized superoperator given in Eq. (30) on the initial density matrix.

To this end we express the density matrix in the following form [16–19]:

$$\hat{\rho}=\frac{1}{2\pi}\int \chi(\vec{z})e^{-\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}}}d^2\vec{z}, \tag{48}$$

where the scalar function $\chi(\vec{z})\equiv\chi(x,p)=\mathrm{tr}\{e^{\imath(p\hat{X}-x\hat{P})}\hat{\rho}\}$, known in the literature as *quantum characteristic function* (QCF) [17–20], satisfies the following properties:

$$\chi(0,0)=1,\quad \chi(x,p)=\chi^*(-x,-p),$$
$$|\chi(x,p)|\leq M\ \ (M\in\mathbb{R}). \tag{49}$$

Having in mind Eqs. (28) and (30), and writing the initial density matrix as follows:

$$\hat{\rho}(0)=\frac{1}{2\pi}\int \chi_0(\vec{z}\,')e^{-\imath\vec{z}\,'\mathfrak{J}\hat{\vec{Z}}}d^2\vec{z}\,', \tag{50}$$

with $\chi_0(\vec{z}\,')=\mathrm{tr}\{e^{\imath\vec{z}\,'\mathfrak{J}\hat{\vec{Z}}}\hat{\rho}(0)\}$, we have

$$\rho(t)=\frac{1}{2\pi}\int \mathbf{T}_{O_s}(t)\mathbf{T}_\Gamma(t)\mathbf{T}_D(t)\chi_0(\vec{z}\,')e^{-\imath\vec{z}\,'\mathfrak{J}\hat{\vec{Z}}}d^2\vec{z}\,'. \tag{51}$$

Let us first introduce two superoperator eigenvalue equations we will use in the following. Indicating with $F(\mathbf{A})$ a generic superoperatorial well defined function of the superoperator $\mathbf{A}$, from Eqs. (4), (8), and (9), we obtain

$$F(\vec{\mathbf{Z}}^S)e^{-\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}}}=F(-\vec{z})e^{-\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}}}, \tag{52}$$

$$F(\mathbf{N})(\vec{z}^t\mathsf{J}\hat{\vec{Z}})^n=F(n)(\vec{z}^t\mathsf{J}\hat{\vec{Z}})^n, \tag{53}$$

where now $F(-\vec{z})$ and $F(n)$ are simple scalar functions.

We begin noting that, using Eqs. (46) and (52), one can write the term $\mathbf{T}_D(t)e^{-\imath\vec{z}\,'\mathfrak{J}\hat{\vec{Z}}}$ appearing in Eq. (51) as follows:

$$\mathbf{T}_D(t)e^{-\imath\vec{z}\,'\mathfrak{J}\hat{\vec{Z}}}=e^{-(\vec{z}\,')^t\mathsf{W}(t)\vec{z}\,'}e^{-\imath\vec{z}\,'\mathfrak{J}\hat{\vec{Z}}}. \tag{54}$$

Moreover, using Eq. (53) it is not difficult to prove that

$$\mathbf{T}_\Gamma(t)e^{-\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}}}=e^{\Gamma(t)}\sum_n\frac{[e^{\Gamma(t)/2}(-\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}})]^n}{n!}$$
$$=e^{\Gamma(t)}\exp[e^{\Gamma(t)/2}(-\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}})]. \tag{55}$$

Finally, remembering Eqs. (41) and (42) we get

$$\mathbf{T}_{Os}(t)e^{-\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}}}=e^{-\imath\vec{z}^t\mathsf{J}\mathbf{T}_{Os}(t)\hat{\vec{Z}}}=e^{-\imath\vec{z}^t\mathsf{J}\mathsf{R}^{-1}(t)\hat{\vec{Z}}}$$
$$=e^{-\imath[\mathsf{R}(t)\vec{z}]^t\mathsf{J}\hat{\vec{Z}}}=e^{-\imath\vec{z}(t)^t\mathsf{J}\hat{\vec{Z}}}. \tag{56}$$

Inserting Eqs. (54)–(56) into Eq. (51), we can express the density-matrix solution as follows:

$$\hat{\rho}(t)=\frac{1}{2\pi}\int e^{-(\vec{z}\,')^t\mathsf{W}(t)\vec{z}\,'}\chi_0(\vec{z}\,')e^{\Gamma(t)}$$
$$\times\exp[e^{\Gamma(t)/2}(-\imath\vec{z}\,'(t)^t\mathsf{J}\hat{\vec{Z}})]d^2\vec{z}\,'. \tag{57}$$

