Quantum limits on noise in dual input-output linear optical amplifiers and attenuators

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The input-output relations for linear amplifiers and attenuators that have two input and two output channels are used to derive inequalities that relate their gain profiles and output noise spectra. The results generalize earlier derivations, which mainly focus their attention on single-channel devices, to the two-ended amplifiers and attenuators often used in practical communications systems. The present inequalities are satisfied by the results of previous calculations for specific model systems. It is shown that; in contrast to single-channel devices, a two-ended system can act as an amplifier for some input signals and an attenuator for others, even when all the signal frequencies are the same. The output from the two-channel amplifier has a minimum noise determined by the sum of the gains for both input channels, even when only one input channel is used and the other is in its vacuum state. The conditions on device construction needed to achieve equal gains for signals that arrive at the two ends of the device are determined. The present results reduce to those of single-channel theory in special cases where the two output channels are each separately fed by only one of the two input channels.

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I. INTRODUCTION

The quantum limits on noise in linear amplifiers were determined by Caves [1] in a seminal paper that consolidated and extended a great deal of previous work. The results apply very generally to amplifiers that are either sensitive or insensitive to the phase of the input signal and have phase-preserving or phase-conjugating characteristics. Separate considerations are given to narrow-band amplifiers, for which single-mode descriptions of the input and output are adequate, and to devices for which multimode theories are needed. The explicit treatment is focused on systems with single input and output channels.

Many amplifiers and attenuators used in practice have linear spatial structures in which signals may be incident on both the left- and the right-hand ends of the device and whose outputs likewise emerge from both ends. A familiar example is the communications link, where the attenuating optical fiber often has amplifying sections inserted periodically along the line. The Caves theory applies in principle to such devices but the results are implicit. The main aim of the present paper is a more explicit account of the two-ended amplifier or attenuator. The generic device is described by linear relations between pairs of input and output signal operators, which also include a pair of noise operators associated with the amplification or attenuation process. The results apply generally to wide ranges of attenuating and amplifying systems.

In Sec. II, the boson commutation properties of the signal operators are used in conjunction with the input-output relations of the device to derive connections between the noise-operator commutators and the four distinct signal gains that occur in reflection and transmission at the two inputs. These are used in Sec. III to obtain minimum values for the noise power spectra that must be added by the device to the amplified or attenuated signal. It is shown in Secs. II and III that the same device may show gain, loss, or passive character at the same frequency depending on the observed output channel and on the forms of signal in the input channels. This feature is not revealed by previous implicit treatments. The conditions for equality of the pairs of reflection and transmission gains are discussed in Sec. IV. Section V treats devices that may be adequately described by models with single input and output channels, where the theory of the dual input-output device reduces to that derived previously [1]. Contact is made with the effective beam-splitter representation of the device. The conclusions of the work are summarized in Sec. VI. Our calculations use a continuous-mode description of the field throughout and the results can therefore be applied to the propagation of traveling-wave optical pulses.

II. INPUT-OUTPUT RELATIONS

The lossy dielectric plate and the lossy beam splitter are examples of the dual input-output linear attenuator that have been considered previously [2–5]. Results have also been derived for the quantum-state transformations by dispersive and absorbing four-port devices [6]. The input-output relations for an amplifying slab [7,8] and the entanglement transformations by absorbing and amplifying four-port devices [9] have been considered. The one-dimensional quantization schemes used in these calculations have been extended to three-dimensional systems [10–12], but the theory presented here is restricted to quasi-one-dimensional devices.

Figure 1 shows the configuration of inputs and outputs for the generic device, with the notations for the mode destruction and noise operators. The shaded rectangle represents a linear attenuator or amplifier, or any arbitrary multilayer succession of such elements. The input and output light beams

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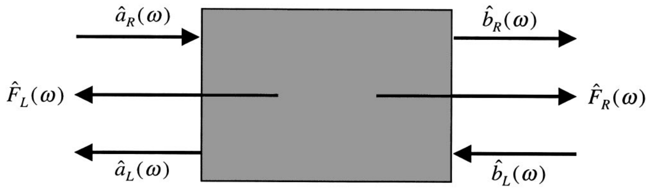

FIG. 1. Configuration of the two inputs and two outputs of an amplifier or attenuator, showing notations for the four input-output mode destruction operators and the two noise operators.

are assumed to propagate in the same material on either side of the device. The corresponding input-output relations for a phase-insensitive, phase-preserving amplifier or attenuator have the forms

$$\hat{a}_L(\omega)=R_{LR}(\omega)\hat{a}_R(\omega)+T_{LL}(\omega)\hat{b}_L(\omega)+\hat{F}_L(\omega) \quad (2.1)$$

and

$$\hat{b}_R(\omega)=T_{RR}(\omega)\hat{a}_R(\omega)+R_{RL}(\omega)\hat{b}_L(\omega)+\hat{F}_R(\omega). \quad (2.2)$$

The terms that include R or T functions describe the frequency-dependent reflection and transmission of the incident signals from and through the amplifier or attenuator. The $\hat{F}$ operators describe the noise added to the input signals by the amplifier or attenuator; the additional noise is associated with the gain or loss of energy by the signal beams as they propagate through the device.

