Retrodiction of generalized measurement outcomes

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If a generalized measurement is performed on a quantum system and we do not know the outcome, are we able to retrodict it with a second measurement? We obtain a necessary and sufficient condition for perfect retrodiction of the outcome of a known generalized measurement, given the final state, for an arbitrary initial state. From this, we deduce that, when the input and output Hilbert spaces have equal (finite) dimension, it is impossible to perfectly retrodict the outcome of any fine-grained measurement [where each positive, operator-valued measure (POVM) element corresponds to a single Kraus operator] for all initial states unless the measurement is unitarily equivalent to a projective measurement. It also enables us to show that every POVM can be realized in such a way that perfect outcome retrodiction is possible for an arbitrary initial state when the number of outcomes does not exceed the output Hilbert space dimension. We then consider the situation where the initial state is not arbitrary, though it may be entangled, and describe the conditions under which unambiguous outcome retrodiction is possible for a fine-grained generalized measurement. We find that this is possible for some state if the Kraus operators are linearly independent. This condition is also necessary when the Kraus operators are nonsingular. From this, we deduce that every trace-preserving quantum operation is associated with a generalized measurement whose outcome is unambiguously retrodictable for some initial state, and also that a set of unitary operators can be unambiguously discriminated iff they are linearly independent. We then examine the issue of unambiguous outcome retrodiction without entanglement. This has important connections with the theory of locally linearly dependent and locally linearly independent operators.

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I. INTRODUCTION

One of the most contentious issues in the development of quantum mechanics was, and continues to be, the measurement process. The fact that measurement appears explicitly in the quantum formalism represents a significant break with the implicit assumption in classical mechanics that all quantities which enter into the description of the state of a physical system are observable and that the measurement process requires no special treatment [1]. It does in quantum mechanics. Among the consequences of the nature of the quantum measurement process as expounded by, for example, von Neumann [2], are indeterminism, the impossibility of measuring the state of a quantum system and uncertainty relations.

However, the projective measurements introduced by von Neumann and defined in full generality by Lüders [3] do retain one significant feature of classical physics. This is the property of repeatability. Simply stated, if we perform such a measurement on a quantum system twice, and if we are able to reverse any evolution of the state between the measurements, then the outcome of the second measurement will be the same as that of the first.

Subsequent developments in quantum measurement theory have shown that the combination of projective measurements with unitary interactions leads to a broader range of state transformations and information-acquisition procedures. These, which are known as quantum operations and generalized measurements respectively, are closely related to each other.

The statistical properties of a generalized measurement are determined by a set of positive operators forming a positive, operator-valued measure (POVM). Generalized measurements enable one to acquire certain kinds of information about quantum states which are unobtainable using only projective measurements, especially if the possible initial states are nonorthogonal [4–7]. However, they do have some disadvantages. One is the fact that they do not possess the aforementioned repeatability property of projective measurements. The repeatability of these measurements is independent of the initial state, which may be arbitrary and unknown. It enables us to predict not only the outcome of a future repetition of the measurement, but also the future post-measurement state, provided that there is no irreversible evolution between the measurements. Furthermore, these predictions will be fulfilled with unit probability.

As well as enabling us to predict the outcome of an identical measurement, repeatability also enables us to *retrodict* the outcome of a projective measurement and also the post-measurement state, given that we know which observable was measured and again, in the absence of subsequent irreversible evolution.

The fact that the repeatability of projective measurements has so many aspects and consequences suggests that, while these may not all hold for generalized measurements, some vestiges of repeatability could be made to hold for these measurements in some circumstances if we are willing to sacrifice others. This is the issue we investigate in this paper. The particular aspect of the repeatability of projective measurements we would like to retain is outcome retrodictability.

As one might expect, when this is possible, the measurement which carries out the retrodiction will, in general, differ from the original measurement in this wider context.

It is well established that the implementation of a generalized measurement will often involve a projective measurement on an extended space [5], for example, a projective measurement on a Cartesian product (Naimark) extension or a unitary-projection scheme on a direct, or tensor product extension. However, it is typically the case that we do not have access to this extension, which is assumed to be the case throughout this paper. When we address the issue of measurement outcome retrodictability, the retrodiction operators will act only on the space of system post-measurement states and not on such an extension. We shall, however, allow for the possibility that the space of post-measurement states differs from that of the preparation states whenever making this distinction is necessary for a fully general analysis.

We should also emphasize the distinction between the idea of retrodicting the outcome of a generalized measurement and the formalism of retrodictive quantum mechanics. The latter was proposed originally by Aharonov and coworkers [8] and has recently been extended and applied in numerous interesting ways by Barnett and co-workers [9]. In retrodictive quantum mechanics, the aim is to use accessible measurement data to retrodict the initial state of a quantum system. The retrodicted information is then quantum information. In the present context, although a measurement has been carried out, the result is not accessible and it is this classical measurement result that we aim to retrodict.

Our motivation for focusing on this particular aspect of repeatability is as follows: if we know the result of a known measurement then in practical situations we would seldom have any reason to carry it out again. The issue of repeatability, or nonrepeatability, will be important in situations when a measurement has been performed and the result has been lost or otherwise made inaccessible to us. If we do not know the measurement result then, in the most favorable scenario, we will at least have access to the final state. This is a mixture of the post-measurement states corresponding to the various possible outcomes weighted by their respective probabilities. When we do have access to the system following the measurement, which we shall assume to be the case, we will be concerned with how its state has been transformed by the measurement process. If the initial state is represented by a density operator ρ, then the final state will be obtained by a completely positive, linear, trace-preserving (CPLTP) map $\Phi:\rho\rightarrow\Phi(\rho)$.

In projective measurements, repeatability and thus perfect outcome retrodiction are possible for an arbitrary, unknown, initial state. At the outset, we make a distinction between two kinds of generalized measurement: fine-grained and coarse-grained measurements. These correspond, respectively, to situations where each POVM element is related to a single, or multiple Kraus transformation operators. The former is clearly a special case of the latter. Section II is devoted to the examination of perfect outcome retrodiction, that is, deterministic, error-free retrodiction of the outcome of a known generalized measurement. We derive a necessary and sufficient condition for such perfect retrodiction to be possible for an arbitrary initial state and show that there is no advantage to be gained if the initial state, though arbitrary, is known. The remainder of this section is devoted to unravelling some implications of this condition. We show that it implies that, when the input and output Hilbert spaces have equal dimension, the only fine-grained measurements with perfectly retrodictable outcomes for arbitrary initial states are those which are unitarily equivalent to projective measurements. However, we also show that there exists a large class of coarse-grained generalized measurements which are highly dissimilar to projective measurements for which perfect outcome retrodiction, with an arbitrary initial state, is possible. We show that a necessary and sufficient condition for a particular POVM to have an associated, typically coarse-grained, generalized measurement whose outcome is perfectly retrodictable for all initial states is that the number of outcomes does not exceed the dimension of the output Hilbert space. We also show how such measurements can be realized in terms of the unitary-projection picture of generalized measurements, when the input and output Hilbert spaces have equal dimensionality.

In Sec. III we drop the condition of perfect retrodiction and require instead that the outcome can be retrodicted, unambiguously, with some probability instead. We also, for the most part, drop the condition that the initial state may be arbitrary, and require only that the outcome is retrodictable for at least one known, initial state. We focus on fine-grained measurements and allow for the possibility of the system being initially entangled with an additional, ancillary system. We show that, when such entanglement is permitted, the measurement operations for which this is possible are closely related to the "canonical" representations of general quantum operations, first studied by Choi [10]. These canonical representations have linearly independent Kraus operators. We find that a general sufficient and, for "finite-strength" measurements [11], which, in the fine-grained case, have nonsingular Kraus operators, necessary condition for unambiguous retrodiction of the outcome of a fine-grained generalized measurement for some, possibly entangled, initial state is that the Kraus operators are linearly independent. Every CPLTP map has a Choi canonical representation, and so every trace-preserving quantum operation has an associated fine-grained, generalized measurement amenable to unambiguous outcome retrodiction. A further consequence of our analysis, relating to unitary operator discrimination, is that a necessary and sufficient condition for unambiguous discrimination among a set of unitary operators is that they are linearly independent.

We finally examine the issue of unambiguous outcome retrodiction without entanglement. We focus on finite-strength, fine-grained measurements. For such measurements, we find that a necessary and sufficient condition for unambiguous outcome retrodiction for some nonentangled initial pure state is that the Kraus operators are not *locally* linearly dependent. We use this, together with some results recently obtained by Šemrl and co-workers [12,13] relating to locally linearly dependent operators, to explore the relationship between unambiguous outcome retrodiction without entanglement and local linear dependence. We then explore the possibility of unambiguous outcome retrodiction for every initial, pure, separable state. For fine-grained, finite-strength measurements, we find that this is possible only when the Kraus operators are locally linearly independent.