After some algebraic manipulation, reported in Appendix C, the previous expression can be recast in the following final form:

$$\hat{\rho}(t)=\frac{1}{2\pi}\int e^{-(\vec{z})^t\bar{\mathsf{W}}(t)\vec{z}}\chi_0[e^{-\Gamma(t)/2}\mathsf{R}^{-1}(t)\vec{z}]e^{-\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}}}d^2\vec{z}$$
$$=\frac{1}{2\pi}\int \chi_t(\vec{z})e^{-\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}}}d^2\vec{z}, \tag{58}$$

where

$$\bar{\mathsf{W}}(t)=e^{-\Gamma(t)}[\mathsf{R}^{-1}(t)]^t\mathsf{W}(t)\mathsf{R}^{-1}(t). \tag{59}$$

Equation (58) constitutes the main result of the paper. It gives the operatorial density-matrix solution of the problem of the dissipative dynamics of a harmonic oscillator interacting with a generic environment satisfying properties (2) and (3) of Sec. II, as for example, a thermal reservoir at temperature T. Our approach to the dynamics of the system relies on the weak-coupling limit but does not invoke the Born-Markov and rotating wave approximations. In more detail, we have solved a non-Markovian generalized master equation for the harmonic oscillator coincident with that deducible using either the time-convolutionless projection operator technique [13] or the superoperatorial approach of Ref. [21].

The exact dissipative dynamics of the reduced density matrix for the damped harmonic oscillator can also be derived in terms of the Wigner function [29] and by means of the path-integral technique [22,23]. Our procedure, however, allows us to obtain an operatorial solution for the density matrix of the system whereas the influence functional method leads to an expression of the density matrix in the coordinate representation. Moreover, as we will see in the following section, once the analytic expression for the QCF is obtained, one can very easily calculate the time evolution of the mean value of a huge class of observables.

In the following section we derive some approximated forms of the solution given by Eq. (58) and we show its usefulness in calculating the analytic expression of many observables of interest and thus in gaining an important insight into the dynamics of one of the most extensively studied physical systems.

V. APPROXIMATED FORMS OF THE SOLUTION AND APPLICATIONS

In this section we derive an approximated form of the operatorial density-matrix solution, given by Eq. (58), valid when

$$\overline{\hat{H}}_0(t) \approx \hat{H}_0, \tag{60}$$

with $\hat{H}_0$ given by Eq. (26). Looking at this equation one sees immediately that Eq. (60) amounts to neglecting the time-dependent frequency renormalization and dephasing terms

$$\frac{r(t)}{\omega_0}, \quad \frac{\gamma^2(t)}{\omega_0^2}. \tag{61}$$

It is worth noting that such an approximation is always justified in the weak-coupling regime $\alpha \ll 1$, provided that the reservoir frequency cutoff remains finite, as one can appreciate with the help of Eqs. (19), (21), and (26). For the same reason it turns out that $\dot{\gamma}(t)/\omega_0 \ll 1$. From Eq. (43), it is not difficult to prove that this last inequality allows us to write the matrix $\mathsf{R}(t)$, defined in Eq. (42), as follows:

$$\mathsf{R}(t) \approx \begin{pmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{pmatrix}. \tag{62}$$

Inserting this expression in the definition of $\overline{\mathsf{W}}(t)$, given by Eq. (59), and exploiting Eqs. (47), (45), and (24) we get

$$\overline{\mathsf{W}}(t) \approx e^{-\Gamma(t)} \int_0^t e^{\Gamma(t_1)} \left[\frac{\overline{\Delta}(t_1)}{2} + \frac{\overline{\Delta}(t_1)}{2} \mathsf{C}_2(t-t_1) - \frac{\Pi(t_1)}{2} \mathsf{S}_2(t-t_1) \right] dt_1, \tag{63}$$

with

$$\mathsf{C}_2(t) = \begin{pmatrix} \cos 2\omega_0 t & -\sin 2\omega_0 t \\ -\sin 2\omega_0 t & -\cos 2\omega_0 t \end{pmatrix}, \quad \mathsf{S}_2(t) = \begin{pmatrix} \sin 2\omega_0 t & \cos 2\omega_0 t \\ \cos 2\omega_0 t & -\sin 2\omega_0 t \end{pmatrix}. \tag{64}$$

The form of the matrices defined in the previous equations suggests a further approximation very common in quantum optics: the rotating wave approximation (RWA). It basically consists in neglecting rapidly oscillating terms as, in our case, those oscillating to frequency $2\omega_0$. This amounts to averaging to zero all the elements of the matrices $\mathsf{C}_2(t)$ and $\mathsf{S}_2(t)$ so that Eq. (63) reduces to

$$\overline{\mathsf{W}}(t) \approx e^{-\Gamma(t)} \int_0^t e^{\Gamma(t_1)} \frac{\overline{\Delta}(t_1)}{2} dt_1 = \frac{\Delta_\Gamma(t)}{2}. \tag{65}$$

Substituting Eq. (65) into Eq. (58) we obtain the following expression for the density-matrix solution in the RWA:

$$\hat{\rho}(t) \approx \hat{\rho}^{\mathrm{RWA}}(t) = \frac{1}{2\pi} \int e^{-[\Delta_\Gamma(t)/2]|\vec{z}|^2} \chi_0[e^{-\Gamma(t)/2} \mathsf{R}^{-1}(t)\vec{z}] e^{-\imath \vec{z}^t \mathsf{J} \hat{\vec{Z}}} d^2\vec{z}. \tag{66}$$

