

Some properties of the computable cross-norm criterion for separability

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(Received 9 December 2002; published 27 March 2003)

The computable cross-norm (CCN) criterion is a powerful analytical and computable separability criterion for bipartite quantum states, which is also known to systematically detect bound entanglement. In certain aspects this criterion complements the well-known Peres positive partial transpose (PPT) criterion. In the present paper we study important analytical properties of the CCN criterion. We show that in contrast to the PPT criterion it is not sufficient in dimension 2×2 . In higher dimensions, theorems connecting the fidelity of a quantum state with the CCN criterion are proved. We also analyze the behavior of the CCN criterion under local operations and identify the operations that leave it invariant. It turns out that the CCN criterion is in general not invariant under local operations.

DOI: 10.1103/PhysRevA.67.032312

PACS number(s): 03.67.Mn, 02.30.Tb

I. INTRODUCTION

Entanglement of composite quantum systems is a key resource in many applications of quantum information technology. However, theoretically entanglement is not yet fully understood and to decide whether or not a given state is entangled or useful for quantum information processing purposes is in general a difficult question. Therefore the characterization and classification of entangled states is an important area of research that has received much attention in the development of quantum information theory. In recent years considerable progress has been made towards developing a general theory of quantum entanglement. In particular, criteria to decide whether or not a given quantum state is entangled are of high theoretical and practical interest. Historically, Bell-type inequalities were the first operational criteria to distinguish between entangled and separable states. Due to the importance of entanglement in quantum information processing, there has been a dramatic increase in our knowledge and understanding of entangled quantum states. Today, we have much more subtle and effective separability criteria than provided by Bell inequalities. Most notably, in Ref. [1] Peres obtained a powerful computable necessary separability criterion, the so-called positive partial transpose (PPT) criterion. The Peres criterion stipulates that the partial transpose of any separable quantum state is again a state. The Horodecki family formulated a necessary and sufficient mathematical characterization of separable states in terms of positive maps [2]. Subsequently, the study of separability criteria and their relation to positive maps attracted a great deal of attention and several new criteria were formulated [3]. By now there exists a sophisticated theory based on so-called entanglement witnesses [3–5]. However, for a long time the PPT criterion remained the most powerful and versatile operational separability criterion. It was only relatively recently that a novel *analytical* separability criterion not based on entanglement witnesses or positive maps was derived in Ref. [6]. The new criterion was derived within the context of an

approach that aims to characterize entanglement by using norms [7]. In Ref. [6], the new criterion was named *computable cross-norm criterion* for reasons to become clear below. In the present paper I shall adopt this terminology and for brevity also use the acronym *CCN criterion*. The CCN criterion is as easy to compute and as versatile as the PPT criterion, but yet independent of it [6]. The new criterion is the first analytical separability criterion that is known to systematically detect bound entanglement as well as genuine multipartite entanglement [8]. The power of the new criterion was already demonstrated in Ref. [6] where a number of examples were discussed. It was shown there that the CCN criterion is necessary and sufficient for pure states while for mixed states the CCN criterion is not sufficient in dimension $d \geq 3$. For dimension 2×2 the question of sufficiency was left open.

Recently, a nonanalytical but computationally tractable generalization of the PPT criterion based on semidefinite programming was presented in Ref. [9]. This powerful method is also able to detect bound entanglement. It is clear, however, that the same ideas can also be applied to the CCN criterion. It is therefore natural to conjecture that the tests described in Ref. [9] together with the analog generalization of the CCN criterion will provide a very powerful hierarchy of numerical separability tests.

The CCN criterion complements the Peres criterion in several aspects. The aim of the present paper is to study and clarify some important analytical properties of the CCN criterion in detail. I shall demonstrate three important results. In Sec. III, we study the CCN criterion in dimension 2×2 . We find that the criterion is in general not sufficient in dimension 2×2 . We also prove that for two-qubit states with maximally disordered subsystems, the CCN criterion is necessary and sufficient. In Sec. IV, we study the CCN criterion in arbitrary dimension and prove theorems relating upper and lower bounds for the fidelity of quantum states to the CCN criterion. Finally, in Sec. V we study the behavior of the CCN criterion under local operations. We show that the CCN criterion is not invariant under local operations and therefore also not under LQCC operations (i.e., quantum operations that can be implemented locally with classical communica-

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tion between the parties). We put forward a generalization of the CCN criterion that is strictly stronger than the CCN criterion. In the course of the present paper, we employ key techniques and methods that I hope will prove useful also for further studies and applications of the CCN criterion.

Throughout the paper we adopt the following notation: the set of bounded operators on \mathbb{C}^d (i.e., $d \times d$ matrices) is denoted by $T(\mathbb{C}^d)$. The canonical real basis of \mathbb{C}^d is denoted by $(|i\rangle)_{i=1}^d$ and the maximally entangled wave function with respect to this basis is denoted by $|\Psi_+\rangle \equiv (1/\sqrt{d})\sum_{i=1}^d |ii\rangle$.