With two inputs and two outputs, four distinct gains can in principle be measured. Thus the reflection gain profiles are defined by

$$G_{LR}(\omega)=|R_{LR}(\omega)|^2 \quad \text{and} \quad G_{RL}(\omega)=|R_{RL}(\omega)|^2 \quad (2.3)$$

and the transmission gains are defined by

$$G_{LL}(\omega)=|T_{LL}(\omega)|^2 \quad \text{and} \quad G_{RR}(\omega)=|T_{RR}(\omega)|^2. \quad (2.4)$$

The term "gain" is used for compactness even when G is less than unity and attenuation occurs.

The detailed functional forms of the coefficients R, T, and of the $\hat{F}$ operators are determined by calculations based on models of specific devices but they are restricted by the commutation properties of the input and output operators. Thus the input operators satisfy the usual continuous-mode commutation relations for independent boson operators

$$[\hat{a}_R(\omega),\hat{a}_R^\dagger(\omega')]=[\hat{b}_L(\omega),b_L^\dagger(\omega')]=\delta(\omega-\omega') \quad \text{and}$$
$$[\hat{a}_R(\omega),b_L^\dagger(\omega')]=0 \quad (2.5)$$

and they also commute with the noise operators,

$$[\hat{a}_R(\omega),\hat{F}_L^\dagger(\omega')]=[\hat{b}_L(\omega),\hat{F}_L^\dagger(\omega')]=[\hat{a}_R(\omega),\hat{F}_R^\dagger(\omega')]$$
$$=[\hat{b}_L(\omega),\hat{F}_R^\dagger(\omega')]=0. \quad (2.6)$$

However, the left- and right-hand noise operators do not commute, as they result from the same noise sources inside the device. The forms of the R and T coefficients must be such as to ensure causality in the relations between outputs and inputs [7,13].

The two output operators are also required to satisfy independent-boson commutation relations, so that

$$[\hat{a}_L(\omega),\hat{a}_L^\dagger(\omega')]=[\hat{b}_R(\omega),\hat{b}_R^\dagger(\omega')]=\delta(\omega-\omega') \quad \text{and}$$
$$[\hat{a}_L(\omega),\hat{b}_R^\dagger(\omega')]=0. \quad (2.7)$$

Substitution of the expressions from Eqs. (2.1) and (2.2) gives the relations

$$[\hat{F}_L(\omega),\hat{F}_L^\dagger(\omega')]=\{1-G_{LR}(\omega)-G_{LL}(\omega)\}\delta(\omega-\omega') \quad (2.8)$$

and

$$[\hat{F}_R(\omega),\hat{F}_R^\dagger(\omega')]=\{1-G_{RR}(\omega)-G_{RL}(\omega)\}\delta(\omega-\omega'), \quad (2.9)$$

where Eqs. (2.3)–(2.6) have been used. It follows that the noise operators $\hat{F}_L(\omega)$ and $\hat{F}_R(\omega)$ have the characters of destruction operators when the sum of the relevant reflection and transmission gains is less than unity and the device acts as an overall attenuator. The noise operators have the characters of creation operators when the sum of the gains is greater than unity and the device acts as an overall amplifier. The final relation in Eq. (2.7) leads similarly to

$$[\hat{F}_L(\omega),\hat{F}_R^\dagger(\omega')]=-\{R_{LR}(\omega)T_{RR}^*(\omega)$$
$$+T_{LL}(\omega)R_{RL}^*(\omega)\}\delta(\omega-\omega'). \quad (2.10)$$

The relations (2.8)–(2.10) reduce to forms derived previously [4,5] when the two reflection coefficients and the two transmission coefficients are the same functions of ω, as in spatially symmetric amplifiers or attenuators, whose structures have reflection symmetry planes at their centers. The noise operator commutators are verified with the explicit expressions derived for a symmetric amplifying or attenuating dielectric slab [8].