It is possible to demonstrate that $\hat{\rho}^{\mathrm{RWA}}(t)$, as given by Eq. (66), satisfies the following master equation:

$$\frac{d}{dt} \hat{\rho}^{\mathrm{RWA}}(t) = \left[\frac{1}{i\hbar} \mathbf{H}_0^S - \frac{\overline{\Delta}(t)}{2} |\vec{\mathbf{Z}}^S|^2 + \gamma(t)(\mathbf{N}+2) \right] \hat{\rho}^{\mathrm{RWA}}(t), \tag{67}$$

which in turn has been derived in Ref. [21].

Once obtained the density matrix we are able, at least in principle, to evaluate the mean value $\langle A \rangle$ of each and every operator A of interest for the system. One of the advantages of having a solution of the density matrix in terms of the characteristic function is the possibility of exploiting the following relations [20]:

$$\langle \hat{X}^n \rangle = (-\imath)^n \left(\frac{\partial^n}{\partial p^n} \chi(x,p) \right)_{x,p=0}, \quad \langle \hat{P}^n \rangle = (\imath)^n \left(\frac{\partial^n}{\partial x^n} \chi(x,p) \right)_{x,p=0}. \tag{68}$$

By using these relations it is not difficult to calculate the time evolution of the mean energy of the oscillatory system:

$$\langle \hat{H}_0 \rangle_t = e^{-\Gamma(t)} \langle \hat{H}_0 \rangle_{t=0} + \hbar \omega_0 \times \begin{cases} \Delta_\Gamma(t) & \text{with the RWA} \\ \mathrm{tr}\{\bar{\mathsf{W}}(t)\} & \text{without the RWA but neglecting renormalization and dephasing terms.} \end{cases} \tag{69}$$

We emphasize that these solutions depend on the initial state of the oscillator only through the term $\langle \hat{H}_0 \rangle_{t=0}$, the second term of the sum being independent of the initial state. Moreover, looking at Eq. (69) one sees that the exponential factor accounting for energy dissipation does not depend on the initial state of the system but only on the characteristic parameters of the reservoir.

These features, characterizing Eq. (69), are directly related to the factorized form of the the QCF $\chi_t(\vec{z})$. In fact, from both Eqs. (58) and (66) it appears evident that such a function is the product of a Gaussian factor and the function $\chi_0[e^{-\Gamma(t)/2}\mathsf{R}^{-1}(t)\vec{z}]$. All the information on the initial state of the system is incorporated in this last function from which the first term into the right-hand side of Eq. (69) comes from. On the contrary, the second term derives exclusively from the Gaussian factor of the QCF, which depends only on the environment functions and not on the initial state of the system.

Note from Eqs. (58) and (66) that, whatever the initial state of the system is,

$$\chi_t(\vec{z}) \overset{t\to\infty}{\longrightarrow} e^{-(\vec{z})^t \bar{\mathsf{W}}(t)\vec{z}} \simeq (\text{in the RWA}) e^{-[\Delta_\Gamma(t)/2]|\vec{z}|^2}. \tag{70}$$

This behavior is easily understandable, when the environment is a thermal reservoir, in the light of the thermalization process [26].

Let us now have a closer look at Eq. (69). If we evaluate $\mathrm{tr}\{\bar{\mathsf{W}}(t)\}$ with the help of Eq. (63), we find that

$$\mathrm{tr}\{\bar{\mathsf{W}}(t)\} = \Delta_\Gamma(t). \tag{71}$$

This means that the time evolution of the oscillator energy is not affected by the contribution of the rapidly oscillating terms neglected in the RWA. This feature comes directly from the particular structure of the free Hamiltonian operator.

To better understand this point we consider the Gaussian factor appearing in the QCF. The superoperator corresponding to such a factor has the form

$$\mathbf{T}_G(t) = \exp[-(\vec{\mathbf{Z}}^S)^t \bar{\mathsf{W}}(t) \vec{\mathbf{Z}}^S]. \tag{72}$$

With some algebraic manipulation one can recast the superoperator appearing in the exponent as follows:

$$(\vec{\mathbf{Z}}^S)^t \bar{\mathsf{W}}(t) \vec{\mathbf{Z}}^S = \frac{\Delta_\Gamma(t)}{2}[(\mathbf{X}^S)^2 + (\mathbf{P}^S)^2] + \frac{\Lambda(t)}{2}[(\mathbf{X}^S)^2 - (\mathbf{P}^S)^2] + \Theta(t)\mathbf{X}^S\mathbf{P}^S, \tag{73}$$

where the time-dependent coefficients appearing in the previous equation are

$$\Delta_\Gamma(t) = \mathrm{tr}\{\bar{\mathsf{W}}(t)\}, \quad \Lambda(t) = \mathrm{tr}\{\hat{\sigma}_z \bar{\mathsf{W}}(t)\}, \quad \Theta(t) = \mathrm{tr}\{\hat{\sigma}_x \bar{\mathsf{W}}(t)\}, \tag{74}$$

with $\hat{\sigma}_z$ and $\hat{\sigma}_x$ being Pauli spin matrices.