II. THE CCN CRITERION

A quantum state ϱ on $\mathbb{C}^d \otimes \mathbb{C}^d$ is called *separable* (disentangled) if it can be expressed as a convex combination of product states [10], i.e., in the form

$$\varrho = \sum_{i=1}^k p_i \varrho_i \otimes \tilde{\varrho}_i.$$

Otherwise ϱ is called *entangled*.

The CCN criterion is a necessary separability criterion. It can be formulated in different equivalent ways. A very useful and instructive way is the following procedure. Consider a quantum state ϱ defined on a tensor product Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$. We denote the canonical real basis in \mathbb{C}^d by $(|i\rangle)_{i=1}^d$ and expand ϱ in terms of the operators $E_{ij} \equiv |i\rangle\langle j|$, i.e., we write

$$\varrho = \sum_{ijkl} \varrho_{ijkl} E_{ij} \otimes E_{kl}. \quad (1)$$

Next, we define an operator $\mathfrak{A}(\varrho)$ that acts on $T(\mathbb{C}^d \otimes \mathbb{C}^d)$ by

$$\mathfrak{A}(\varrho) \equiv \sum_{ijkl} \varrho_{ijkl} |E_{ij}\rangle\langle E_{kl}|. \quad (2)$$

Here $|E_{ij}\rangle$ denotes the ket vector with respect to Hilbert-Schmidt inner product $\langle A, B \rangle \equiv \text{tr}(A^\dagger B)$ in $T(\mathbb{C}^d)$. We also write $\|A\|_2 \equiv \langle A, A \rangle^{1/2}$. The norm $\|A\|_2$ is often called the *Hilbert-Schmidt norm* or the *Frobenius norm* of A and is equal to the sum of the squares of the singular values of A . The sum of the absolute values of the singular values of A is called the *trace class norm*, or simply *trace norm*, and is denoted by $\|A\|_1$.

Criterion 1. The CCN criterion asserts that if ϱ is separable, then the trace class norm of $\mathfrak{A}(\varrho)$ is less than or equal to 1. Whenever a quantum state ϱ satisfies $\|\mathfrak{A}(\varrho)\|_1 > 1$, this signals that ϱ is entangled.

In Ref. [6] it has been shown that the criterion is independent of the basis of \mathbb{C}^d chosen. In fact, there is the following representation for $\|\mathfrak{A}(\varrho)\|_1$:

$$\tau(\varrho) \equiv \|\mathfrak{A}(\varrho)\|_1 = \inf \left\{ \sum_i \|x_i\|_2 \|y_i\|_2 : \varrho = \sum_i x_i \otimes y_i \right\}, \quad (3)$$

where the infimum runs over all decompositions of ϱ into finite sums of simple tensors. It is easy to see that the norm τ satisfies the inequality

$$\tau(\sigma_1 \otimes \sigma_2) \leq \|\sigma_1\|_1 \|\sigma_2\|_1.$$

This inequality is called the *subcross property* in the mathematical literature, which justifies the name *computable cross-norm criterion*. From Eqs. (1) and (2) it is a straightforward and trivial exercise to determine the matrix representation for $\mathfrak{A}(\varrho)$ in the canonical basis. It turns out that $\mathfrak{A}(\varrho)$ is equal to the so-called *Oxenrider-Hill matrix reordering* of ϱ that was studied in Ref. [11].

We conclude this section by remarking that also the Peres criterion can be written in the form of a norm criterion, i.e., the Peres criterion is equivalent to the following statement: if a state ϱ satisfies $\|\varrho^{T_2}\|_1 > 1$, then ϱ is entangled. Here T_2 denotes the partial transpose with respect to the second subsystem.

III. THE CCN CRITERION FOR TWO QUBITS

In Ref. [6] the CCN criterion was computed for several examples, including Werner states, isotropic and Bell diagonal states. In dimension 2×2 the CCN criterion turned out to be necessary and sufficient for all these examples. It is the purpose of this section to study the CCN criterion in dimension 2×2 in more detail. It is known that any two-qubit state ϱ can be expressed in terms of Hilbert-Schmidt operators,

$$\varrho = \frac{1}{4} \left(\mathbb{1} \otimes \mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{s} \cdot \boldsymbol{\sigma} + \sum_{m,n=1}^3 t_{mn} \sigma_m \otimes \sigma_n \right). \quad (4)$$

Here $\mathbb{1}$ stands for the identity operator, $\{\sigma_{ij}\}_{i,j=1}^3$ are the standard Pauli matrices, $\mathbf{r}, \mathbf{s} \in \mathbb{R}^3$ and $\mathbf{r} \cdot \boldsymbol{\sigma} = \sum_{i=1}^3 r_i \sigma_i$. We denote the real matrix formed by the coefficients t_{mn} by $T(\varrho)$. The separability and distillability properties of two qubit states in the Hilbert-Schmidt space formalism have been discussed in detail in Refs. [12,13]. Here we built on these results to study properties of the CCN criterion. First we note that \mathbf{r} and \mathbf{s} equal the Bloch vectors of the reductions $\varrho_1 \equiv \text{tr}_2 \varrho$ and $\varrho_2 \equiv \text{tr}_1 \varrho$ of ϱ , respectively. A state with maximally disordered subsystems thus has $\mathbf{r} = \mathbf{s} = \mathbf{0}$ in Eq. 4. We prove that the CCN criterion is necessary and sufficient for two-qubit states with maximally disordered subsystems.