The noise operators are allowed to vanish only for a passive device that neither adds nor subtracts energy to or from the propagating light beams. The quantities in the curly brackets in Eqs. (2.8)–(2.10) then all vanish. The resulting relations

$$G_{LR}(\omega)+G_{LL}(\omega)=G_{RR}(\omega)+G_{RL}(\omega)=1 \quad (2.11)$$

and

$$R_{LR}(\omega)T_{RR}^*(\omega)+T_{LL}(\omega)R_{RL}^*(\omega)=0 \quad (2.12)$$

are the same as those ordinarily found for a lossless slab or beam splitter [3,14]. They lead to the conditions

$$|R_{LR}(\omega)|=|R_{RL}(\omega)|, \quad |T_{LL}(\omega)|=|T_{RR}(\omega)|, \quad \text{and}$$

$$\phi_{LR}+\phi_{RL}-\phi_{LL}-\phi_{RR}=\pm\pi, \tag{2.13}$$

where the ϕ are the phase angles of the reflection and transmission coefficients in an obvious notation. The reflection and transmission gains defined in Eqs. (2.3) and (2.4) are thus related by

$$G_{LR}(\omega)=G_{RL}(\omega) \quad \text{and} \quad G_{LL}(\omega)=G_{RR}(\omega). \tag{2.14}$$

These same relations for a passive device can also be derived from the condition for energy conservation for incoming and outgoing light beams at frequency ω, in the form

$$a_L^\dagger(\omega)\hat{a}_L(\omega)+\hat{b}_R^\dagger(\omega)\hat{b}_R(\omega)=\hat{a}_R^\dagger(\omega)\hat{a}_R(\omega)+\hat{b}_L^\dagger(\omega)\hat{b}_L(\omega). \tag{2.15}$$

Substitution of Eqs. (2.1) and (2.2) with the noise operators removed on the left-hand side then leads to the relations

$$G_{LR}(\omega)+G_{RR}(\omega)=G_{LL}(\omega)+G_{RL}(\omega)=1 \tag{2.16}$$

and

$$R_{LR}(\omega)T_{LL}^*(\omega)+T_{RR}(\omega)R_{RL}^*(\omega)=0. \tag{2.17}$$

These differ slightly from Eqs. (2.11) and (2.12) but it is readily verified that they are equivalent and they lead to the same conditions as in Eq. (2.13). It is emphasized that the equalities in Eqs. (2.13) and (2.14) are valid, in general, for passive devices with arbitrary structures. They also hold for the spatially symmetric attenuators or amplifiers mentioned above.

The energy conservation relation (2.15) is no longer valid in the presence of loss or gain. For a specific pair of input signals, the device acts as an overall amplifier when the expectation value of the left-hand side of Eq. (2.15) exceeds that of the right-hand side and as an attenuator when the expectation value of the right-hand side exceeds that of the left-hand side. We adopt these criteria as the definitions of the amplifying or attenuating character of the device and we emphasize that the distinction depends, in principle, not only on the device construction but also on the nature of the input signals. It is instructive to consider the common practical system with coherent signals in both input arms of the device. We therefore take expectation values of the input and output energies for coherent-state signals $|\alpha_R(\omega)\rangle$ and $|\beta_L(\omega)\rangle$ [14]. The input-output relations hold for all values of the input amplitudes and we consider a classical limit where the input signals are sufficiently intense that the attenuated or amplified signal components in the outputs greatly exceed the noise contributions. Substitution of Eqs. (2.1) and (2.2) in the output energy on the left-hand side of Eq. (2.15), with neglect of the noise operators, then leads to the inequalities

$$|R_{LR}\alpha_R+T_{LL}\beta_L|^2+|T_{RR}\alpha_R+R_{RL}\beta_L|^2 \lesseqgtr |\alpha_R|^2+|\beta_L|^2, \tag{2.18}$$

where the ω dependences are omitted for simplicity and the upper and lower symbols refer to the attenuator and the amplifier, respectively. The special cases of these relations for $\beta_L=0$ and $\alpha_R=0$ then lead to the inequalities

$$G_{LR}(\omega)+G_{RR}(\omega)\lesseqgtr 1 \quad \text{and} \quad G_{LL}(\omega)+G_{RL}(\omega)\lesseqgtr 1, \tag{2.19}$$

which must be satisfied for net overall attenuation or amplification of the single input signals. Another special case is that of equal-amplitude signals with $|\alpha_R|=|\beta_L|$ in the two inputs, when Eq. (2.18) leads to

$$|R_{LR}(\omega)T_{LL}^*(\omega)+T_{RR}(\omega)R_{RL}^*(\omega)| < \left|1-\tfrac{1}{2}\{G_{LR}(\omega)+G_{RR}(\omega)+G_{LL}(\omega)+G_{RL}(\omega)\}\right|, \tag{2.20}$$

which is valid both for joint attenuation and amplification of the two coherent inputs.