II. PERFECT OUTCOME RETRODICTION FOR ARBITRARY INITIAL STATES

Consider a quantum system $\mathcal{Q}$. Its initial state lies in a Hilbert space which we will denote by $\mathcal{H}_{\mathcal{Q}}$. Except where explicitly stated otherwise, this will have finite-dimension $D_{\mathcal{Q}}$. A generalized measurement $\mathcal{M}_{\mathcal{Q}}$ is carried out on this system. We assume that the number of possible outcomes of this measurement is also finite and shall denote this by N.

The possible outcomes of the measurement $\mathcal{M}_{\mathcal{Q}}$ will be labeled by the index $k\in\{1,\ldots,N\}$. Associated with the kth outcome is a linear, positive, quantum detection operator, or POVM element $\Pi_k:\mathcal{H}_{\mathcal{Q}}\to\mathcal{H}_{\mathcal{Q}}$. These satisfy

$$\sum_{k=1}^{N}\Pi_k=1_{\mathcal{Q}}, \tag{2.1}$$

where $1_{\mathcal{Q}}$ is the identity operator on $\mathcal{H}_{\mathcal{Q}}$. The probability of outcome k when the initial state is described by the density operator ρ is

$$P(k|\rho)=\mathrm{Tr}(\Pi_k\rho). \tag{2.2}$$

Suppose that the measurement $\mathcal{M}_{\mathcal{Q}}$ is carried out on $\mathcal{Q}$ and that the outcome is withheld from us. We do, nevertheless, have access to the final state of the system. On the basis of this, can we retrodict the measurement outcome?

To proceed, we must account for the manner in which the state of the system is transformed by the measurement process. Let $\tilde{\mathcal{H}}_{\mathcal{Q}}$ be the Hilbert space of post-measurement states. These definitions enable us to allow for the possibility that the initially prepared system and the system corresponding to the space of post-measurement states, which will subsequently be subjected to a retrodiction attempt, may be different. For example, the initial state may be that of an atom, yet the final state that of an electromagnetic field mode. However, for the sake of notational convenience, we shall denote both the initially prepared system and the final, interrogated system by the symbol $\mathcal{Q}$, as it will be clear from the context which system is being referred to.

We distinguish between two kinds of generalized measurement. We will refer to these as fine-grained measurements and coarse-grained measurements. In a fine-grained measurement, corresponding to each detection operator Π_k, there is a single Kraus operator $A_k:\mathcal{H}_{\mathcal{Q}}\to\tilde{\mathcal{H}}_{\mathcal{Q}}$ such that

$$\Pi_k=A_k^{\dagger}A_k \tag{2.3}$$

and the final, normalized state of the system when the outcome is k is given by the transformation

$$\rho\to\rho_k=\frac{A_k\rho A_k^{\dagger}}{P(k|\rho)}. \tag{2.4}$$

In a coarse-grained measurement, corresponding to the operator Π_k, there is a set of R Kraus operators A_{kr}, where $r\in\{1,\ldots,R\}$, some of which may be zero, such that

$$\Pi_k=\sum_{r=1}^{R}A_{kr}^{\dagger}A_{kr}. \tag{2.5}$$

The final, normalized state of the system when the outcome is k is given by the transformation

$$\rho\to\rho_k=\frac{\sum_{r=1}^{R}A_{kr}\rho A_{kr}^{\dagger}}{P(k|\rho)}, \tag{2.6}$$

where, in both cases, $P(k|\rho)$ is given by Eq. (2.2). We can easily see from these definitions that fine-grained measurements are a special case of coarse-grained measurements.

Given the post-measurement system, to retrodict the measurement outcome we must be able to distinguish between the k possible post-measurement states ρ_k. We will say that the retrodiction is perfect if the probability of error is zero and the retrodiction is deterministic, i.e., the probability of the attempt at retrodiction giving an inconclusive result is also zero. Perfect retrodiction will be possible only if the ρ_k are orthogonal, that is

$$\mathrm{Tr}(\rho_{k'}\rho_k)=\mathrm{Tr}(\rho_k^2)\,\delta_{kk'}, \tag{2.7}$$

or equivalently, that

$$\rho_{k'}\rho_k=\rho_k^2\delta_{kk'}. \tag{2.8}$$

Even if, for every initial state ρ, the final states ρ_k are orthogonal, it could be the case that a different measurement is required to distinguish between the final states for each initial state. So, it would appear that there are two distinct cases to consider when examining the issue of whether the outcome of a generalized measurement can be perfectly retrodicted for an arbitrary initial state, corresponding to whether the initial state is known or unknown. The former case is clearly at least as favorable as the latter, since in the former there is the possibility of tailoring the retrodicting measurement to suit the possible final states, and by implication the initial state, which we cannot do in the latter case. It follows that if perfect retrodiction of the outcome of a generalized measurement $\mathcal{M}_{\mathcal{Q}}$ is possible for an arbitrary, known, initial state, then it is also possible if the initial state is unknown. The following theorem gives a necessary and sufficient condition for perfect outcome retrodiction for all initial states, and moreover shows that there is, in fact, no advantage to be gained when the initial state, though arbitrary, is known.

Theorem 1. A quantum system $\mathcal{Q}$ is initially prepared in the state $|\psi\rangle\in\mathcal{H}_{\mathcal{Q}}$. A generalized measurement $\mathcal{M}_{\mathcal{Q}}$ with N POVM elements Π_k and Kraus operators A_{kr} satisfying Eq. (2.5) is carried out on $\mathcal{Q}$. The Hilbert space of the post-measurement states $\tilde{\mathcal{H}}_{\mathcal{Q}}$ has dimension $\tilde{D}_{\mathcal{Q}}$. A necessary and sufficient condition for the outcome of $\mathcal{M}_{\mathcal{Q}}$ to be perfectly retrodictable for every initial state $|\psi\rangle\in\mathcal{H}_{\mathcal{Q}}$ is

$$A_{k'r'}^{\dagger}A_{kr}=\delta_{kk'}A_{kr'}^{\dagger}A_{kr} \tag{2.9}$$

for all $r,r'\in\{1,\ldots,R\}$ and irrespective of whether or not $|\psi\rangle$ is known.

Proof. We will prove this theorem by establishing the necessity of condition (2.9) when the initial state is arbitrary and known. Subsequently, we will show that this condition is sufficient when the initial state is arbitrary and unknown. Thus, knowing the state confers no benefits in the context of this problem. To prove necessity, we will make use of the unnormalized final density operators

$$\tilde{\rho}_k=\sum_{r=1}^{R}A_{kr}|\psi\rangle\langle\psi|A_{kr}^{\dagger}. \tag{2.10}$$

We do this to avoid unnecessary complications which arise when the probability of one of the outcomes is zero. When this is so, the corresponding unnormalized final density operator will also be zero, but shall see that this causes no problems.

From Eq. (2.7), we see that the necessary condition for perfect outcome retrodiction given the initial state $|\psi\rangle$ is

$$\mathrm{Tr}(\tilde{\rho}_{k'}\tilde{\rho}_k)=0, \tag{2.11}$$

when $k\neq k'$ and for all $|\psi\rangle\in\mathcal{H}_Q$. This is the sole condition for perfect retrodictability we will impose in order to establish the necessity of Eq. (2.9). It says that the final states are orthogonal, which must be true if we can distinguish between them perfectly (using a projective measurement). We do not require that the same distinguishing measurement is suitable for all initial states, so we take the initial state to be known, and assume that the appropriate distinguishing measurement can always be carried out.

Substituting (2.10) into (2.11), we find that

$$\mathrm{Tr}\left(\sum_{r,r'=1}^{R}A_{k'r'}|\psi\rangle\langle\psi|A_{k'r'}^{\dagger}A_{kr}|\psi\rangle\langle\psi|A_{kr}^{\dagger}\right)=\sum_{r,r'=1}^{R}|\langle\psi|A_{k'r'}^{\dagger}A_{kr}|\psi\rangle|^2=0 \tag{2.12}$$

for $k\neq k'$. From this, we see that

$$\langle\psi|A_{k'r'}^{\dagger}A_{kr}|\psi\rangle=\delta_{kk'}\langle\psi|A_{kr'}^{\dagger}A_{kr}|\psi\rangle$$
$$\Rightarrow\langle\psi|(A_{k'r'}^{\dagger}A_{kr}-\delta_{kk'}A_{kr'}^{\dagger}A_{kr})|\psi\rangle=0 \tag{2.13}$$

for all $r,r'\in\{1,\ldots,R\}$ and all $|\psi\rangle\in\mathcal{H}_Q$, which implies Eq. (2.9). This proves necessity.