Proposition 2. Let ϱ be a two-qubit state with maximally disordered subsystems. Then $\|\mathfrak{A}(\varrho)\|_1 = (1 + \|T(\varrho)\|_1)/2$, i.e., $\|\mathfrak{A}(\varrho)\|_1 \leq 1$ if and only if ϱ is separable.

Proof. Since the Hilbert-Schmidt norm $\|\cdot\|_2$ is invariant under unitaries, it is obvious from the variational expression for $\|\mathfrak{A}(\varrho)\|_1$ given above that $\|\mathfrak{A}(\varrho)\|_1$ is invariant under local unitary operations of the form $U_1 \otimes U_2$ acting on ϱ . As shown in Ref. [13] we can always choose local unitaries U_1, U_2 such that $T(U_1 \otimes U_2 \varrho U_1^\dagger \otimes U_2^\dagger)$ is diagonal. These two facts imply that without loss of generality we can assume that $T(\varrho)$ is diagonal. Then ϱ is of the form $\varrho = \frac{1}{4} (\mathbb{1} \otimes \mathbb{1} + \sum_{m=1}^3 t_m \sigma_m \otimes \sigma_m)$. Correspondingly, we find $\mathfrak{A}(\varrho) = \frac{1}{4} |\mathbb{1}\rangle\langle \mathbb{1}| + \sum_{m=1}^3 (t_m/4) |\sigma_m\rangle\langle \sigma_m^*|$. Here $*$ denotes complex

conjugation. Note that $(1/\sqrt{2}|1\rangle, (1/\sqrt{2})|\sigma\rangle)$ is an orthonormal basis with respect to the Hilbert-Schmidt inner product. $\|T(\varrho)\|_1$ is invariant under local unitary operations acting on ϱ . Thus $\|\mathfrak{A}(\varrho)\|_1 = (1 + \sum_{m=1}^3 |t_m|)/2 = (1 + \|T(\varrho)\|_1)/2$. Clearly if ϱ is separable, then $\|\mathfrak{A}(\varrho)\|_1 \leq 1$. If ϱ is not separable, then it follows from Proposition 4 in Ref. [13] that $\|T\|_1 > 1$. This implies $\|\mathfrak{A}(\varrho)\|_1 > 1$. Alternatively, the last implication also follows from Theorem 2 in Ref. [14]. ■

We now wish to relate the CCN criterion with the fidelity of two-qubit states. The fidelity of a state ϱ is defined as $f(\varrho) \equiv \max_{\Psi} \langle \Psi | \varrho | \Psi \rangle$ where the maximum is over all maximally entangled pure states Ψ . The fidelity is an important quantity that is often employed as a measure of the efficiency of quantum communication protocols. We have the following

Proposition 3. For any two-qubit state ϱ we have $f(\varrho) \leq \|\mathfrak{A}(\varrho)\|_1/2$.

The proof of Proposition 3 can be found in Appendix A.

Proposition 4. Let ϱ be an entangled two-qubit state with maximally disordered subsystems. Then $\|\mathfrak{A}(\varrho)\|_1 = 2f(\varrho)$.

Proof. Let ϱ be an entangled two-qubit state with maximally disordered subsystems. Since $f(\varrho)$ and $\|\mathfrak{A}(\varrho)\|_1$ are both invariant under local unitary operations, we can assume again that $T(\varrho)$ is diagonal. From Proposition 2 we know that $\|\mathfrak{A}(\varrho)\|_1 = \frac{1}{2}[1 + \|T(\varrho)\|_1]$. On the other hand, an argument similar to the proof of Eq. (A1) in Appendix A leads to

$$f(\varrho) = \frac{1}{4} + \max_U \sum_{n=1}^3 \frac{t_n}{8} \text{tr}(\sigma_n^T U \sigma_n U^\dagger), \quad (5)$$

where the maximum is over all unitaries U on \mathbb{C}^2 and T denotes transposition. We observe that for any entangled two-qubit state ϱ with maximally disordered subsystems, the number of negative eigenvalues of the matrix $T(\varrho)$ is either exactly one or exactly three. The latter statement is an immediate consequence of the geometric representation for such states given in Proposition 3 and Proposition 4 in Ref. [13]. From Proposition 3 above and the proof of Proposition 1 in Ref. [13] [in particular Eq. (13) there], it follows that there exists a maximally entangled pure state that compensates the signs of the negative eigenvalues of $T(\varrho)$. More precisely, if the signature of $T(\varrho)$ is $(-, -, -)$, then in Eq. (5) choose $U = e^{i\phi} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Moreover, for the signatures $(+, +, -)$, $(+, -, +)$, and $(-, +, +)$ choose $U = e^{i\phi} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $U = e^{i\phi} \mathbb{1}$, and $U = e^{i\phi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, respectively. This shows that $2f(\varrho) = \|\mathfrak{A}(\varrho)\|_1$. ■