Exploiting the properties of the trace and putting $\hat{\rho}'(t) = \mathbf{T}_G(t)^{-1}\hat{\rho}(t)$, one can easily show that the following chain of equalities holds:

$$\langle \hat{A} \rangle_t = \mathrm{tr}\{\hat{\rho}(t)\hat{A}\} = \mathrm{tr}\{\mathbf{T}_G(t)\hat{\rho}'(t)\hat{A}\} = \mathrm{tr}\{\hat{\rho}'(t)\mathbf{T}_G(t)\hat{A}\} = \mathrm{tr}\{\hat{\rho}'(t)\hat{A}^G(t)\}, \tag{75}$$

where we have defined

$$\hat{A}^G(t) \equiv \mathbf{T}_G(t)\hat{A}. \tag{76}$$

Having these equations in mind it is not difficult to convince oneself that calculating $\langle \hat{A} \rangle_t$ using $\chi_t(x,p)$ is equivalent to calculating $\langle \hat{A}^G \rangle_t$ using $\chi_0[e^{-\Gamma(t)/2}\mathsf{R}^{-1}(t)\vec{z}]$. In the case $\hat{A} = \hat{H}_0 = \frac{1}{2}(\hat{P}^2 + \hat{X}^2)$ we have that

$$[(\mathbf{X}^S)^2 + (\mathbf{P}^S)^2]\hat{H}_0 = -2\hbar\omega_0, \quad [(\mathbf{X}^S)^2 - (\mathbf{P}^S)^2]\hat{H}_0 = 0, \quad \mathbf{X}^S\mathbf{P}^S\hat{H}_0 = 0, \tag{77}$$

and thus

$$\hat{H}_0^G(t) = \hat{H}_0 + \hbar\omega_0\Delta_\Gamma(t). \tag{78}$$

Note that this equation is not affected by the RWA as well as $\chi_0[e^{-\Gamma(t)/2}\mathsf{R}^{-1}(t)\vec{z}]$. This explains why we obtain for the mean value of the oscillator energy the same result with or without the RWA.

The previous procedure suggests a sufficient condition to single out operators which "do not suffer the RWA approximation," indeed we have that

$$[(\mathbf{X}^S)^2 - (\mathbf{P}^S)^2]\hat{A} = 0, \ \ \mathbf{X}^S\mathbf{P}^S\hat{A} = 0 \Rightarrow \langle \hat{A} \rangle_t = \langle \hat{A} \rangle_t^{\mathrm{RWA}}. \tag{79}$$

With the expression operators which do not suffer the RWA approximation we indicate operators having the property that the time evolution of their mean value is not influenced by the RWA. In other words, the counter-rotating terms do not contribute to the dynamics of this class of observables. Examples of operators belonging to such a class are $\hat{X}$, $\hat{P}$, $\hat{X}^2 + \hat{P}^2$ and all linear combinations of such operators, as one can easily verify. On the contrary $\hat{X}^2$ and $\hat{P}^2$ do not satisfy condition (79). For such operators, indeed, exploiting Eq. (75), one gets

$$\langle \hat{X}^2 \rangle_t - \langle \hat{X}^2 \rangle_t^{\mathrm{RWA}} = -\Lambda(t), \quad \langle \hat{P}^2 \rangle_t - \langle \hat{P}^2 \rangle_t^{\mathrm{RWA}} = \Lambda(t). \tag{80a}$$

Similarly, for the mean value of the "correlation" operator $(\hat{X}\hat{P}+\hat{P}\hat{X})$ one has

$$\langle \hat{X}\hat{P}+\hat{P}\hat{X} \rangle_t - \langle \hat{X}\hat{P}+\hat{P}\hat{X} \rangle_t^{\mathrm{RWA}} = 2\Theta(t). \tag{80b}$$