We now prove that Eq. (2.9) is a sufficient condition for perfect outcome retrodiction when the initial state is both arbitrary and unknown. We show that there exists a projective measurement which is independent of the initial state and can be used to distinguish perfectly between the post-measurement states ρ_k. Consider the following subspaces of $\tilde{\mathcal{H}}_Q$:

$$\tilde{\mathcal{H}}_{Qk}=\mathrm{supp}\left\{\sum_{r=1}^{R}A_{kr}A_{kr}^{\dagger}\right\}, \tag{2.14}$$

that is, $\tilde{\mathcal{H}}_{Qk}$ is the support of the operator $\Sigma_r A_{kr}A_{kr}^{\dagger}:\tilde{\mathcal{H}}_Q\to\tilde{\mathcal{H}}_Q$. Let $P_k:\tilde{\mathcal{H}}_Q\to\tilde{\mathcal{H}}_Q$ be the projector onto $\tilde{\mathcal{H}}_{Qk}$. We will prove that when Eq. (2.9) is satisfied, these projectors are orthogonal and form a projective measurement which can always be used to distinguish perfectly between the post-$\mathcal{M}_Q$ states.

To show that they form a projective measurement, define

$$G_k=\sum_{r=1}^{R}A_{kr}A_{kr}^{\dagger}. \tag{2.15}$$

Equation (2.9) implies that

$$G_kG_{k'}=\delta_{kk'}G_k^2. \tag{2.16}$$

It follows from this, and the positivity of the G_k, that, when $k\neq k'$, every eigenvector of G_k corresponding to a nonzero eigenvalue is orthogonal to every eigenvector of $G_{k'}$ corresponding to a nonzero eigenvalue. Let $\tilde{\mathcal{H}}_G$ be the support of the operator $\Sigma_{kr}A_{kr}A_{kr}^{\dagger}$, having dimension D_G. It follows from Eq. (2.16) that $\tilde{\mathcal{H}}_G$ has an orthonormal basis $\{|g_j\rangle\}$ in terms of which we can write

$$G_k=\sum_{j=1}^{D_G}g_{jk}|g_j\rangle\langle g_j|, \tag{2.17}$$

where

$$g_{jk}g_{j'k'}=\delta_{kk'}g_{jk}g_{j'k}\quad\forall\ \ j,j'\in\{1,\ldots,D_G\}. \tag{2.18}$$

It follows from Eq. (2.17) that

$$P_k=\sum_{j:g_{jk}\neq0}|g_j\rangle\langle g_j|. \tag{2.19}$$

Making use of Eq. (2.18), we see that these projectors are orthogonal, i.e.,

$$P_kP_{k'}=\delta_{kk'}P_k. \tag{2.20}$$

They are also complete on the space $\tilde{\mathcal{H}}_G$. To prove that a projective measurement based on these projectors can distinguish perfectly between the ρ_k, we make use of the fact that the support of $\tilde{\rho}_k$ is a subspace of $\tilde{\mathcal{H}}_{Qk}$. To prove this, we make use of the fact that the positivity of $1_Q-\rho$ implies that

$$\tilde{\rho}_k\leqslant G_k. \tag{2.21}$$

In other words,

$$\langle\phi|\tilde{\rho}_k|\phi\rangle\leqslant\langle\phi|G_k|\phi\rangle\quad\forall\ \ |\phi\rangle\in\tilde{\mathcal{H}}_Q. \tag{2.22}$$

Hence, every state $|\phi\rangle$ which is in the support of $\tilde{\rho}_k$ is also in $\tilde{\mathcal{H}}_{Qk}$, the support of G_k. Furthermore, for any final state ρ_k with nonzero outcome probability, the support of ρ_k is the same as that of $\tilde{\rho}_k$. The fact that the subspaces $\tilde{\mathcal{H}}_{Qk}$ are orthogonal and can thus be perfectly distinguished using a projective measurement on the space $\tilde{\mathcal{H}}_Q$ based on the projectors P_k enables us to distinguish between the states ρ_k with the same projective measurement, irrespective of the initial state $|\psi\rangle$. This completes the proof□.

The fact that Eq. (2.19) is a sufficient condition for perfect outcome retrodiction when $|\psi\rangle$ is an arbitrary, unknown pure state $|\psi\rangle$ can easily be seen to imply that it is also sufficient when the initial state is an arbitrary mixed state ρ.

Theorem 1 implies the following for fine-grained measurements.

Theorem 2. A quantum system $\mathcal{Q}$ is initially prepared in the state $|\psi\rangle\in\mathcal{H}_Q$. A fine-grained generalized measurement $\mathcal{M}_Q$ is carried out on $\mathcal{Q}$. If $\tilde{D}_Q = D_Q$, the outcome of $\mathcal{M}_Q$ is perfectly retrodictable for all $|\psi\rangle\in\mathcal{H}_Q$, irrespective of whether or not $|\psi\rangle$ is known, if and only if $\mathcal{M}_Q$ is a projective measurement followed by a unitary transformation from $\mathcal{H}_Q$ to $\tilde{\mathcal{H}}_Q$, that is

$$\Pi_{k'}\Pi_k = \delta_{kk'}\Pi_k, \tag{2.23}$$

where each POVM element is related to its corresponding Kraus operator in the following way:

$$A_k = U\Pi_k \tag{2.24}$$

and U is a unitary transformation from $\mathcal{H}_Q$ to $\tilde{\mathcal{H}}_Q$.

Proof. For a fine-grained measurement, we see that it follows from Eq. (2.9) that a necessary and sufficient condition for perfect outcome retrodiction with an arbitrary, known or unknown, initial state $|\psi\rangle\in\mathcal{H}_Q$ is

$$A_{k'}^\dagger A_k = \delta_{kk'}A_k^\dagger A_k. \tag{2.25}$$

Sufficiency is easily proven. When Eqs. (2.23) and (2.24) are satisfied, we see that $A_{k'}^\dagger A_k = \Pi_{k'}\Pi_k = \delta_{kk'}\Pi_k = \delta_{kk'}A_k^\dagger A_k$. This proves sufficiency. To prove necessity, we notice that, for fine-grained measurements, Eqs. (2.3) and (2.9) imply

$$A_{k'}^\dagger A_k = \Pi_k\delta_{kk'}. \tag{2.26}$$

If we sum both sides of this with respect to k and k', and make use of the resolution of the identity (2.1), we find that

$$\left(\sum_{k'=1}^{N} A_{k'}^\dagger\right)\left(\sum_{k=1}^{N} A_k\right) = 1_Q, \tag{2.27}$$

which implies that $\Sigma_{k=1}^N A_k$ is an isometry, which, if $\tilde{D}_Q = D_Q$, is necessarily unitary. We will write

$$\sum_{k=1}^{N} A_k = U. \tag{2.28}$$

Summing both sides of Eq. (2.26) over k', and making use of the adjoint of Eq. (2.28), we obtain

$$A_k = U\Pi_k. \tag{2.29}$$

Substituting this into Eq. (2.26) gives

$$\Pi_{k'}\Pi_k = \Pi_k\delta_{kk'}. \tag{2.30}$$

So, the POVM elements of the measurement $\mathcal{M}_Q$ form a set of orthogonal projectors. Thus, if perfect retrodiction of the outcome of a fine-grained measurement is possible for every initial state, even if the actual state is known, then when the input and output Hilbert spaces have the same dimension, the measurement is a projective measurement followed by a unitary transformation. This completes the proof. ■

It is natural to examine in more detail the issue of outcome retrodictability for more general, coarse-grained measurements. As we shall see, there do exist coarse-grained measurements which are highly dissimilar to projective measurements for which perfect outcome retrodiction is possible. Prior to showing this, we make the following observation which will put our findings in context. The statistical properties of a generalized measurement are determined solely by the POVM elements Π_k. These operators can always be decomposed in the manner of Eq. (2.5). This decomposition is nonunique, so a POVM with elements Π_k defines an *equivalence class* $\mathcal{E}(\{\Pi_k\})$ of measurements, each element of which corresponds to a particular coarse-grained operator-sum decomposition of the form Eq. (2.6) with fine-grained decompositions being special cases. Having these ideas in mind, we can ask the following question: under what circumstances does the equivalence class associated with a particular POVM contain a generalized measurement whose outcome is perfectly retrodictable for an arbitrary pure initial state? For generalized measurements with a finite number of outcomes, this is answered by the following theorem:

Theorem 3. Let $\mathcal{E}(\{\Pi_k\})$ be the equivalence class of generalized measurements associated with a particular POVM with $N<\infty$ elements Π_k, where these operators act on the Hilbert space $\mathcal{H}_Q$ of a quantum system $\mathcal{Q}$. This space has dimension $\mathcal{D}_Q$. The Hilbert space of the post-measurement states, $\tilde{\mathcal{H}}_Q$, has dimension $\tilde{D}_Q$. A necessary and sufficient condition for the existence of a measurement $\mathcal{M}_Q \in\mathcal{E}(\{\Pi_k\})$ whose outcome is perfectly retrodictable for an arbitrary pure initial state is

$$N\leqslant\tilde{D}_Q. \tag{2.31}$$

Proof. To prove the necessity, we make use of the fact that for every generalized measurement with $N<\infty$ outcomes, there exists a state vector $|\psi\rangle\in\mathcal{H}_Q$ such that $P(k|\psi) > 0\,\forall k\in\{1,\ldots,N\}$. To see why this is so, let $\mathcal{K}_k$ be the kernel of Π_k. None of the Π_k are equal to the zero operator, so the space $\mathcal{K}_k$ is at most D_Q-1 dimensional. It follows that if there is no vector $|\psi\rangle\in\mathcal{H}_Q$ such that $\langle\psi|\Pi_k|\psi\rangle > 0\,\forall k\in\{1,\ldots,N\}$, then every $|\psi\rangle\in\mathcal{H}_Q$ is an element of at least one of the $\mathcal{K}_k$. We conclude that $\mathcal{H}_Q=\cup_{k=1}^N\mathcal{K}_k$. This statement, that the D_Q-dimensional Hilbert space $\mathcal{H}_Q$ is the union of a finite set of Hilbert spaces of strictly lower dimension, is clearly false. For example, a two-dimensional plane is not the union of a finite set of one-dimensional rays. Hence, for each generalized measurement with a finite number of potential outcomes, there exists a pure initial state for which all of these outcomes have nonzero probability of occurrence [14].

Suppose that $\mathcal{Q}$ is initially prepared such a state. The final state corresponding to the kth outcome is ρ_k. If Eq. (2.31) is not satisfied, then the number of final states will exceed the dimension $\tilde{D}_{\mathcal{Q}}$ of $\tilde{\mathcal{H}}_{\mathcal{Q}}$. To retrodict the outcome of the measurement perfectly, we must be able to distinguish between the states ρ_k perfectly. The supports of these states must be orthogonal, which is clearly impossible if their number exceeds the dimension of $\tilde{\mathcal{H}}_{\mathcal{Q}}$. This proves necessity.

We will prove sufficiency constructively, which is to say that we will explicitly derive a measurement in the equivalence class corresponding to any POVM which satisfies Eq. (2.31) for which the outcome is perfectly retrodictable for an arbitrary pure initial state. To begin, we write the Π_k in spectral decomposition form

$$\Pi_k = \sum_{r=1}^{D_{\mathcal{Q}}} \pi_{kr} |\pi_{kr}\rangle\langle\pi_{kr}|, \tag{2.32}$$

where the π_{kr} are real and non-negative and, for each k, the set $\{|\pi_{kr}\rangle\}$ is an orthonormal basis for $\mathcal{H}_{\mathcal{Q}}$. We require a set of Kraus operators $A_{kr}:\mathcal{H}_{\mathcal{Q}} \to \tilde{\mathcal{H}}_{\mathcal{Q}}$ satisfying

$$\Pi_k = \sum_{r=1}^{D_{\mathcal{Q}}} A_{kr}^{\dagger} A_{kr} \tag{2.33}$$

for the Π_k defined by Eq. (2.32) and which satisfy the perfect retrodiction condition in Eq. (2.9). To this end, consider

$$A_{kr} = \sqrt{\pi_{kr}} |x_k\rangle\langle\pi_{kr}|, \tag{2.34}$$

where the set $\{|x_k\rangle\}$ is any set of N orthonormal states in $\tilde{\mathcal{H}}_{\mathcal{Q}}$. Notice that this construction is possible only if Eq. (2.31) is satisfied. The orthonormality of the $|x_k\rangle$ implies that the A_{kr} satisfy the perfect outcome retrodictability condition Eq. (2.9). One can also easily verify that they are related to the Π_k in Eq. (2.33) through Eq. (2.34). This completes the proof. ■

The forgoing discussion has been somewhat abstract. It would be helpful to have a concrete physical understanding of how these measurements can be implemented. Generalized measurements are commonly understood as resulting from a unitary interaction with an ancillary system, followed by a projective measurement on the latter. For $\tilde{D}_{\mathcal{Q}} = D_{\mathcal{Q}}$, we shall see here how to form a unitary-projection implementation of any POVM which satisfies Eq. (2.31) whose outcome is perfectly retrodictable given what we shall shortly refer to as a standard implementation.

We begin with the following well-known fact about generalized measurements, as described, for example, by Kraus [15]. Suppose that we have a POVM Π_k, with $k \in \{1, \ldots, \mathcal{D}_{\mathcal{Q}}\}$ which we wish to measure. This POVM may be factorized as

$$\Pi_k = B_k^{\dagger} B_k \tag{2.35}$$

for some operators $B_k:\mathcal{H}_{\mathcal{Q}} \to \tilde{\mathcal{H}}_{\mathcal{Q}}$. Let us introduce a $\mathcal{D}_{\mathcal{Q}}$-dimensional ancilla $\mathcal{A}_1$ with Hilbert space $\mathcal{H}_{\mathcal{A}_1}$, initially prepared in the state $|\chi\rangle$. For any operators B_k satisfying Eq. (2.35) and the resolution of the identity (2.1), there exists a unitary transformation $U_{\mathcal{Q}\mathcal{A}}:\mathcal{H}_{\mathcal{Q}} \otimes \mathcal{H}_{\mathcal{A}_1} \to \tilde{\mathcal{H}}_{\mathcal{Q}} \otimes \mathcal{H}_{\mathcal{A}_1}$ such that

$$U_{\mathcal{Q}\mathcal{A}_1} |\psi\rangle_{\mathcal{Q}} \otimes |\chi\rangle_{\mathcal{A}_1} = \sum_{k=1}^{\mathcal{D}_{\mathcal{Q}}} (B_k|\psi\rangle)_{\mathcal{Q}} \otimes |x_k\rangle_{\mathcal{A}_1}, \tag{2.36}$$

where $\{|x_k\rangle\}$ is an orthonormal basis set for $\mathcal{H}_{\mathcal{A}_1}$. A measurement on $\mathcal{A}_1$ in this basis, yielding the result k, transforms the state of $\mathcal{Q}$ from $|\psi\rangle$ into $B_k|\psi\rangle/\sqrt{P(k|\psi)}$, with probability $P(k|\psi) = \langle\psi|\Pi_k|\psi\rangle$. We will refer to this construction as a standard implementation of a POVM.

To obtain from this measurement a perfectly retrodictable one which is also in the equivalence class of the same POVM, we introduce a further ancilla $\mathcal{A}_2$ with $\tilde{\mathcal{D}}_{\mathcal{Q}}$-dimensional Hilbert space $\mathcal{H}_{\mathcal{A}_2}$, also initially prepared in the state $|\chi\rangle$. Following the action of $U_{\mathcal{Q}\mathcal{A}}$, we apply a unitary copying transformation on $\mathcal{A}_1\mathcal{A}_2$ which perfectly copies the orthogonal states $|x_k\rangle$, that is,

$$\mathrm{COPY}_{\mathcal{A}_1\mathcal{A}_2} |x_k\rangle_{\mathcal{A}_1} \otimes |\chi\rangle_{\mathcal{A}_2} = |x_k\rangle_{\mathcal{A}_1} \otimes |x_k\rangle_{\mathcal{A}_2}. \tag{2.37}$$

Since $\tilde{D}_{\mathcal{Q}} = D_{\mathcal{Q}}$, we can carry out the SWAP operation on $\mathcal{Q}\mathcal{A}_1$, which exchanges the states of these two systems. The entire unitary interaction between $\mathcal{Q}$ and the ancilla $\mathcal{A}_1\mathcal{A}_2$ is then

$$\begin{aligned}\mathrm{SWAP}_{\mathcal{Q}\mathcal{A}_1} \mathrm{COPY}_{\mathcal{A}_1\mathcal{A}_2} U_{\mathcal{Q}\mathcal{A}_1} |\psi\rangle_{\mathcal{Q}} \otimes |\chi\rangle_{\mathcal{A}_1} \otimes |\chi\rangle_{\mathcal{A}_2} \\ = \sum_{k=1}^{\mathcal{D}_{\mathcal{Q}}} |x_k\rangle_{\mathcal{Q}} \otimes (B_k|\psi\rangle)_{\mathcal{A}_1} \otimes |x_k\rangle_{\mathcal{A}_2}.\end{aligned} \tag{2.38}$$