It is worthwhile to note that Proposition 4 is in general not true for separable states. To see this consider a separable state with maximally disordered subsystems for which $T(\varrho)$ has two nonpositive eigenvalues. Such a state exists by the results of Ref. [13]. To achieve $2f(\varrho) = 2\langle \psi | \varrho | \psi \rangle = \|\mathfrak{A}(\varrho)\|_1$ for some maximally entangled pure state $|\psi\rangle$, we need to have, say, $T(|\psi\rangle\langle\psi|) = \text{diag}(-1, -1, 1)$. However, by the results of Ref. [13] there is no state with such a T matrix.

Notice that all the main examples for two-qubit states for which the CCN criterion was explicitly computed in Ref. [6] have maximally disordered subsystems. Thus by Proposition 2—and in accordance with the results of Ref. [6]—the CCN criterion is necessary and sufficient for these states. It is

worthwhile to note that there are also families of two-qubit states without maximally disordered subsystems for which the CCN criterion is a necessary and sufficient condition for separability. An example is the family of states

$$\varrho_p = p|\psi\rangle\langle\psi| + \frac{1-p}{4}\mathbb{1} \otimes \mathbb{1},$$

where $|\psi\rangle$ is a (not necessarily maximally entangled) pure state and where $p \in [-\frac{1}{3}, 1]$. It is straightforward to check that for this family of states $\|\mathfrak{A}(\varrho_p)\|_1 \leq 1$ if and only if $p \leq 1/(4\sqrt{\alpha_1\alpha_2} + 1)$ where (α_1, α_2) denote the Schmidt coefficients of $|\psi\rangle$. Invoking the PPT criterion shows that $\|\mathfrak{A}(\varrho_p)\|_1 \leq 1$ iff ϱ_p is separable. In view of these examples one may thus conjecture that the CCN criterion is necessary and sufficient for two qubits. However, it turns out that this conjecture is not true. A counterexample can easily be constructed along the lines of Ref. [15]. Consider a two-qubit state that can be expressed in the form $\varrho = \frac{1}{4}[\mathbb{1} \otimes \mathbb{1} + s(\mathbb{1} \otimes \sigma_3) + r(\sigma_3 \otimes \mathbb{1}) + t(\sigma_1 \otimes \sigma_1) - t(\sigma_2 \otimes \sigma_2) + (1+r-s)(\sigma_3 \otimes \sigma_3)]$ where $r, s, t \in \mathbb{R}$ and where we assume $s > r$. A straightforward calculation shows that the partial transpose of this state is positive if and only if $t = 0$. On the other hand, $\|\mathfrak{A}(\varrho)\|_1 = g(s, r) + |t|$ where $g(s, r)$ is a non-negative function of s and r . Therefore if we pick appropriate values for s, r , and t such that $g(s, r) < 1$ and such that $0 < |t| \leq 1 - g(s, r)$, then the resulting two-qubit state is entangled (as the PPT criterion is necessary and sufficient in dimension 2×2) but is not detected by the CCN criterion. A possible choice would be, for instance, $s = \frac{1}{2}, r = \frac{1}{4}$, and $t = \frac{1}{16}$. Details of the calculations and the precise form of g can be found in Appendix B. Our example proves the following.

Proposition 5. The CCN criterion is not a sufficient criterion for separability in dimension 2×2 .

IV. THE CCN CRITERION IN ARBITRARY DIMENSION

The aim of the present section is to prove generalized versions of the Propositions 2–4 in arbitrary dimensions. In particular, we prove that $(1/d)\text{tr}[\mathfrak{A}(\varrho)]$ and $(1/d)\|\mathfrak{A}(\varrho)\|_1$ are lower and upper bounds for the fidelity $f(\varrho)$, respectively. The examples studied in Ref. [6] imply that the CCN criterion is not sufficient for separability in dimension greater than 2. In this section, we use the generalized d -level spin matrices that were studied in Refs. [16,17]. If we denote the canonical basis by $(|i\rangle)_{i=1}^d$, then the d -level spin matrices are given by

$$S_{jk} \equiv \sum_{r=0}^{d-1} \exp(2\pi i jr/d) |r\rangle\langle r \oplus k|,$$