VI. CONCLUSIONS

In this paper, we have developed a method to solve the weak-coupling generalized master equation for the QBM. In particular, we have considered the case in which the system interacting with the environment is a quantum harmonic oscillator. The master equation we have solved, given by Eq. (25), although non-Markovian is local in time. Such a master equation has been derived in Ref. [21] by using a superoperatorial technique and in Refs. [9,14] by means of the time-convolutionless method and is the weak-coupling approximated form of the exact master equation for the QBM derived by Paz and Zhang in Ref. [27]. The main result of the paper is the derivation of the analytic solution of Eq. (25) for the density matrix of the reduced system. To this aim, we have used an approach based on the algebra of superoperators. Our method is independent of the specific form of the environment and does not rely on any approximation apart from the weak-coupling one. We have also studied simpler forms of the density-matrix solution obtained by neglecting frequency renormalization terms and/or performing the RWA. We have demonstrated the existence of a class of superoperators whose mean value is not affected by the presence of the counter-rotating terms at any time t and we have given a sufficient condition to verify if a given operator belongs to such a class. This circumstance simplifies substantially the calculations, since, for operators belonging to such a class, one can use the approximated density-matrix solution, given by Eq. (66), in order to calculate their mean value. The analytic solution we derive and discuss in the paper is given in terms of the QCF [see Eqs. (58)] by means of which one can calculate the expectation value of many observables of physical interest in a very direct way, as suggested by Eqs. (68). For example, thanks to the simplicity of the analytic solution we have derived, we succeed in calculating the dissipative time evolution of the mean energy of the system. Finally, it is worth noting that from the QCF it is easy to derive the Wigner function characterizing the state of the dissipative system.

Concluding, we believe that the non-Markovian analytic approach we have derived in this paper for the QBM can be generalized to other fundamental dissipative systems, such as, for example, the Jaynes-Cummings model with losses. As for the QBM, we think that the analytic solution of the density matrix may be used for studying important aspects of such a basic model, both for fundamental and for applicative research, under conditions in which, up to now, only numerical approaches were possible.

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APPENDIX A: PROPERTIES OF THE DISSIPATION SUPEROPERATOR

In this appendix, we demonstrate some useful properties of the superoperator $\mathbf{N}$. Such properties, allowing us to factorize the time-evolution superoperator given in Eq. (29), are the following:

$$[\mathbf{N},\overline{\mathbf{H}}_0^S(t)]=0, \;\; [\mathbf{N},\overline{\mathbf{D}}(t)]=-2\overline{\mathbf{D}}(t). \tag{A1}$$

The first one is the most surprising one: it can be considered as a "dynamical invariance" of the dissipation process. As we shall see in the following, the dynamical invariance originates from the particular structure of the Hamiltonian term $\hat{\overline{H}}_0(t)$, defined by Eq. (26).

Independently on the form of the time-dependent functions present in such a term, it can be written, in general, as

$$\hat{\overline{H}}_0(t)=\hat{\vec{Z}}^t \mathsf{S}(t)\hat{\vec{Z}} \quad \text{with} \quad \mathsf{S}^t(t)=\mathsf{S}(t), \tag{A2}$$

where $\mathsf{S}(t)$ is a 2×2 real symmetric matrix. Remembering that

$$\mathbf{N}=-\frac{1}{2}\imath(\vec{\mathbf{Z}}^\Sigma)^t \mathsf{J}\vec{\mathbf{X}}^S, \tag{A3}$$

one gets

$$\begin{aligned}[\mathbf{N},\overline{\mathbf{H}}_0^S(t)] &= -\frac{1}{2}\imath[(\vec{\mathbf{Z}}^\Sigma)^t \mathsf{J}\vec{\mathbf{Z}}^S,(\hat{\vec{Z}}^t \mathsf{S}(t)\hat{\vec{Z}})^S] \\ &= \frac{1}{2}\imath\{(\vec{\mathbf{Z}}^\Sigma)^t \mathsf{J}[\hat{\vec{Z}}^t \mathsf{S}(t)\hat{\vec{Z}},\hat{\vec{Z}}]^S \\ &\quad +([\hat{\vec{Z}}^t \mathsf{S}(t)\hat{\vec{Z}},\hat{\vec{Z}}]^\Sigma)^t \mathsf{J}\vec{\mathbf{Z}}^S\}. \end{aligned} \tag{A4a}$$

It is possible to show that [28]

$$[\hat{\vec{Z}}^t \mathsf{S}(t)\hat{\vec{Z}},\hat{\vec{Z}}]=\imath \mathsf{J}\mathsf{S}(t)\hat{\vec{Z}}. \tag{A4b}$$

Substituting this equation into Eq. (A4a) and using the properties of the matrices $\mathsf{S}(t)$ and J [cf. Eq. (13)] one obtains

$$\begin{aligned}[\mathbf{N},\overline{\mathbf{H}}_0^S(t)] &= -\frac{1}{2}[(\vec{\mathbf{Z}}^\Sigma)^t \mathsf{J}\mathsf{J}\mathsf{S}(t)\vec{\mathbf{Z}}^S+(\vec{\mathbf{Z}}^\Sigma)^t \mathsf{S}^t(t)\mathsf{J}^t \mathsf{J}\vec{\mathbf{Z}}^S] \\ &= -\frac{1}{2}[-(\vec{\mathbf{Z}}^\Sigma)^t \mathsf{S}(t)\vec{\mathbf{Z}}^S+(\vec{\mathbf{Z}}^\Sigma)^t \mathsf{S}(t)\vec{\mathbf{Z}}^S]=0. \end{aligned} \tag{A4c}$$

The generality of the matrix $\mathsf{S}(t)$ ensures the validity of this relation for a generic quadratic Hamiltonian and thus, in particular, for $\hat{\hat{H}}_0(t)$.