Following this unitary interaction, we carry out a projective measurement on $\mathcal{A}_1\mathcal{A}_2$, with the projection operators

$$P_k = 1_{\mathcal{A}_1} \otimes (|x_k\rangle\langle x_k|)_{\mathcal{A}_2}. \tag{2.39}$$

The probability $P(k|\psi)$ of the kth outcome is easily shown to be $\langle\psi|\Pi_k|\psi\rangle$. The final state of $\mathcal{Q}$ is obtained by tracing the entire final state over the ancilla. If we write

$$V = \mathrm{SWAP}_{\mathcal{Q}\mathcal{A}_1} \mathrm{COPY}_{\mathcal{A}_1\mathcal{A}_2} U_{\mathcal{Q}\mathcal{A}_1}, \tag{2.40}$$

where V is clearly unitary, then when outcome k is obtained for the measurement based on the projectors P_k in Eq. (2.39), the state of $\mathcal{Q}$ is transformed by the following completely positive, linear, trace non-increasing map:

$$\begin{aligned}\Phi_k(\rho_{\mathcal{Q}}) &= \mathrm{Tr}_{\mathcal{A}_1\mathcal{A}_2}(P_k V(\rho_{\mathcal{Q}} \otimes |\chi\rangle\langle\chi|_{\mathcal{A}_1} \otimes |\chi\rangle\langle\chi|_{\mathcal{A}_2}) V^{\dagger}), \\ &= |x_k\rangle\langle x_k|.\end{aligned} \tag{2.41}$$

So, there is a one-to-one correspondence between the measurement outcomes and the orthonormal states $|x_k\rangle$. This implies that the result of the measurement is perfectly retrodictable for an arbitrary initial quantum state.

It is often helpful to make use of the fact that every such map has an operator-sum decomposition. In this case, we have

$$\Phi_k(\rho_{\mathcal{Q}})=\sum_{r=1}^{D_{\mathcal{Q}}} A_{kr}\rho_{\mathcal{Q}}A_{kr}^{\dagger} \tag{2.42}$$

for some operators A_{kr}. After some algebra, we find that we may write

$$A_{kr}=|x_k\rangle\langle x_r|B_k. \tag{2.43}$$

These are given by Eq. (2.34) if we take

$$B_k=\sum_{r=1}^{\mathcal{D}_{\mathcal{Q}}} \sqrt{\pi_{kr}}|x_r\rangle\langle \pi_{kr}|. \tag{2.44}$$

One can show, using Eqs. (2.32) and (2.33), that these operators satisfy Eq. (2.35). We have thus shown how to form from a standard implementation of a POVM one whose outcome is perfectly retrodictable for an arbitrary initial state when $\tilde{D}_{\mathcal{Q}}=D_{\mathcal{Q}}$.

III. UNAMBIGUOUS OUTCOME RETRODICTION

A. With entanglement

In the preceding section, we addressed the issue of perfectly retrodicting the outcome of a generalized measurement $\mathcal{M}_{\mathcal{Q}}$ on a quantum system $\mathcal{Q}$ by examining the final state when the initial state is arbitrary. Here we impose the less stringent condition that for some known, initial state, the outcome can always be retrodicted, unambiguously, which is to say with zero probability of error, with some nonzero probability instead. We allow for the possibility that the retrodiction attempt gives an inconclusive result.

The issues that we discuss in this subsection are insensitive to the dimension $\tilde{D}_{\mathcal{Q}}$ of $\tilde{\mathcal{H}}_{\mathcal{Q}}$, provided that $\tilde{D}_{\mathcal{Q}}\geqslant D_{\mathcal{Q}}$. For maximum generality, we should assume, and take advantage of the fact that $\mathcal{Q}$ can be initially entangled with some ancillary system $\mathcal{A}$, with corresponding Hilbert space $\mathcal{H}_{\mathcal{A}}$, having finite-dimension $D_{\mathcal{A}}$. These systems are initially prepared in a joint state with corresponding density operator $\rho_{\mathcal{QA}}$. The measurement $\mathcal{M}_{\mathcal{Q}}$ is carried out on $\mathcal{Q}$. For the sake of simplicity, we will consider only fine-grained measurements. Here, the final, normalized state corresponding to outcome k is obtained by the transformation

$$\rho_{\mathcal{QA}}\rightarrow\rho_{\mathcal{QA}k}=\frac{(A_k\otimes 1_{\mathcal{A}})\rho_{\mathcal{QA}}(A_k^{\dagger}\otimes 1_{\mathcal{A}})}{P(k|\rho_{\mathcal{QA}})}. \tag{3.1}$$

Our aim is to retrodict the outcome of the measurement $\mathcal{M}_{\mathcal{Q}}$ by distinguishing between the states $\rho_{\mathcal{QA}k}$. To do this, we must perform a second measurement $\mathcal{M}_{\mathcal{QA}}$ on $\mathcal{QA}$. This will be tailored so that its outcome matches that of $\mathcal{M}_{\mathcal{Q}}$ as closely as possible. As we are interested in situations where the outcome is retrodicted unambiguously, the measurement $\mathcal{M}_{\mathcal{QA}}$ will have $(N+1)$ outcomes: N of these correspond to the possible outcomes of $\mathcal{M}_{\mathcal{Q}}$ and a further signals the failure of the retrodiction attempt, making this result inconclusive. So, we may represent this measurement by an $(N+1)$-element POVM $(\Xi_0,\Xi_1,\ldots,\Xi_N)$ for which

$$\sum_{k=0}^{N}\Xi_k=1_{\mathcal{QA}}. \tag{3.2}$$

The condition for error-free unambiguous outcome retrodiction may be written as

$$\mathrm{Tr}(\Xi_{k'}\rho_{\mathcal{QA}k})=\mathrm{Tr}(\Xi_k\rho_{\mathcal{QA}k})\,\delta_{kk'} \tag{3.3}$$

for $\mathrm{Tr}(\Xi_k\rho_{\mathcal{QA}k})>0\ \forall\, k,k'\in\{1,\ldots,N\}$. The probability that the retrodiction attempt gives an inconclusive result is

$$P(?|\rho_{\mathcal{QA}})=\mathrm{Tr}\left(\Xi_0\sum_{k=0}^{N}(A_k\otimes 1_{\mathcal{A}})\rho_{\mathcal{QA}}(A_k^{\dagger}\otimes 1_{\mathcal{A}})\right). \tag{3.4}$$

Under what conditions does there exist an initial state $\rho_{\mathcal{QA}}$ for which the outcome of the fine-grained measurement $\mathcal{M}_{\mathcal{Q}}$ is unambiguously retrodictable? To address this question, we may, without loss of generality take the initial state to be a pure state $\rho_{\mathcal{QA}}=|\psi_{\mathcal{QA}}\rangle\langle\psi_{\mathcal{QA}}|$, since any mixed state can be purified by considering a sufficiently large ancilla $\mathcal{A}$. The Schmidt decomposition theorem implies that we can always take the dimensionality of $\mathcal{H}_{\mathcal{A}}$ to be at most $D_{\mathcal{Q}}$. We will now prove:

Theorem 4. A sufficient condition for the existence of an initial state $|\psi_{\mathcal{QA}}\rangle\in\mathcal{H}_{\mathcal{QA}}$ for which the outcome of a fine-grained measurement $\mathcal{M}_{\mathcal{Q}}$ is unambiguously retrodictable is that the corresponding Kraus operators are linearly independent. When this is the case, the outcome of $\mathcal{M}_{\mathcal{Q}}$ is unambiguously retrodictable for any known $|\psi_{\mathcal{QA}}\rangle\in\mathcal{H}_{\mathcal{QA}}$ with maximum Schmidt rank. When the Kraus operators are nonsingular, linear independence is also a necessary condition for the existence of an initial state $|\psi_{\mathcal{QA}}\rangle\in\mathcal{H}_{\mathcal{QA}}$ for which the outcome of $\mathcal{M}_{\mathcal{Q}}$ can be unambiguously retrodicted.