where \oplus denotes addition modulo d . It was shown in Ref. [17] that

$$\left(\frac{1}{\sqrt{d}} S_{jk} \right)_{jk},$$

forms an orthonormal basis of the Hilbert-Schmidt space in d dimensions. Moreover, for $(j,k) \neq (0,0)$ the matrix $(1/\sqrt{d})S_{jk}$ has vanishing trace. We arrange the matrices $(S_{jk})_{(j,k) \neq (0,0)}$ into a (d^2-1) -vector $\mathbf{S} = (S_{01}, S_{02}, \dots, S_{d-1, d-1}) =: (S_1, S_2, \dots, S_{d^2-1})$. With this notation, we can easily generalize the representation in Eq. (4). We arrive at that every bipartite quantum state ϱ on $\mathbb{C}^d \otimes \mathbb{C}^d$ can be expanded in Hilbert-Schmidt space as

$$\varrho = \frac{1}{d^2} \left(\mathbb{1} \otimes \mathbb{1} + \mathbf{r} \cdot \mathbf{S} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{s} \cdot \mathbf{S}^* + \sum_{m,n=1}^{d^2-1} t_{mn} S_n \otimes S_m^* \right). \quad (6)$$

Here \mathbf{r} and \mathbf{s} are complex vectors in \mathbb{C}^{d^2-1} and the t_{mn} form a $(d^2-1) \times (d^2-1)$ complex matrix $T(\varrho)$. * denotes complex conjugation. The reduced states of the subsystems of ϱ are given by $\varrho_1 = \text{tr}_2(\varrho) = (\mathbb{1} + \mathbf{r} \cdot \mathbf{S})/d$ and $\varrho_2 = \text{tr}_1(\varrho) = (\mathbb{1} + \mathbf{s} \cdot \mathbf{S}^*)/d$. From Eq. (6) we infer

$$\mathfrak{A}(\varrho) = \frac{1}{d^2} \left(|\mathbb{1}\rangle\langle\mathbb{1}| + \sum_i r_i |S_i\rangle\langle\mathbb{1}| + \sum_i s_i |\mathbb{1}\rangle\langle S_i| + \sum_{m,n=1}^{d^2-1} t_{mn} |S_n\rangle\langle S_m| \right). \quad (7)$$

We now wish to relate the operator $\mathfrak{A}(\varrho)$ to the fidelity $f(\varrho)$ of ϱ . The results of Refs. [6,18] imply that if ϱ is pure or an isotropic state, then $df(\varrho) = \|\mathfrak{A}(\varrho)\|_1$. In general we will see that equality does not hold. However, Proposition 3 immediately generalizes to arbitrary dimension, i.e., we have the following.

Proposition 6. Let $\varrho \in T(\mathbb{C}^d \otimes \mathbb{C}^d)$ be a bipartite state. Then $df(\varrho) \leq \|\mathfrak{A}(\varrho)\|_1$.

For a proof we refer to Appendix A. Moreover, we have the following proposition.

Proposition 7. Let ϱ be a bipartite state on $\mathbb{C}^d \otimes \mathbb{C}^d$. Then

$$df(\varrho) = \max_U |\text{tr} \mathfrak{A}((\mathbb{1} \otimes U)\varrho(\mathbb{1} \otimes U^\dagger))|,$$

where the maximum is over all unitary operators U on \mathbb{C}^d . Moreover, $d\langle\Psi_+|\varrho|\Psi_+\rangle = \text{tr}[\mathfrak{A}(\varrho)] \leq df(\varrho)$. If in addition $\mathfrak{A}(\varrho) \geq 0$, then $df(\varrho) = |\text{tr}[\mathfrak{A}(\varrho)]| = \|\mathfrak{A}(\varrho)\|_1$.

Proof. From Eq. (7) we infer $\text{tr}[\mathfrak{A}(\varrho)] = (1/d)(1 + \sum_n t_{nn})$. On the other hand, note that [19] $f(\varrho) = \max_{\psi} \langle\psi|\varrho|\psi\rangle = \max_U \langle\Psi_+|(\mathbb{1} \otimes U)\varrho(\mathbb{1} \otimes U^\dagger)|\Psi_+\rangle$. The first maximum is with respect to all maximally entangled states ψ while the second is with respect to all unitary operators on \mathbb{C}^d . Moreover, $|\Psi_+\rangle = (1/\sqrt{d})\sum_i |i,i\rangle$ where $(|i\rangle)$ denotes the canonical real basis of \mathbb{C}^d . A straightforward calculation shows that

$$f(\varrho) = \frac{1}{d^2} \left(1 + \max_U \frac{1}{d} \sum_{mn} t_{mn} \langle S_m | U^T S_n U^* \rangle \right). \quad (8)$$

This implies the variational expression in Proposition 7 and also that $\text{tr}[\mathfrak{A}(\varrho)] \leq df(\varrho)$. Moreover, we find

$\text{tr}[\mathfrak{A}(\varrho)] = d\langle\Psi_+|\varrho|\Psi_+\rangle$ (corresponding to $U=\mathbb{1}$). If $\mathfrak{A}(\varrho) \geq 0$, then $\text{tr}[\mathfrak{A}(\varrho)] \leq df(\varrho) \leq \|\mathfrak{A}(\varrho)\|_1 = \text{tr}[\mathfrak{A}(\varrho)]$. ■

Corollary 8. Let ϱ be a bipartite state on $\mathbb{C}^d \otimes \mathbb{C}^d$. If $\text{tr}[\mathfrak{A}(\varrho)] > 1$, then ϱ is distillable.