We should note that there exist devices for which a particular superposition of coherent input signals experiences net gain but the orthogonal superposition experiences net loss. In this case, the different values of α_R and β_L corresponding to these superpositions satisfy Eq. (2.18) with different inequality signs. A simple example is provided by the special case in which both reflection coefficients equal R, both transmission coefficients equal T, and α_R and β_L are both real. The device then amplifies the input with $\alpha_R=\beta_L$ and attenuates the input with $\alpha_R=-\beta_L$ if

$$|R+T|^2>1 \quad \text{and} \quad |R-T|^2<1. \tag{2.21}$$

Values of R and T that satisfy these conditions are, of course, readily found. The device may also show passive characteristics for some superpositions, with equality of the two sides of Eq. (2.18), and an example of such hybrid behavior is given in Sec. III. The relation (2.18) applies with a fixed inequality sign only for fully attenuating (amplifying) devices, in which net loss (gain) occurs for all possible input states.

III. OUTPUT NOISE POWER SPECTRA

We follow the method of Caves [1] to derive minimum values for the noise outputs of the amplifying or attenuating device. We consider systems for which the expectation values of the noise operators can be set equal to zero,

$$\langle\hat{F}_L(\omega)\rangle=\langle\hat{F}_R(\omega)\rangle=0. \tag{3.1}$$

We also restrict attention to devices for which the added noise is time stationary. This condition restricts the moments of the noise operators in the operating state to satisfy [1]

$$\langle\hat{F}_I(\omega)\hat{F}_I^\dagger(\omega')+\hat{F}_I^\dagger(\omega')\hat{F}_I(\omega)\rangle=N_I(\omega)\delta(\omega-\omega'), \tag{3.2}$$

where subscript I denotes L or R. The spectra $N_I(\omega)$ are dimensionless real and positive semidefinite functions that determine the amount of noise added by the amplifier or attenuator.

The commutators of the noise operators from Eqs. (2.8) and (2.9) have the form

$$[\hat{F}_I(\omega),\hat{F}_I^\dagger(\omega')]=\{1-G_{IR}(\omega)-G_{IL}(\omega)\}\delta(\omega-\omega')$$
$$\equiv C_I(\omega)\delta(\omega-\omega'), \tag{3.3}$$

where the $C_I(\omega)$ are real functions that are positive for an attenuator but negative for an amplifier. The combination of Eq. (3.2) with Eq. (3.3) gives

$$\langle\hat{F}_I(\omega)\hat{F}_I^\dagger(\omega')\rangle=\tfrac{1}{2}[N_I(\omega)+C_I(\omega)]\delta(\omega-\omega') \tag{3.4}$$

and

$$\langle\hat{F}_I^\dagger(\omega')\hat{F}_I(\omega)\rangle=\tfrac{1}{2}[N_I(\omega)-C_I(\omega)]\delta(\omega-\omega'). \tag{3.5}$$

It is straightforward to show that the quantities in the square brackets in Eqs. (3.4) and (3.5) must be positive semidefinite. Thus, consider the operator

$$\hat{f}_{I\alpha}=\int d\omega\hat{F}_I(\omega)\alpha(\omega), \tag{3.6}$$

where $\alpha(\omega)$ is any complex function of ω. It follows from Eqs. (3.4) and (3.5) that

$$\langle\hat{f}_{I\alpha},\hat{f}_{I\alpha}^\dagger\rangle=\tfrac{1}{2}d\omega[N_I(\omega)+C_I(\omega)]|\alpha(\omega)|^2 \tag{3.7}$$

and

$$\langle\hat{f}_{I\alpha}^\dagger\hat{f}_{I\alpha}\rangle=\tfrac{1}{2}\int d\omega[N_I(\omega)-C_I(\omega)]|\alpha(\omega)|^2. \tag{3.8}$$

These moments correspond to the norms of the states $\hat{f}_{I\alpha}^\dagger|\psi\rangle$ and $\hat{f}_{I\alpha}|\psi\rangle$, where $|\psi\rangle$ is the operating state, and they are therefore necessarily positive semidefinite for all functions $\alpha(\omega)$. It follows that the added noise spectra satisfy

$$N_I(\omega)\geqslant|C_I(\omega)|=|1-G_{IR}(\omega)-G_{IL}(\omega)|, \tag{3.9}$$

where the equality sets the minimum level of added noise. Thus, from Eq. (3.2),

$$\langle\hat{F}_I(\omega)\hat{F}_I^\dagger(\omega')+\hat{F}_I^\dagger(\omega')\hat{F}_I(\omega)\rangle$$
$$\geqslant|1-G_{IR}(\omega)-G_{IL}(\omega)|\delta(\omega-\omega'). \tag{3.10}$$

This relation between the output noise spectra and the gain profiles generalizes Eqs. (4.19b) and (4.21) of Ref. [1], which refer to a system with a single input and a single output. They have previously been quoted without proof for the special case of a spatially symmetric system [8], where the two reflection gains and the two transmission gains are equal, as in Eq. (2.14); the explicit expressions derived for the gains and the noise operators in a symmetric dielectric slab indeed satisfy these noise inequalities [8].