Proof. We will first prove necessity for nonsingular Kraus operators. Consider the final states

$$|\psi_{\mathcal{QA}k}\rangle=P(k|\psi_{\mathcal{QA}})^{-1/2}(A_k\otimes 1_{\mathcal{A}})|\psi_{\mathcal{QA}}\rangle. \tag{3.5}$$

If the A_k are nonsingular, then the corresponding probabilities $P(k|\psi_{\mathcal{QA}})$ will be nonzero for all $|\psi_{\mathcal{QA}}\rangle\in\mathcal{H}_{\mathcal{QA}}$. If the A_k are linearly dependent, then there exist coefficients α_k, not all of which are zero, such that $\Sigma_k\alpha_kA_k=0$. It is then a simple matter to show that $\Sigma_k\beta_k|\psi_{\mathcal{QA}k}\rangle=0$, where $\beta_k=\alpha_kP(k|\psi_{\mathcal{QA}})^{1/2}$ and that these are not all zero. Hence the final states are linearly dependent and cannot be unambiguously distinguished [7], so for no initial state can the outcome of the measurement $\mathcal{M}_{\mathcal{Q}}$ be unambiguously retrodicted.

We now prove, again by contradiction, that linear independence of the A_k is a sufficient condition for being able to unambiguously retrodict the outcome of $\mathcal{M}_{\mathcal{Q}}$ when the initial state $|\psi_{\mathcal{QA}}\rangle\in\mathcal{H}_{\mathcal{QA}}$ has maximum Schmidt rank. To do this, we make use of the fact that linear independence of the final states is a sufficient condition for them being amenable to unambiguous discrimination [7]. We write $|\psi_{\mathcal{QA}}\rangle$ in Schmidt decomposition form

$$|\psi_{\mathcal{QA}}\rangle=\sum_{j=1}^{D_{\mathcal{Q}}} c_j|x_j\rangle_{\mathcal{Q}}\otimes|y_j\rangle_{\mathcal{A}}, \tag{3.6}$$

where $\{|x_j\rangle\}$ is an orthonormal basis for $\mathcal{H}_{\mathcal{Q}}$ and $\{|y_j\rangle\}$ is an orthonormal subset of $\mathcal{H}_{\mathcal{A}}$. When outcome k is obtained, the post-measurement state is

$$|\psi_{\mathcal{QA}k}\rangle=P(k|\psi_{\mathcal{QA}})^{-1/2}\sum_{j=1}^{D_{\mathcal{Q}}} c_j(A_k|x_j\rangle_{\mathcal{Q}})\otimes|y_j\rangle_{\mathcal{A}}, \tag{3.7}$$

where the probability of outcome k is

$$P(k|\psi_{\mathcal{QA}})=\sum_{j=1}^{D_{\mathcal{Q}}} |c_j|^2\langle x_j|\Pi_k|x_j\rangle. \tag{3.8}$$

We will assume that $|\psi_{\mathcal{QA}}\rangle$ has maximum Schmidt rank, that is, that all of the c_j are nonzero. For any initial state with this property, all of the outcome probabilities $P(k|\psi_{\mathcal{QA}})$ are nonzero, even if some of the A_k are singular. To prove this, let $c>0$ be the smallest of the $|c_j|$. Then $P(k|\psi_{\mathcal{QA}})\geqslant c^2\Sigma_{j=1}^{D_{\mathcal{Q}}}\langle x_j|\Pi_k|x_j\rangle=c^2\mathrm{Tr}(\Pi_k)$. The Π_k are positive operators, which, while not necessarily being positive definite, are nevertheless nonzero. Hence, $\mathrm{Tr}(\Pi_k)>0\ \ \forall k\in\{1,\ldots,N\}$. From this, it follows that $P(k|\psi_{\mathcal{QA}})>0\ \ \forall k\in\{1,\ldots,N\}$.

Suppose now that unambiguous outcome retrodiction is impossible, that is, that the final states $|\psi_{\mathcal{QA}k}\rangle$ are linearly dependent. There would then exist coefficients α_k, not all of which are zero, such that

$$\sum_{k=1}^{N} \alpha_k|\psi_{\mathcal{QA}k}\rangle=0. \tag{3.9}$$

If we again let $\beta_k=\alpha_kP(k|\psi_{\mathcal{QA}})^{1/2}$, then we see that these are not all zero and that, with the help of Eq. (3.7), this linear dependence condition can be written as

$$\sum_{k=1}^{N} \beta_k\sum_{j'=1}^{D_{\mathcal{Q}}} c_{j'}(A_k|x_{j'}\rangle_{\mathcal{Q}})\otimes|y_{j'}\rangle_{\mathcal{A}}=0. \tag{3.10}$$

Taking the partial inner product of this with $\langle y_j|$ and dividing the result by c_j, we find

$$\sum_{k=1}^{N} \beta_kA_k|x_j\rangle=0\ \forall\ j\in\{1,\ldots,D_{\mathcal{Q}}\}. \tag{3.11}$$

Finally, we make use of the completeness of the $|x_j\rangle$ and see that this, when combined with Eq. (3.11), gives

$$\sum_{k=1}^{N} \beta_kA_k=\sum_{k=1}^{N} \beta_kA_k\sum_{j=1}^{D_{\mathcal{Q}}} |x_j\rangle\langle x_j|=0, \tag{3.12}$$

that is, the A_k must be linearly dependent. So, for an initial state which is pure with maximum Schmidt rank, if the final states are unamenable to unambiguous discrimination, which is to say that they are linearly dependent, then the Kraus operators are also linearly dependent. This completes the proof. ■

This theorem has some interesting consequences that we shall now describe. The first is in relation to general quantum operations. These are described by completely positive, linear, trace nonincreasing maps $\rho\to\Phi(\rho)=\Sigma_{k=1}^{N}A_k\rho A_k^{\dagger}$, where $\Sigma_{k=1}^{N}A_k^{\dagger}A_k\leqslant 1_{\mathcal{Q}}$. In a well-known theorem, Choi [10] showed that every such map has an operator-sum decomposition in terms of linearly independent Kraus operators A_k. Combining this fact with Theorem 4, we see that for each trace-preserving quantum operation Φ, there exists a fine-grained generalized measurement whose Kraus operators form an operator-sum decomposition of Φ and whose outcome is unambiguously retrodictable for all pure initial states with maximum Schmidt rank.

A second consequence of this theorem relates to the problem of distinguishing between unitary operators. Childs *et al.* [16] and Acín [17] have addressed the problem of distinguishing between a pair of unitary operators. Theorem 4 enables us to say something about the more general problem of distinguishing between N unitary operators.

The problem is this: a quantum system $\mathcal{Q}$ and an ancilla $\mathcal{A}$ are initially prepared in the possibly entangled state $\rho_{\mathcal{QA}}$. With probability p_k, $\mathcal{Q}$ is subjected to one of the N unitary operators U_k. The entire state undergoes the transformation

$$\rho_{\mathcal{QA}}\to\rho_{\mathcal{QA}k}=(U_k\otimes 1_{\mathcal{A}})\rho_{\mathcal{QA}}(U_k^{\dagger}\otimes 1_{\mathcal{A}}) \tag{3.13}$$

with probability p_k. The aim is to determine which unitary operator has been applied. This is done by distinguishing between the final states $\rho_{\mathcal{QA}k}$.

Comparison of Eq. (3.13) with Eq. (3.1) shows that this procedure can be regarded as a particular example of retrodiction of the outcome of a fine-grained generalized measurement, specifically one which has the Kraus operators

$$A_k=\sqrt{p_k}U_k. \tag{3.14}$$

Clearly, when all of the p_k are nonzero, then linear independence of the A_k is equivalent to that of the U_k. It follows from this and the nonsingularity of unitary operators that a necessary and sufficient condition for being able to unambiguously discriminate between N unitary operators U_k for some, possibly entangled, initial state is that they are linearly independent.

Theorem 4 gives a special status to generalized measurements with nonsingular Kraus operators. Measurements of this kind might appear to be somewhat artificial constructions. After all, neither projective measurements nor many of the optimal generalized measurements for the various kinds of state discrimination have this property [4–7]. However, it has recently been suggested by Fuchs and Jacobs [11] that such measurements may, in practice, be the rule rather than the exception. They argue that a measurement for which a particular outcome is impossible to achieve for some initial state is an idealization that would require infinite resources to implement (infinite precision in tuning interactions, timings etc.) Accordingly, realistic, *finite-strength* measurements do not possess this property and have nonsingular POVM elements or equivalently, for fine-grained measurements, Kraus operators.

Of course, this reasoning also applies to the measurement which retrodicts the outcome of $\mathcal{M}_{\mathcal{Q}}$. Unambiguous outcome retrodiction will, in general, require that the Kraus operators of the retrodicting measurement are highly singular. While, for the reasons given above, this is difficult, even impossible to achieve in practice, there are, as far as we are aware, no fundamental limitations on how well these idealized measurements can be approximately implemented. Finite-strength measurements will have a special status with regard to unambiguous outcome retrodiction if the measurement whose outcome we are trying to retrodict is not as strong as the retrodicting measurement.