As far as the second commutation relation is concerned, remembering that $\overline{\mathbf{D}}(t)=(\vec{\mathbf{Z}}^S)^t\overline{\mathsf{M}}(t)\vec{\mathbf{Z}}^S$ one has

$$[\mathbf{N},\overline{\mathbf{D}}(t)]=[\mathbf{N},(\vec{\mathbf{Z}}^S)^t\overline{\mathsf{M}}(t)\vec{\mathbf{Z}}^S]$$
$$=[\mathbf{N},\vec{\mathbf{Z}}^S]^t\overline{\mathsf{M}}(t)\vec{\mathbf{Z}}^S+(\vec{\mathbf{Z}}^S)^t\overline{\mathsf{M}}(t)[\mathbf{N},\vec{\mathbf{Z}}^S]. \tag{A5a}$$

Exploiting the superoperatorial commutation rules given in Eq. (3) yields

$$[\mathbf{N},\vec{\mathbf{Z}}^S]=-\frac{\imath}{2}[(\mathbf{P}^\Sigma\mathbf{X}^S-\mathbf{X}^\Sigma\mathbf{P}^S),\vec{\mathbf{Z}}^S]=-\vec{\mathbf{Z}}^S. \tag{A5b}$$

Finally, substituting into Eq. (A5a) one gets

$$[\mathbf{N},\overline{\mathbf{D}}(t)]=-(\vec{\mathbf{Z}}^S)^t\overline{\mathsf{M}}(t)\vec{\mathbf{Z}}^S-(\vec{\mathbf{Z}}^S)^t\overline{\mathsf{M}}(t)\vec{\mathbf{Z}}^S=-2\overline{\mathbf{D}}(t). \tag{A5c}$$

APPENDIX B: TIME EVOLUTION

In this appendix, we shall demonstrate that the general time evolution of a "S- or "Σ"-type superoperator is equivalent to the "S"- or "Σ"-type superoperator of the time evolution of the corresponding operator. In formulas this amounts to demonstrating Eq. (41). First of all let us define $T_c(t)$ as solution of the equation

$$\frac{d}{dt}T_c(t)=L(t)T_c(t), \tag{B1}$$

where in general $[L(t),L(t_1)]\neq 0$. $L(t)$, as well as $T_c(t)$, can be either an operator or a superoperator.

The previous equation can be solved in an iterative way and its solution is

$$T_c(t)=\sum_{n=0}\int_0^t\cdots\int_0^{t_{n-1}}L(t_1)\cdots L(t_n)dt_1\cdots dt_n$$
$$\equiv\exp_c\left[\int_0^t L(t_1)dt_1\right]. \tag{B2}$$

It can be shown that $T_c^{-1}(t)$ satisfies the equation [25]

$$\frac{d}{dt}T_c^{-1}(t)=-T_c^{-1}(t)L(t), \tag{B3}$$

which can be again solved in an iterative way giving the following form of the solution:

$$T_c^{-1}(t)=\sum_{n=0}(-1)^n\int_0^t\cdots\int_0^{t_{n-1}}L(t_n)\cdots L(t_1)dt_1\cdots dt_n$$
$$\equiv\exp_a\left[-\int_0^t L(t_1)dt_1\right]. \tag{B4}$$

Let us define

$$A_c(t)=T_c(t)AT_c^{-1}(t)\equiv\mathcal{T}_c(t)A. \tag{B5}$$

From the previous equation we have

$$\frac{d}{dt}\mathcal{T}_c(t)A=[L(t),A_c(t)]=L^S(t)A_c(t)=L^S(t)\mathcal{T}_c(t)A$$

$$\Rightarrow\frac{d}{dt}\mathcal{T}_c(t)=L^S(t)\mathcal{T}_c(t). \tag{B6}$$

This means that $\mathcal{T}_c(t)$ satisfies an equation similar to the one given by Eq. (B1). The form of $\mathcal{T}_c(t)$ can be obtained from Eq. (B2) by replacing $T_c(t)\rightarrow\mathcal{T}_c(t)$ and $L(t)\rightarrow L^S(t)$.

Now, let us suppose that A is S- or Σ-type superoperator and L is an S-type superoperator. From the properties given in Eq. (3) we have that