It is often more convenient to work with the noise operator moment defined in Eq. (3.5), instead of the symmetrized noise operator combination in Eq. (3.10); the former satisfies

$$\langle\hat{F}_I^\dagger(\omega')\hat{F}_I(\omega)\rangle\geqslant\begin{cases}0 \quad \text{for} \quad G_{IR}(\omega)+G_{IL}(\omega)\leqslant 1\\ \{G_{IR}(\omega)+G_{IL}(\omega)-1\}\delta(\omega-\omega') \quad \text{for} \quad G_{IR}(\omega)+G_{IL}(\omega)\geqslant 1.\end{cases} \tag{3.11}$$

The first inequality refers to devices whose individual outputs $I=L$ or R show attenuation of the input signals, while the second inequality refers to devices whose individual outputs show amplification of the input signals. The noise power added by an attenuator is allowed to vanish but an amplifying device inevitably adds some noise to the output signal. The minimum noise is achieved, for example, in an inverted-population amplifier when all of the population is in the upper level of the active pair [14]. Note that the minimum noise for the two-channel amplifying device includes the sum of the gains for the two input signals, even when only one input channel is excited. This represents an increase in the output noise over that for a comparable single-channel amplifier.

An example of a non-spatially-symmetric system is provided by a gas laser below threshold with mirrors of different intensity transmission rates γ_L and γ_R at the left- and right-hand ends of the cavity [15,16]. This is one of the few systems with complete available expressions for the gain parameters. When total population inversion of the active levels is assumed, the expressions for the R and T functions are obtained from Sec. IV A of [16] as

$$T_{LL}(\omega)=T_{RR}(\omega)=\frac{(\gamma_L\gamma_R)^{1/2}}{-i\omega+\frac{1}{2}(\gamma_L+\gamma_R)(1-C)} \tag{3.12}$$

and

$$R_{\substack{LR\\RL}}(\omega)=\frac{i\omega\pm\frac{1}{2}(\gamma_L-\gamma_R)+\frac{1}{2}(\gamma_L+\gamma_R)C}{-i\omega+\frac{1}{2}(\gamma_L+\gamma_R)(1-C)}. \tag{3.13}$$

The corresponding gains, defined as in Eqs. (2.3) and (2.4), are thus

$$G_{LL}(\omega)=G_{RR}(\omega)=\frac{\gamma_L\gamma_R}{\omega^2+\frac{1}{4}(\gamma_L+\gamma_R)^2(1-C)^2} \tag{3.14}$$

and

$$G_{\substack{LR\\RL}}(\omega)=\frac{\omega^2+\frac{1}{4}[\pm(\gamma_L-\gamma_R)+(\gamma_L+\gamma_R)C]^2}{\omega^2+\frac{1}{4}(\gamma_L+\gamma_R)^2(1-C)^2}. \tag{3.15}$$

Here C is the cooperation parameter of the laser, in the range $0\leqslant C\leqslant 1$, where $C=0$ corresponds to zero pumping and $C=1$ to threshold pumping. The device has equal transmission gains but the reflection gains are generally different. They become equal only for a spatially symmetric laser cavity with $\gamma_L=\gamma_R$. The quantity that occurs in Eq. (2.10) is given by

$$R_{LR}(\omega)T^*_{RR}(\omega)+T_{LL}(\omega)R^*_{RL}(\omega) =\frac{(\gamma_L\gamma_R)^{1/2}(\gamma_L+\gamma_R)C}{\omega^2+\frac{1}{4}(\gamma_L+\gamma_R)^2(1-C)^2} \tag{3.16}$$

and the similar quantity on the left of Eq. (2.20) is given by the same expression. The corresponding output noise spectra on the left and right of the laser cavity are obtained on multiplication of Eq. (4.24) of Ref. [16] by γ_L and γ_R, respectively, to give

$$\langle\hat{F}^\dagger_L(\omega')\hat{F}_L(\omega)\rangle=\frac{\gamma_L(\gamma_L+\gamma_R)C}{\omega^2+\frac{1}{4}(\gamma_L+\gamma_R)^2(1-C)^2}\,\delta(\omega-\omega'). \tag{3.17}$$

The expression for the noise spectrum on the right of the cavity is the same but with γ_L and γ_R interchanged.