It should also be noted that when some of the Kraus operators are singular, linear independence is not, in general, a necessary condition for unambiguous outcome retrodiction for some initial state. As a counter example, consider the case of $\mathcal{H}_{\mathcal{Q}}$ being three dimensional and spanned by the orthonormal vectors $|x\rangle,|y\rangle$ and $|z\rangle$. Consider now a fine-grained measurement with the singular, linearly dependent Kraus operators

$$A_1=\frac{|x\rangle\langle x|}{\sqrt{2}}, \tag{3.15}$$

$$A_2=\frac{|y\rangle\langle y|}{\sqrt{2}}, \tag{3.16}$$

$$A_3=|z\rangle\langle z|, \tag{3.17}$$

$$A_4=\frac{|x\rangle\langle x|+|y\rangle\langle y|}{\sqrt{2}}. \tag{3.18}$$

If the initial state is $|z\rangle$, then we know *a priori* that the only possible outcome is 3, so knowing that this state was prepared enables us to perfectly retrodict the outcome without having access to the measurement record.

B. Without entanglement

The final issue we shall investigate is unambiguous outcome retrodiction without entanglement. For the sake of simplicity, we again confine our attention to fine-grained measurements. When $\mathcal{Q}$ is initially prepared in the pure state $|\psi\rangle$, then the final state corresponding to the kth outcome is, up to a phase

$$|\psi_k\rangle=\frac{A_k|\psi\rangle}{\sqrt{P(k|\psi)}}, \tag{3.19}$$

when the probability $P(k|\psi)$ of the kth outcome is nonzero. Unambiguous retrodiction of the outcome of $\mathcal{M}_{\mathcal{Q}}$ with the initial state $|\psi\rangle$ is possible only if the final states which have nonzero probability are linearly independent. Actually, in what follows it will, for reasons that will become apparent, be more convenient to enquire as to when unambiguous retrodiction is *impossible* for every pure initial state in $\mathcal{H}_{\mathcal{Q}}$. Let $\sigma(\psi)$ be the subset of $\{1,\ldots,N\}$ for which $A_k|\psi\rangle\neq 0$ when $k\in\sigma(\psi)$. Then unambiguous retrodiction of the outcome of $\mathcal{M}_{\mathcal{Q}}$ is impossible for every pure initial state $|\psi\rangle\in\mathcal{H}_{\mathcal{Q}}$ iff there exist coefficients $\alpha_k(\psi)$, not all of which are zero for $k\in\sigma(\psi)$, such that

$$\left(\sum_{k\in\sigma(\psi)}\alpha_k(\psi)A_k\right)|\psi\rangle=0 \tag{3.20}$$

for all $|\psi\rangle\in\mathcal{H}_{\mathcal{Q}}$, which is to say iff the possible final states are linearly dependent for all initial states. In particular, for a finite-strength measurement, the A_k are nonsingular and so $\sigma(\psi)=\{1,\ldots,N\}$. In this case, the impossibility condition is that for each $|\psi\rangle\in\mathcal{H}_{\mathcal{Q}}$, there exist coefficients $\alpha_k(\psi)$, not all of which are zero, such that

$$\left(\sum_{k=1}^{N}\alpha_k(\psi)A_k\right)|\psi\rangle=0. \tag{3.21}$$

Operators A_k with this property are said to be *locally linearly dependent*.

Locally linearly dependent sets of operators have been investigated in detail by Šemrl and co-workers [12,13]. Notice that local linear dependence is weaker than linear dependence, which is the special case of the α_k being independent of $|\psi\rangle$.

Equivalently, it is necessary, though not sufficient for a set of operators to be linearly independent to not be locally linearly dependent. Consequently, it is sufficient for the outcome of a finite-strength, fine-grained measurement to be unambiguously retrodictable for a single unentangled pure state for it to be unambiguously retrodictable for all maximum Schmidt rank entangled states, but not vice versa. Consider, for example, the four, nonsingular, Pauli operators $(1,\sigma_x,\sigma_y,\sigma_z)$. Though linearly independent, these operators are locally linearly dependent. So, as far as pure states are concerned, an entangled initial state is required to unambiguously determine which operator has been implemented, as in dense coding [18]. More generally, the nonsingularity of unitary operators implies that a necessary and sufficient condition for a set of unitary operators to be unambiguously distinguishable with a pure, nonentangled initial state is that they are not locally linearly dependent.

Having made the distinction between linear dependence and local linear dependence, which is responsible for the fact that there exist finite-strength measurements whose outcomes are unambiguously retrodictable for some entangled but no unentangled, pure, initial states, one particular question forces itself upon us: given that the outcome of a measurement is unambiguously retrodictable with an entangled, pure initial state, what subsidiary conditions must the measurement satisfy for its outcome to be unambiguously retrodictable for some nonentangled, pure initial state? For finite-strength measurements, this question is equivalent to: under what conditions is a linearly independent set of Kraus operators, subject, of course, to the resolution of the identity, not a locally linearly dependent set?

The problem of determining when a linearly independent set of operators is not a locally linearly dependent set has been solved for the special cases $N=2,3$ [12]. The solutions for $N\geqslant 4$ are not known at this time. Progress has, however, been made with regard to this problem. It has been shown by Brešar and Šemrl [12] that the solution for arbitrary N can be deduced from that of the problem of classifying the maximal vector spaces of $N\times N$ matrices with zero determinant. However, this is also currently unknown.

We will examine here the solution for $N=2$ and unravel its implications. Here, we are considering a fine-grained measurement with two outcomes having corresponding Kraus operators A_1 and A_2. Brešar and Šemrl [12] have shown that the following two statements are equivalent:

(i) A_1 and A_2 are locally linearly dependent.
(ii) (a) A_1 and A_2 are linearly dependent or (b) there exists a vector $|\phi\rangle\in\tilde{\mathcal{H}}_Q$ such that $\mathrm{span}\{A_1|\psi\rangle:|\psi\rangle\in\mathcal{H}_Q\}=\mathrm{span}\{A_2|\psi\rangle:|\psi\rangle\in\mathcal{H}_Q\}=\tilde{\mathcal{H}}_\phi$, where $\tilde{\mathcal{H}}_\phi$ is the one-dimensional subspace of $\tilde{\mathcal{H}}_Q$ spanned by $|\phi\rangle$.

It follows that if A_1 and A_2 are linearly independent and also locally linearly dependent, then condition (iib) must be satisfied. This condition, when combined with the resolution of the identity, implies that $D_Q=2$ and that $\mathcal{H}_Q$ has an orthonormal basis $\{|x\rangle,|y\rangle\}$ such that

$$A_1=|\phi\rangle\langle x|, \tag{3.22}$$

$$A_2=|\phi\rangle\langle y|. \tag{3.23}$$

These operators are clearly singular. It follows that for every two-outcome, fine-grained, finite-strength measurement, if the Kraus operators are not linearly dependent, then they are not locally linearly dependent either. So, for such measurements, if the outcome can be unambiguously retrodicted for some entangled initial state, then it can also be unambiguously retrodicted for some nonentangled initial pure state.

Let us conclude with an examination of the possibility of unambiguous outcome retrodiction for all initial states $|\psi\rangle\in\mathcal{H}_Q$. For a finite-strength, fine-grained measurements, the necessary and sufficient condition for this to be possible is that for every pure, initial state, the set of N pure, post-measurement states is a linearly independent set. Formally, this requirement can be expressed as

$$\left(\sum_{k=1}^{N}\alpha_k A_k\right)|\psi\rangle\neq 0, \tag{3.24}$$

for all nonzero $|\psi\rangle\in\mathcal{H}_Q$ and all complex coefficients α_k unless $\alpha_k=0\ \ \forall k\in\{1,\ldots,N\}$. A set of operators A_k with this property can be said to be *locally linearly independent*. Local linear dependence and local linear independence are not, like linear dependence and independence, complementary concepts. For example, no set of two Pauli operators is either locally linearly dependent or locally linearly independent.

Local linear independence is a considerably stronger condition than linear independence. So strong, in fact, that if $\mathcal{H}_Q$ and $\tilde{\mathcal{H}}_Q$ are finite dimensional and $\tilde{D}_Q\leqslant D_Q$, then it cannot be satisfied (except in the trivial case of the equality and a single, nonsingular operator). It is easy to see that locally linearly independent operators must be nonsingular, so that this condition cannot be satisfied if $\tilde{D}_Q<D_Q$. To prove that it cannot be satisfied when $\tilde{D}_Q=D_Q$ either, we make use of the fact that any subset of a locally linearly independent set must also be locally linearly independent. Let us then consider just two operators, A_1 and A_2. These operators must be nonsingular. This implies, in the finite-dimensional case, that if $\tilde{D}_Q=D_Q$, each of them has a unique left and right inverse. These are, of course, also nonsingular.