Proof. This follows immediately from Proposition 7 and the results of Ref. [18]. ■

Corollary 9. Let ϱ be a bipartite state on $\mathbb{C}^d \otimes \mathbb{C}^d$. Then $\text{tr}[\mathfrak{A}(\varrho)] \geq 0$.

Note that $\mathfrak{A}(\varrho)$ is in general not Hermitian. The following proposition is our generalization of Proposition 2.

Proposition 10. Let ϱ a bipartite state with maximally disordered subsystems. Then $\|\mathfrak{A}(\varrho)\|_1 = (1/d)[1 + \|T(\varrho)\|_1]$. If $T(\varrho) \geq 0$ and $\|\mathfrak{A}(\varrho)\|_1 > 1$, then ϱ is distillable.

Proof. Let ϱ be a bipartite state with maximally disordered subsystems. Then as in the proof of Proposition 2,

$$\mathfrak{A}(\varrho) = \frac{1}{d^2} \left(|\mathbb{1}\rangle\langle\mathbb{1}| + \sum_{m,n=1}^{d^2-1} t_{mn} |S_n\rangle\langle S_m| \right).$$

Since

$$\left(\frac{1}{\sqrt{d}}|\mathbb{1}\rangle, \frac{1}{\sqrt{d}}|S_n\rangle \right)$$

forms an orthonormal basis of the Hilbert-Schmidt space in dimension d , we find that $\|\mathfrak{A}(\varrho)\|_1 = (1/d)[1 + \|T(\varrho)\|_1]$. This proves the first half of Proposition 10. From Eq. (7) we see that for states ϱ with maximally disordered subsystems $T(\varrho) \geq 0$ if and only if $\mathfrak{A}(\varrho) \geq 0$. Now if $T(\varrho) \geq 0$, then by Proposition 7 $df(\varrho) = \|\mathfrak{A}(\varrho)\|_1$. Thus $\|\mathfrak{A}(\varrho)\|_1 > 1$ is equivalent to $f(\varrho) > 1/d$. By the results of Ref. [18] this implies that ϱ is distillable. This proves the proposition. ■

V. THE CCN CRITERION UNDER LOCAL OPERATIONS

In the paradigmatic situation studied in quantum information theory two parties, traditionally called Alice and Bob, share parts of composite quantum systems and are able to perform local operations on their respective parts and communicate classically. An essential requirement for measures of entanglement is to be nonincreasing under LQCC operations, i.e., operations that can be implemented locally with classical communication between the parties. In the present section, we study the behavior of the quantity $\|\mathfrak{A}(\varrho)\|_1$ under local operations. An operation is a completely positive linear map Λ that is trace nonincreasing for positive operators. In the following we are only interested in trace preserving operations. Such quantum operations are all those operations that can be composed out of the following elementary operations [19]: (O1) adding an uncorrelated ancilla system, (O2) tracing out part of the system, (O3) unitary transformations, (O4) Lüders-von Neumann measurements: $\Lambda_{L\vee N}: T(H) \rightarrow T(H), \Lambda_{L\vee N}(\rho) = \sum_{i=1}^r P_i \rho P_i$ where $(P_i)_{i=1}^r$ is a complete sequence of pairwise orthogonal projection operators on H .

Proposition 11. The quantity $\|\mathfrak{A}(\varrho)\|_1$ remains invariant under local operations of the type (O3). It is nonincreasing under local operations of type (O1) and (O4). $\|\mathfrak{A}(\varrho)\|_1$ may

increase, decrease or stay invariant under local operations of type (O2).

Corollary 12. The CCN criterion is not invariant under local operations.

The statement of the Corollary means that if ϱ is a state satisfying, say, $\|\mathfrak{A}(\varrho)\|_1 \leq 1$, then there may be a state $\Lambda(\varrho)$ obtained from ϱ by a local trace nonincreasing operation Λ such that $\|\mathfrak{A}(\Lambda(\varrho))\|_1 > 1$.