The individual gains in Eqs. (3.14) and (3.15) are larger or smaller than unity depending on the values of the various parameters but the sums of the pairs of gains that occur in Eqs. (2.19) and (3.11) are always greater than unity, except for $C=0$ when they equal unity. The left-hand side of Eq. (2.18) gives

$$|R_{LR}\alpha_R+T_{LL}\beta_L|^2+|T_{RR}\alpha_R+R_{RL}\beta_L|^2 =\left\{2+\frac{(\gamma_L+\gamma_R)[\gamma_L+\gamma_R+2(\gamma_L\gamma_R)^{1/2}\cos\varphi]C}{\omega^2+\frac{1}{4}(\gamma_L+\gamma_R)^2(1-C)^2}\right\}|\alpha_R|^2, \tag{3.18}$$

for equal-amplitude input signals, $|\alpha_R|=|\beta_L|$, with phase difference φ. The inequalities (2.18) and (2.20) are thus satisfied by the expressions in Eqs. (3.12)–(3.15), except for a symmetric cavity with $\gamma_L=\gamma_R$, where the symmetric superposition of input coherent states with $\varphi=0$ sees net gain but the antisymmetric superposition with $\varphi=\pi$ sees neither gain nor loss. Thus Eq. (2.18) holds as an equality for the symmetric cavity when $\alpha_R(\omega)=-\beta_L(\omega)$ and, more trivially, for zero pumping with $C=0$. It is also readily verified that Eq. (3.11) is satisfied as an equality, corresponding to the minimum added noise.

Note that the same noise power expressed by Eq. (3.17) is present in measurements of the output on the left of the device for inputs that are either reflected or transmitted, even though the respective gains are generally different. A similar remark applies to the noise in the output on the right of the device. Again, although the two transmission gains are equal, the noise that accompanies the transmitted signal is generally different for the two directions of propagation. The derivation of the expressions in Eqs. (3.12)–(3.17) assumes complete population inversion of the active levels. The formal expressions for the gains remain the same in the presence of some population in the lower of the two active levels but the value of the cooperation parameter is reduced for a given pumping of the upper level and the noise is increased [16] so that the inequality (3.11) is no longer satisfied as an equality. The laser above threshold, with cooperation parameter $C>1$, also acts as an amplifier over a limited range of C but the behavior of the device is complicated by the generation of two or more output frequencies for a monochromatic input, and we do not consider it here.

IV. SYMMETRY PROPERTIES

Considerations of time-reversal symmetry were applied by Stokes [17] to the transmission and reflection of light at an interface. His analysis of the effects of reversal of the directions of propagation of the input and output light beams is valid for the transmission and reflection of light at a passive or lossless slab and, in the notation of the present paper, it shows that

$$R_{LR}(\omega)T^*_{RR}(\omega)+T_{RR}(\omega)R^*_{RL}(\omega)=0 \tag{4.1}$$

and

$$|R_{LR}(\omega)|^2+T_{LL}(\omega)T^*_{RR}(\omega)=1. \tag{4.2}$$

These relations lead to conditions additional to Eq. (2.13) on the phase angles of the reflection and transmission coefficients, given by

$$\phi_{LR}+\phi_{RL}-2\phi_{LL}=\pm\pi \quad\text{and}\quad \phi_{LL}=\phi_{RR}. \tag{4.3}$$

It follows that

$$T_{LL}=T_{RR} \tag{4.4}$$

and the transmission coefficients are equal in both amplitude and phase.

The symmetry operation of time reversal can, in principle, also be applied to the amplifying or attenuating device. However, reversal of the propagation directions of the inputs and outputs interchanges the natures of amplifiers and attenuators, and the study of this phenomenon lies in the realm of retrodiction theory [18], which endeavours to determine the nature of the input to a communication channel from a knowledge of its output. Thus time reversal does not lead to any restrictions on the phase angles of the reflection and transmission coefficients for the amplifier or the attenuator as such. Furthermore, in the absence of energy conservation for the incoming and outgoing light beams of an amplifier or attenuator, the relations in Eq. (2.14) between the reflection and transmission gains cannot be established on this basis.

The explicit example considered in Sec. III indeed shows a case of unequal reflection gains when the two cavity mirrors are different, with $\gamma_L\neq\gamma_R$. It is easy to understand how the reflection gains can be different. Thus, for example, insertion of a two-sided partial mirror in the interior of a device, with different amplifying and/or attenuating sections on its two sides, ensures that the outputs in reflection on the left and on the right have experienced regions of different gain characteristics within the device. The effect of a perfect reflector, considered below in Sec. V, provides a simple example of this.