Given that A_1 and A_2 are nonsingular, it follows that $A_1^{-1}A_2$ must also be nonsingular. It then has D_Q linearly independent eigenvectors with nonzero eigenvalues. Let $\lambda\neq 0$ be an eigenvalue of $A_1^{-1}A_2$ with corresponding eigenvector $|\psi\rangle$. Now consider

$$A_1^{-1}(-\lambda A_1+A_2)|\psi\rangle=(-\lambda+\lambda)|\psi\rangle=0. \tag{3.25}$$

Operating throughout this equation with A_1, we find that

$$(-\lambda A_1+A_2)|\psi\rangle=0, \tag{3.26}$$

and so the operators A_1 and A_2 cannot be locally linearly independent. From this, it follows that, for finite-dimensional quantum systems, if the dimension of the output Hilbert space does not exceed that of the input Hilbert space, then fine-grained measurements with locally linearly independent Kraus operators are impossible. However, one can devise examples of such measurements for finite-dimensional quantum systems if the output Hilbert space has higher dimension than the input Hilbert space. Let $D_Q=2$ and $\tilde{D}_Q=4$. Also, let $\{|x_1\rangle,|x_2\rangle\}$ and $\{|\tilde{x}_1\rangle,|\tilde{x}_2\rangle,|\tilde{x}_3\rangle,|\tilde{x}_4\rangle\}$ be orthonormal basis sets for $\mathcal{H}_Q$ and $\tilde{\mathcal{H}}_Q$, respectively. Consider now a two-outcome, fine-grained measurement whose Kraus operators have the following matrix representations in these bases:

$$A_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix},\quad A_2=\frac{1}{\sqrt{2}}\begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}, \tag{3.27}$$

that is, the row-j', column-j element of A_k is $\langle\tilde{x}_{j'}|A_k|x_j\rangle$. One can easily verify that $A_1^\dagger A_1+A_2^\dagger A_2=1_Q$, and so these operators constitute a fine-grained measurement. To prove that they are locally linearly independent, let us write an arbitrary pure initial state in $\mathcal{H}_Q$ as $|\psi\rangle=c_1|x_1\rangle+c_2|x_2\rangle$. Then,

$$(\alpha_1A_1+\alpha_2A_2)|\psi\rangle = \frac{c_1(\alpha_1|\tilde{x}_1\rangle+\alpha_2|\tilde{x}_2\rangle)+c_2(\alpha_1|\tilde{x}_3\rangle+\alpha_2|\tilde{x}_4\rangle)}{\sqrt{2}}. \tag{3.28}$$

As a consequence of the orthonormality of the $|\tilde{x}_{j'}\rangle$, when either or both c_1 and c_2 are nonzero, this expression is never equal to the zero vector unless α_1 and α_2 are equal to 0. Hence, the operators A_1 and A_2 are locally linearly independent. In fact, these operators satisfy the condition in Eq. (2.9) for *perfect* retrodiction for an arbitrary initial state condition in $\mathcal{H}_Q$.

There also exist interesting examples of measurements with locally linearly independent Kraus operators on infinite-dimensional quantum systems. Consider a bosonic mode with Hilbert space spanned by the orthonormal occupation number states $|n\rangle$, $n=0,1,2,\ldots$. Now consider a two-outcome generalized measurement with the Kraus operators $A_1=\mu\Sigma_{n=0}^{\infty}|n+1\rangle\langle n|$ and $A_2=\sqrt{1-|\mu|^2}\Sigma_{n=0}^{\infty}|n\rangle\langle n|$, where $0<|\mu|<1$. It is a simple matter to show that $A_1^{\dagger}A_1+A_2^{\dagger}A_2=\Sigma_{n=0}^{\infty}|n\rangle\langle n|=1$, so that these operators do indeed form a fine-grained generalized measurement. To show that these operators are locally linearly independent, let the initial state of the system be the pure state $|\psi\rangle=\Sigma_{n=0}^{\infty}c_n|n\rangle$, where at least one of the c_n is nonzero. The operators A_1 and A_2 will be locally linearly independent iff, for every such state, and for every pair of complex coefficients α_1 and α_2, at least one of which is nonzero,

$$(\alpha_1 A_1+\alpha_2 A_2)|\psi\rangle\neq 0. \tag{3.29}$$

To show that this is so, let n_0 be the smallest value of n for which $c_n\neq 0$. It follows then that $\langle n_0|A_1|\psi\rangle=0$ and $\langle n_0|A_2|\psi\rangle=\sqrt{1-|\mu|^2}c_{n_0}$. Hence

$$\langle n_0|(\alpha_1 A_1+\alpha_2 A_2)|\psi\rangle=\alpha_2\sqrt{1-|\mu|^2}c_{n_0}, \tag{3.30}$$

which is nonzero for nonzero α_2, implying that when $\alpha_2\neq 0$, Eq. (3.29) is satisfied. To show that it is also satisfied when $\alpha_2=0$, we simply make use of the fact that if this were not the case, then we would have $A_1|\psi\rangle=0$, which is not true. We can see this, for example, by making use of the fact that $\langle n_0+1|A_1|\psi\rangle=\mu c_{n_0}\neq 0$.

The key property which makes the operators A_1 and A_2 defined above a locally linearly independent set is the fact that A_1 has no eigenvalues/eigenvectors. In fact, it is straightforward to prove that, for any pair of nonsingular operators A_1 and A_2, if A_2 is proportional to the identity, then local linear independence of A_1 and A_2 is equivalent to the condition that A_1 has no eigenvalues/eigenvectors [19].

IV. DISCUSSION

In this paper, we have addressed the following problem: suppose that a generalized measurement has been carried out on a quantum system. We do not know the outcome of the measurement. We do, however, know which measurement has been carried out and have access to the system following the measurement. We are free to interrogate the final state in any way which is physically possible. Our aim is to devise a suitable ''retrodicting'' measurement which will reveal the outcome of the first measurement.

This task is simple if the initial measurement is a projective measurement; if there is no irreversible evolution following this measurement, then we can simply reverse any evolution that occurs and perform the same measurement again. Generalized measurements do not, however, possess the repeatability property which is responsible for the straightforward nature of outcome retrodiction for projective measurements. In Sec. II, we derived a necessary and sufficient condition on the Kraus transformation operators for the outcome of a generalized measurement to be perfectly retrodictable for an arbitrary initial state. We also showed that there is no advantage to be gained if the initial state, though arbitrary, is known.

When the input and output Hilbert spaces have the same dimension, the only fine-grained measurements which satisfy this condition are projective measurements, possibly followed by an outcome-independent unitary transformation. We also showed that every POVM can be realized by a measurement whose outcome is perfectly retrodictable for all initial states iff the number of outcomes does not exceed the output Hilbert space dimension. We also described an algorithm by which such an implementation can be constructed using a standard implementation. This applies when the input and output Hilbert spaces have equal dimensionality and essentially involves swapping the information contained in the measuring apparatus and the system following the measurement.

We then addressed the problem of unambiguously retrodicting the outcome of a generalized measurement, with zero probability of error but with a possible nonzero probability of the retrodiction attempt giving an inconclusive result. We addressed this issue in Sec. III, focusing on fine-grained measurements. The fact that only linearly independent pure, final states can be unambiguously discriminated places constraints on the Kraus operators of such measurements. We showed that if entanglement with an ancillary system is possible, then a sufficient and, for finite-strength measurements, necessary condition is that the Kraus operators are linearly independent. This result has interesting connections with a theorem due to Choi and also with the problem of unambiguously discriminating between unitary operators.

When the initial state is pure and entanglement is not permitted, we have shown that the issue of unambiguous outcome retrodiction is closely related to the concepts of operator local linear dependence and local linear independence. While being interesting in their own right, our demonstration that these concepts are relevant to quantum measurement theory gives a further incentive to explore them.

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[19] If $(\alpha_1 A_1+\alpha_2 A_2)|\psi\rangle=0$, then $\alpha_1\neq 0$ and $A_1|\psi\rangle=-(\alpha_2/\alpha_1)\sqrt{1-|\mu|^2}|\psi\rangle$ and so $|\psi\rangle$ is an eigenvector of A_1 with eigenvalue $-(\alpha_2/\alpha_1)\sqrt{1-|\mu|^2}$. Conversely, if $A_1|\psi\rangle=\lambda|\psi\rangle$ then $[A_1-(\lambda/\sqrt{1-|\mu|^2})A_2]|\psi\rangle=0$ and so A_1 and A_2 are not locally linearly independent.