$$\mathbf{A}_c^{S(\Sigma)}(t)=\mathbf{T}_c(t)\mathbf{A}^{S(\Sigma)}\mathbf{T}_c^{-1}(t)=\mathcal{T}_c(t)\mathbf{A}^{S(\Sigma)}$$
$$=\sum_{n=0}\int_0^t\cdots\int_0^{t_{n-1}}[\mathbf{L}^S(t_1)\cdots[\mathbf{L}^S(t_n),\mathbf{A}^{S(\Sigma)}]\cdots]$$
$$\times dt_1\cdots dt_n$$
$$=\sum_{n=0}\int_0^t\cdots\int_0^{t_{n-1}}[\hat{L}(t_1)\cdots[\hat{L}(t_n),\hat{A}]\cdots]^{S(\Sigma)}$$
$$\times dt_1\cdots dt_n$$
$$=\left(\sum_{n=0}\int_0^t\cdots\int_0^{t_{n-1}}\mathbf{L}^S(t_1)\cdots\mathbf{L}^S(t_n)\right.$$
$$\left.\times\hat{A}dt_1\cdots dt_n\right)^{S(\Sigma)}$$
$$=\left(\exp_c\left[\int_0^t\mathbf{L}^S(t_1)dt_1\right]\hat{A}\right)^{S(\Sigma)}$$
$$=(\mathbf{T}_c(t)\hat{A})^{S(\Sigma)}$$
$$=(\hat{T}_c(t)\hat{A}\hat{T}_c^{-1}(t))^{S(\Sigma)}=(\hat{A}_c(t))^{S(\Sigma)}, \tag{B7}$$

where we have put $\hat{T}_c(t)\equiv\exp_c[\int_0^t\hat{L}^S(t_1)dt_1]$.

Now, let us consider

$$\mathbf{A}_a^{S(\Sigma)}(t)=\mathbf{T}_c^{-1}(t)\mathbf{A}^{S(\Sigma)}\mathbf{T}_c(t)\equiv\mathcal{T}_c^{-1}(t)\mathbf{A}^{S(\Sigma)}. \tag{B8}$$

The last definition comes directly from the definition of $\mathcal{T}_c(t)$ and can be verified applying $\mathcal{T}_c(t)$ on $\mathbf{A}_a^S(t)$ and, vice versa, applying the previous definition of $\mathcal{T}_c^{-1}(t)$ on $\mathbf{A}_c^S(t)$.

Exploiting Eq. (B4) and following the same lines of the derivation of Eq. (B7), one can show that

$$\mathbf{A}_a^{S(\Sigma)}(t)=(\hat{T}_c^{-1}(t)\hat{A}\hat{T}_c(t))^{S(\Sigma)}=(\hat{A}_a(t))^{S(\Sigma)}. \tag{B9}$$

Let us now derive Eqs. (42). Using Eq. (A4b), (B1), (B3) and remembering the definition of $\hat{\hat{Z}}(t)$,

$$\hat{\hat{Z}}(t)=\hat{T}_{Os}^{-1}(t)\hat{\hat{Z}}\hat{T}_{Os}(t), \tag{B10}$$

one has

$$\frac{d}{dt}\hat{\hat{Z}}(t)=-\frac{1}{\imath\hbar}\hat{T}_{Os}^{-1}(t)[\hat{\tilde{H}}_0(t),\hat{\hat{Z}}]\hat{T}_{Os}(t)=-\mathsf{J}\tilde{\mathsf{S}}(t)\hat{\hat{Z}}(t),$$

$$\tilde{\mathsf{S}}(t)=\frac{1}{\hbar}\mathsf{S}(t). \tag{B11}$$

Deriving the previous equation once more we get

$$\frac{d^2}{dt^2}\hat{\hat{Z}}(t)=\{-\mathsf{J}\dot{\tilde{\mathsf{S}}}(t)+[\mathsf{J}\tilde{\mathsf{S}}(t)]^2\}\hat{\hat{Z}}(t)$$

$$=\{-\mathsf{J}\dot{\tilde{\mathsf{S}}}(t)-\det[\tilde{\mathsf{S}}(t)]\}\hat{\hat{Z}}(t). \tag{B12}$$

In our case

$$\tilde{\mathsf{S}}(t)=\frac{1}{2}\begin{pmatrix}\omega_0-r(t) & \gamma(t)\\ \gamma(t) & \omega_0\end{pmatrix}, \tag{B13}$$

so that the differential equation for the operator $\hat{X}(t)$ has the following form:

$$\frac{d^2}{dt^2}\hat{X}(t)=-\omega_0^2\left(1-\frac{r(t)}{\omega_0}-\frac{\gamma^2(t)}{\omega_0^2}-\frac{\dot{\gamma}(t)}{\omega_0^2}\right)\hat{X}(t). \tag{B14}$$

A solution of the previous equation can be written as

$$\hat{X}(t)=\hat{X}c(t)+\hat{P}s(t), \tag{B15}$$

with $c(t)$ and $s(t)$ solutions of Eq. (43) with the same initial conditions. Note that, from Eq. (B11), it follows that

$$\frac{d}{dt}\hat{X}(t)=\omega_0\hat{P}(t)+\gamma(t)\hat{X}(t). \tag{B16}$$

With the help of Eq. (B15) we have for $\hat{P}(t)$,

$$\hat{P}(t)=\frac{1}{\omega_0}\left(\frac{d}{dt}\hat{X}(t)-\gamma(t)\hat{X}(t)\right)$$

$$=\hat{P}\frac{\dot{s}(t)-\gamma(t)s(t)}{\omega_0}+\hat{X}\frac{\dot{c}(t)-\gamma(t)s(t)}{\omega_0}$$

$$=\hat{P}c_r(t)-\hat{X}s_r(t), \tag{B17}$$

where we have defined, as in Eq. (44),

$$c_r(t)=\frac{\dot{c}(t)-\gamma(t)s(t)}{\omega_0},\quad -s_r(t)=\frac{\dot{c}(t)-\gamma(t)s(t)}{\omega_0}. \tag{B18}$$