Proof of Proposition 11. The invariance of $\|\mathfrak{A}(\varrho)\|_1$ under local unitary operations is an immediate consequence of the representation in Eq. (3). Similarly, it is immediate from Eq. (3) that $\|\mathfrak{A}(\varrho)\|_1$ is nonincreasing under adding a local ancilla (O1). To see that $\|\mathfrak{A}(\varrho)\|_1$ is nonincreasing under operations of type (O4), let $(P_k)_k$ be a complete family of mutually orthogonal projectors on \mathbb{C}^d and let $\Lambda_{\text{LVN}}(\varrho) \equiv \sum_k (P_k \otimes 1) \varrho (P_k \otimes 1)$. Then using Eq. (3) yields

$$\begin{aligned} \|\mathfrak{A}(\Lambda_{\text{LVN}}(\varrho))\|_1 &\leq \inf \left\{ \sum_i \left\| \sum_k P_k x_i P_k \right\|_2 \|y_i\|_2 : \varrho = \sum_i x_i \otimes y_i \right\} \\ &\leq \inf \left\{ \sum_i \|x_i\|_2 \|y_i\|_2 : \varrho = \sum_i x_i \otimes y_i \right\} \\ &= \|\mathfrak{A}(\varrho)\|_1, \end{aligned}$$

where in the second line we used that the Hilbert-Schmidt norm is nonincreasing under *pinching*, i.e., $\|\sum_k P_k \sigma P_k\|_2 \leq \|\sigma\|_2$ for all families of mutually orthogonal projectors with $\sum_k P_k = 1$ and all σ . Finally, consider two bipartite states ϱ_1 and ϱ_2 that satisfy $\|\mathfrak{A}(\varrho_1)\|_1 < 1$ and $\|\mathfrak{A}(\varrho_2)\|_1 > 1$. Then $\|\mathfrak{A}(\varrho_1 \otimes \varrho_2)\|_1 = \|\mathfrak{A}(\varrho_1)\|_1 \|\mathfrak{A}(\varrho_2)\|_1$. It is immediate that if Alice and Bob locally trace out ϱ_1 , then the value of $\|\mathfrak{A}(\varrho)\|_1$ will increase, while tracing out ϱ_2 decreases $\|\mathfrak{A}(\varrho)\|_1$. [If ϱ_1 would satisfy $\|\mathfrak{A}(\varrho_1)\|_1 = 1$, tracing out ϱ_1 would obviously leave the value of $\|\mathfrak{A}(\varrho)\|_1$ invariant.] ■

Proof of Corollary 12. The argument in the proof of Proposition 11 also implies that the CCN criterion is not invariant under local operations. To see this, choose ϱ_1 and ϱ_2 such that $\|\mathfrak{A}(\varrho_1 \otimes \varrho_2)\|_1 = \|\mathfrak{A}(\varrho_1)\|_1 \|\mathfrak{A}(\varrho_2)\|_1 < 1$ and $\|\mathfrak{A}(\varrho_2)\|_1 > 1$, i.e., the state $\varrho_1 \otimes \varrho_2$ satisfies the CCN criterion. Tracing out ϱ_1 leaves Alice and Bob with ϱ_2 , i.e., with a state that violates the CCN criterion. ■

Proposition 11 and Corollary 12 show that an entangled state ϱ that satisfies the CCN criterion may be transformed into a state violating it by locally tracing out part of the system. This suggests the following extension of the CCN criterion.

Criterion 13. Consider the quantity

$$\Lambda(\varrho) := \sup_{K_A, K_B} \|\mathfrak{A}(\text{tr}_{K_A \otimes K_B}(\varrho))\|_1,$$

where the supremum is over all local spaces K_A and K_B (on Alice's and Bob's side, respectively) that can be traced out locally. The extended CCN criterion asserts that if ϱ is separable, then $\Lambda(\varrho) \leq 1$. Whenever a quantum state ϱ satisfies $\Lambda(\varrho) > 1$, this signals that ϱ is entangled.

This criterion is stronger than the CCN criterion. A trivial example has been given above in the proof of Corollary 12.

Since there are infinitely many ways of realizing an isomorphism $\mathbb{C}^{d_1 d_2} \simeq \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ the quantity $\Lambda(\varrho)$ will in general not be computable and thus Criterion 13 is not fully operational. By fixing an isomorphism it is obviously always possible to pass to a weaker but operational criterion. However, we have not yet identified a nontrivial example where the extended criterion detects entanglement that is not already detected by the CCN criterion. This problem is thus left as an open problem.

ACKNOWLEDGMENTS

The author would like to thank Shashank Virmani for stimulating discussions. Funding by the EC project ATESIT (Contract No. IST-2000-29681) is gratefully acknowledged.