Consider, however, the corresponding gains in transmission, where the situation is more complicated. Each input beam must now propagate through the entire device to reach the output on its far side. Both beams thus experience all of the amplifying and attenuating sections in a composite device and two cases can be distinguished. The first case includes devices where the gain or loss in each section is independent of the direction of signal propagation. The overall transmission gains in the two directions are thus derived from products of the same gain or loss factors and their magnitudes are expected to be the same. The equality can be proved rigorously for a multilayer stack of attenuating materials [19,20] and the proof can be extended to a combination of amplifying and attenuating sections for stacks in which the gain in no part of the system exceeds lasing threshold. The two transmission gains are then the same, as in the second equality in Eq. (2.14), and the below-threshold laser gains in Eq. (3.14) provide an example.

The second case includes amplifiers and attenuators where the gain or loss in at least one section of the composite device depends on the direction of signal propagation. The overall transmission gains are then clearly different. Examples are provided by devices driven by a traveling-wave pump, as in Raman and Brillouin amplifiers. For the example of the Brillouin amplifier, phase-matching requirements produce zero gain for signals that propagate in the same direction as the pump and amplification occurs only for signal propagation in the opposite direction [21]. This is an extreme case of the inequality of the two transmission gains. Time-reversal symmetry relates the transmission gains for signals incident on opposite ends of a pair of amplifiers for which the directions of the pump propagation are also reversed [22], but there are no overall restrictions on the two transmission gains for the same amplifier.

V. LIMIT OF SINGLE INPUT AND OUTPUT CHANNELS

The results derived in previous sections are readily specialized to amplifiers and attenuators that have single input and output channels. Thus, if the right-hand boundary of the device is made perfectly reflecting, we can set

$$R_{RL}=1, \quad T_{LL}=T_{RR}=0, \quad \text{and} \quad \hat{F}_R=0. \tag{5.1}$$

The input-output relations then separate into two independent equations, where Eq. (2.2) merely expresses the perfect reflection on the right and the remaining nontrivial input-output relation (2.1) reduces to

$$\hat{a}_L(\omega)=R_{LR}(\omega)\hat{a}_R(\omega)+\hat{F}_L(\omega). \tag{5.2}$$

The commutation relation from Eq. (2.8) is

$$[\hat{F}_L(\omega),\hat{F}_L^\dagger(\omega')]=\{1-G_{LR}(\omega)\}\delta(\omega-\omega'); \tag{5.3}$$

the inequality (3.9) for the added noise spectrum reduces to

$$N_L(\omega)\geqslant|1-G_{LR}(\omega)| \tag{5.4}$$

and the limits (3.11) to

$$\langle\hat{F}_L^\dagger(\omega')\hat{F}_L(\omega)\rangle \geqslant \begin{cases} 0 \quad \text{for} \quad G_{LR}(\omega)\leqslant 1 \\ \{G_{LR}(\omega)-1\}\delta(\omega-\omega') \quad \text{for} \quad G_{LR}(\omega)\geqslant 1. \end{cases} \tag{5.5}$$

These expressions reproduce the standard results [1] for a phase-insensitive phase-preserving linear amplifier with one input channel and one output channel.

An example of these conditions is provided by the below-threshold laser amplifier of Sec. III when $\gamma_R=0$. Then Eqs. (3.14), (3.15), and (3.17) reduce to

$$G_{LL}(\omega)=G_{RR}(\omega)=0, \tag{5.6}$$

$$G_{LR}(\omega)=\frac{\omega^2+\frac{1}{4}\gamma_L^2(1+C)^2}{\omega^2+\frac{1}{4}\gamma_L^2(1-C)^2}, \quad G_{RL}(\omega)=1, \tag{5.7}$$

and

$$\langle\hat{F}_L^\dagger(\omega')\hat{F}_L(\omega)\rangle=\frac{\gamma_L^2 C}{\omega^2+\frac{1}{4}\gamma_L^2(1-C)^2}\,\delta(\omega-\omega'). \tag{5.8}$$

It is readily verified that Eqs. (5.4) and (5.5) are satisfied with $N_L(\omega)$ taking its minimum possible value.