Grouping Eqs. (B15) and (B17) and using the matrix representation, we obtain

$$\begin{matrix}\hat{X}(t)=\hat{X}c(t)+\hat{P}s(t)\\ \hat{P}(t)=-\hat{X}s_r(t)+\hat{P}c_r(t)\end{matrix}\quad\Rightarrow\hat{\hat{Z}}(t)=\mathsf{R}(t),\quad \mathsf{R}(t)=\begin{pmatrix}c(t) & s(t)\\ -s_r(t) & c_r(t)\end{pmatrix}. \tag{B19}$$

APPENDIX C: FINAL FORM OF THE DENSITY MATRIX

In this appendix, we sketch the main steps of the derivation of the final form of the density-matrix solution, given by Eq. (58), from Eq. (57). Let us remind the generic expression for the density matrix in terms of the QCF [See Eq. (48)]

$$\hat{\rho}(t)=\frac{1}{2\pi}\int\chi_t(\vec{z})e^{-\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}}}d^2\vec{z}, \tag{C1}$$

where $\chi_t(\vec{z})=\mathrm{tr}\{e^{\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}}}\hat{\rho}(t)\}$. From Eq. (57) we have

$$\hat{\rho}(t)=\frac{1}{2\pi}\int e^{-(\vec{z}')^t\mathsf{W}(t)\vec{z}'}\chi_0(\vec{z}')e^{\Gamma(t)}$$

$$\times\exp\{e^{\Gamma(t)/2}[-\imath\vec{z}'(t)^t\mathsf{J}\hat{\hat{\vec{Z}}}]\}d^2\vec{z}'. \tag{C2}$$

Comparing these last two equations one gets

$$\chi_t(\vec{z})=\mathrm{tr}\{e^{\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}}}\hat{\rho}(t)\}=\frac{1}{2\pi}\int e^{-(\vec{z}')^t\mathsf{W}(t)\vec{z}'}\chi_0(\vec{z}')e^{\Gamma(t)}$$

$$\times\mathrm{tr}\{e^{\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}}}\exp[e^{\Gamma(t)/2}(-\imath\vec{z}'(t)^t\mathsf{J}\hat{\hat{\vec{Z}}})]\}d^2\vec{z}'. \tag{C3}$$

Using some properties of the δ function we obtain

$$\frac{1}{2\pi}\mathrm{tr}\{e^{\imath\vec{z}'^t\mathsf{J}\hat{\vec{Z}}}\exp[e^{\Gamma(t)/2}(-\imath\vec{z}(t)^t\mathsf{J}\hat{\hat{\vec{Z}}})]\}$$

$$=\delta[\vec{z}-e^{\Gamma(t)/2}\vec{z}'(t)]=e^{-\Gamma(t)}\delta[\vec{z}'-e^{-\Gamma(t)/2}\mathsf{R}^{-1}(t)\vec{z}]. \tag{C4}$$

Introducing this equation into Eq. (C3) yields

$$\chi_t(\vec{z})=\int e^{-(\vec{z}')^t\mathsf{W}(t)\vec{z}'}\chi_0(\vec{z}')\,\delta[\vec{z}'-e^{-\Gamma(t)/2}\mathsf{R}^{-1}(t)\vec{z}]d^2\vec{z}'$$

$$=\exp[-e^{-\Gamma(t)}(\vec{z})^t(\mathsf{R}^{-1}(t))^t\mathsf{W}(t)\mathsf{R}^{-1}(t)\vec{z}]$$

$$\times\chi_0[e^{-\Gamma(t)/2}\mathsf{R}^{-1}(t)\vec{z}]. \tag{C5}$$

Finally, substituting this equation into Eq. (C1) gives

$$\hat{\rho}(t)=\frac{1}{2\pi}\int e^{-(\vec{z})^t\bar{\mathsf{W}}(t)\vec{z}}\chi_0[e^{-\Gamma(t)/2}\mathsf{R}^{-1}(t)\vec{z}]e^{-\imath\vec{z}^t\mathsf{J}\hat{\vec{Z}}}d^2\vec{z}, \tag{C6}$$

where

$$\bar{\mathsf{W}}(t)=e^{-\Gamma(t)}(\mathsf{R}^{-1}(t))^t\mathsf{W}(t)\mathsf{R}^{-1}(t), \tag{C7}$$

which is the form of $\hat{\rho}(t)$ given in Eq. (58).

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