APPENDIX A: PROOF OF PROPOSITIONS 3 AND 6

First we extend the definition of fidelity to arbitrary trace class operators on $\mathfrak{w} \in T(\mathbb{C}^d \otimes \mathbb{C}^d)$ by

$$f(\mathfrak{w}) := \max_{\Psi} |\langle \Psi | \mathfrak{w} | \Psi \rangle|,$$

where the maximum is over all maximally entangled pure states $|\Psi\rangle$. Every maximally entangled wave function is of the form $|\Psi\rangle = (1 \otimes U) |\Psi_+\rangle$ for some unitary U [19]. It is straightforward to check that for all operators of the form $\mathfrak{w}_1 \otimes \mathfrak{w}_2$ we have

$$f(\mathfrak{w}_1 \otimes \mathfrak{w}_2) = \frac{1}{d} \max_U |\text{tr}(\mathfrak{w}_1^T U \mathfrak{w}_2 U^\dagger)|, \quad (\text{A1})$$

where T denotes transposition. This implies that

$$f(\mathfrak{w}_1 \otimes \mathfrak{w}_2) \leq \frac{1}{d} \|\mathfrak{w}_1\|_2 \|\mathfrak{w}_2\|_2.$$

In other words, df satisfies the *subcross property* with respect to the Hilbert-Schmidt norm $\|\cdot\|_2$. This implies immediately that $df(\mathfrak{w}) \leq \|\mathfrak{A}(\mathfrak{w})\|_1$, as $\|\mathfrak{A}(\mathfrak{w})\|_1$ is the greatest cross norm with respect to the Hilbert-Schmidt norm. (Note that this should be carefully distinguished from the greatest cross norm with respect to the trace class norm that was studied in Ref. [7]). Namely, let $\mathfrak{w} = \sum_{i=1}^k x_i \otimes y_i$ be a decomposition of \mathfrak{w} into a finite sum of simple tensors, then

$$f(\mathfrak{w}) \leq \sum_{i=1}^k f(x_i \otimes y_i) \leq \frac{1}{d} \sum_{i=1}^k \|x_i\|_2 \|y_i\|_2.$$

Taking the infimum over all possible finite decompositions on the right-hand side yields [compare Eq. (3)]

$$df(\mathfrak{w}) \leq \|\mathfrak{A}(\mathfrak{w})\|_1.$$

**APPENDIX B: THE COUNTEREXAMPLE
IN DIMENSION 2×2**

The matrix representation of the state $\varrho = \frac{1}{4}[1 \otimes 1 + s(1 \otimes \sigma_3) + r(\sigma_3 \otimes 1) + t(\sigma_1 \otimes \sigma_1) - t(\sigma_2 \otimes \sigma_2) + (1+r-s)(\sigma_3 \otimes \sigma_3)]$ in the canonical basis is given by

$$\varrho = \frac{1}{2} \begin{pmatrix} 1+r & 0 & 0 & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & s-r & 0 \\ t & 0 & 0 & 1-s \end{pmatrix}.$$

The eigenvalues are given by $\lambda_1=0, \lambda_2=(s-r)/2$, and

$$\lambda_{3,4} = \frac{1}{2} + (r-s)/4 \pm \frac{1}{2} \sqrt{t^2 + \frac{(s+r)^2}{4}}.$$

ϱ is a state if the parameters s, r, t are chosen such that each $\lambda_i \geq 0$. We assume that $s > r$. By considering the subsystems of ϱ , we see that $|s| \leq 1$ and $|r| \leq 1$. The eigenvalues of the partial transpose of ϱ are easily confirmed to be $\lambda_1=(1+r)/2, \lambda_2=(1-s)/2$,

$$\lambda_{3,4} = \frac{s-r}{4} \pm \frac{1}{2} \sqrt{\frac{(s-r)^2}{4} + t^2}.$$

Therefore, ϱ^{T_2} has a negative eigenvalue if and only if $t \neq 0$. Now $\mathfrak{A}(\varrho)$ is given by

$$\mathfrak{A}(\varrho) = \frac{1}{4} [|1\rangle\langle 1| + s|\sigma_3\rangle\langle 1| + r|1\rangle\langle \sigma_3| + (1+r-s)|\sigma_3\rangle\langle \sigma_3| + t|\sigma_1\rangle\langle \sigma_1| + t|\sigma_2\rangle\langle \sigma_2|].$$

The matrix representation of $\mathfrak{A}(\varrho)$ is

$$\mathfrak{A}(\varrho) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & r \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ s & 0 & 0 & 1+r-s \end{pmatrix}.$$

The trace class norm of this operator is easily computed. We set $\psi(s, r) \equiv (1+r)^2 + (s-r)^2 + (1-s)^2$. The eigenvalues of the operator $\mathfrak{A}(\varrho)^\dagger \mathfrak{A}(\varrho)$ are then

$$\lambda_1 = \frac{1}{8} [\psi(s, r) + \sqrt{\psi(s, r)^2 - 4(1+r)^2(1-s)^2}],$$

$$\lambda_4 = \frac{1}{8} [\psi(s, r) - \sqrt{\psi(s, r)^2 - 4(1+r)^2(1-s)^2}],$$

and $\lambda_2 = \lambda_3 = t^2/4$. Therefore, if we set $g(s, r) := \sqrt{\lambda_1} + \sqrt{\lambda_4}$, we arrive at

$$\|\mathfrak{A}(\varrho)\|_1 = g(s, r) + |t|.$$

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