The basic input-output relation (5.2) can be put in a form identical to that for a lossless beam splitter when $G_{LR}(\omega) <1$ and the device acts as an attenuator. Thus, defining effective transmission and reflection coefficients $t(\omega)$ and $r(\omega)$ by

$$t(\omega)=R_{LR}(\omega) \quad \text{and} \quad r(\omega)\hat{f}_L(\omega)=\hat{F}_L(\omega), \tag{5.9}$$

with a normalized noise operator $\hat{f}_L(\omega)$ that satisfies the commutation relation

$$[\hat{f}_L(\omega),\hat{f}_L^\dagger(\omega')]=\delta(\omega-\omega'), \tag{5.10}$$

the commutator (5.3) gives

$$1=|t(\omega)|^2+|r(\omega)|^2, \tag{5.11}$$

which is the standard beam-splitter condition [14]. The input-output relation (5.2) takes the form

$$\hat{a}_L(\omega)=t(\omega)\hat{a}_R(\omega)+r(\omega)\hat{f}_L(\omega) \tag{5.12}$$

appropriate to a beam splitter whose output acquires contributions from a signal input $\hat{a}_R(\omega)$ and a noise input $\hat{f}_L(\omega)$. In this commonly used representation of a single-channel attenuator, the beam splitter is itself passive or lossless and the noise associated with the loss enters via a noise oscillator mode at the second input.

An analogous effective beam-splitter representation applies when $G_{LR}(\omega)>1$ and the device acts as an amplifier. The relations (5.9) and (5.10) continue to apply, except that the normalized noise operator now has the nature of a creation operator defined by

$$r(\omega)\hat{f}_L^\dagger(\omega)=\hat{F}_L(\omega) \tag{5.13}$$

and the commutator (5.3) gives

$$1=|t(\omega)|^2-|r(\omega)|^2. \tag{5.14}$$

This condition has been used previously for the effective amplifying beam splitter [14,23,24]. The input-output relation (5.2) takes the form

$$\hat{a}_L(\omega)=t(\omega)\hat{a}_R(\omega)+r(\omega)\hat{f}_L^\dagger(\omega) \tag{5.15}$$

appropriate to a beam splitter whose output acquires contributions from an input signal $\hat{a}_R(\omega)$ and a noise input $\hat{f}_L^\dagger(\omega)$. This representation of a single-channel amplifier again has a passive effective beam splitter and the noise associated with the gain enters via an inverted harmonic oscillator mode at the second input.

Other examples of devices with single inputs and outputs are produced when the generic system shown in Fig. 1 is assumed to have antireflection coatings, so that Eqs. (2.1) and (2.3) reduce to

$$\hat{a}_L(\omega)=T_{LL}(\omega)\hat{b}_L(\omega)+\hat{F}_L(\omega) \tag{5.16}$$

and

$$\hat{b}_R(\omega)=T_{RR}(\omega)\hat{a}_R(\omega)+\hat{F}_R(\omega). \tag{5.17}$$

The effective beam-splitter theory outlined above can be applied to the separate devices represented by these two relations.

VI. CONCLUSIONS

The main results of the paper are the limits expressed by Eqs. (3.9)–(3.11) for the added output noise properties of amplifiers or attenuators that are modeled by the input-output relations (2.1) and (2.2). The model applies to the typical components of optical communication systems, where each linear element may have input and output signals at both ends. As compared to a device that has a single input and output, the minimum noise for an attenuator retains its zero value but that for an amplifier is increased by the gain at the measured output end associated with the second input channel. This increase in the minimum noise occurs, of course, even when only the first input channel has a nonzero signal incident on the amplifier. It is also shown that the same dual input-output device may display passive, amplifying, or attenuating behavior for different combinations of signals at the two inputs. The limits on the noise power spectra are illustrated by their application to model results for a gas laser in an unsymmetrical Fabry-Perot cavity.

The two reflection gains of the model gas laser are different when the cavity mirrors have different reflectivities. This inequality of the reflection gains is a general property of systems that are spatially unsymmetric around the center of the device, as signals incident on the two ends experience the optical components within the amplifier or attenuator to different extents. The two transmission gains, on the other hand, can be equal or unequal even for devices that are spatially unsymmetric. *Equality* occurs for devices whose components have gains or losses that are independent of the direction of propagation of the input signal, as the overall effect of the device is determined by a product of the same factors for the two directions; the gas laser, with common transmission gains given by Eq. (3.14), is an example. *Inequality* of the transmission gains occurs for devices whose components include at least one for which the gain or loss is different for the two directions of propagation; the Brillouin amplifier driven by a traveling-wave pump provides an extreme example, where the gain is zero for signal propagation parallel to the pump and nonzero for antiparallel propagation.

Finally, it is shown how the results derived for the amplifier and attenuator with two inputs and two outputs reduce to the more familiar relations that apply for devices with a single input and output. As is well known, the simpler devices can be represented by effective beam splitters, with one input and output devoted to the signal and the other input devoted to the noise. No such simple beam-splitter model can represent the more commonly used devices with pairs of inputs and outputs, but the general results for the noise power spectra derived here are easily applied to the specific amplifiers and attenuators that are used in practical communication systems.

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