Jordan blocks and Gamow-Jordan eigenfunctions associated with a degeneracy of unbound states

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An accidental degeneracy of unbound states gives rise to a double pole in the scattering matrix, a double zero in the Jost function, and a Jordan chain of length 2 of generalized Gamow-Jordan eigenfunctions of the radial Schrödinger equation. The generalized Gamow-Jordan eigenfunctions are basis elements of an expansion in bound- and resonant-state energy eigenfunctions plus a continuum of scattering wave functions of a complex wave number. In this biorthonormal basis, any operator $f(H_r^{(\ell)})$ which is a regular function of the Hamiltonian is represented by a complex matrix that is diagonal except for a Jordan block of rank 2. The occurrence of a double pole in the Green's function, as well as the non exponential time evolution of the Gamow-Jordan generalized eigenfunctions are associated with the Jordan block in the complex energy representation.

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I. INTRODUCTION

For many years now, it has been appreciated that there are distinct advantages in describing quantum resonances and the quantum phenomena associated with the production, evolution, and decay of unbound quantum states in terms of resonant or Gamow eigenstates, since many effects are then readily expressed and evaluated [1].

The Gamow eigenstates represent unbound decaying states of a physical system in a situation in which there are no particles incident. Hence, the resonant or Gamow eigenfunctions are eigenfunctions of a self-adjoint Hamiltonian which are regular at the origin of coordinates and behave as purely outgoing waves at infinity. The corresponding energy eigenvalues are complex, $\mathcal{E}_n = E_n - i\Gamma_n/2$, with $E_n > \Gamma_n/2 > 0$. These resonance energy eigenvalues are precisely equal to the complex resonance energies of the system which occur as poles of the scattering matrix located in the lower half of the second or unphysical sheet of energy. Accordingly, a degeneracy of resonances, that is, the exact coincidence of two (or more) simple resonance poles that merge to produce one double (or higher rank) pole of the scattering matrix results from the degeneracy of two (or more) resonance energy eigenvalues of the Hamiltonian.

In this paper we are concerned with the degeneracy of resonance eigenstates in the absence of symmetry, the concomitant double (or higher order) poles of the scattering matrix, and the Gamow-Jordan generalized eigenfunctions and nondiagonal Jordan blocks in the complex energy representation of the Hamiltonian which are associated with them.

The possibility of multiple resonance poles in partial-wave scattering amplitudes and nonexponential decays of the associated unstable particles was already explicitly mentioned in the classical paper by Goldberger and Watson [2]. Afterwards, the possible occurrence of double poles in the scattering matrix was discussed in connection with problems in nuclear [3] and hadron physics [4,5]. Some interesting examples of interfering unbound two-level systems are the $S=1, T=0,1, J^{\pi}=2^{+}$ doublet in ^{8}Be [6–8], the $T=0,1$ doublet of ρ and ω mesons, and the σ-K_s doublet of neutral sigma and K mesons [7,9–12]. However, the initial interest declined because, although the phenomenon of resonance overlapping might be related to the occurrence of double-pole resonances, at present, there is little empirical evidence for the existence of naturally occurring higher-order pole resonance states in nuclear or hadronic systems.

Later, it was realized that, when the resonant-states can be manipulated by external parameters, i.e., the application of external fields, degeneracies of resonances can be made to occur by simply adjusting the parameters. This interesting possibility brought about a renewed interest in the interference effects of resonances, the crossing and anticrossing properties of the energies and widths of two unbound levels, and the occurrence of double poles of the scattering matrix. For example, it was pointed out that Stark mixing in an atom [13] and the decay of Rabi oscillations in a two-level system [14] can induce degeneracies that lead to a double pole decay.

Similarly, it was shown that degeneracies occur in the two-color ionization of atoms using commensurate frequencies [15]. Hernández and Mondragón showed that the doublet of nuclear unbound states in ^{8}Be with $S=1$ and $T=0,1$ is analytically connected with a degeneracy of resonances [8]. It was also shown [16] that degeneracies of resonances occur when the ground (or excited) state of an atom is strongly coupled to an autoionizing resonance. Kylstra and Joachain [17] demonstrated double-pole degeneracies in the S matrix of laser-assisted electron-atom scattering.

The problem of the degeneracy of resonances also arises naturally in connection with the Berry phase of resonant states [15,18–20], which was recently measured by Dembowski *et al.* [21]. A detailed discussion of the geometric and topological properties of the Berry phase acquired by two closely spaced unbound states when they are adiabatically transported in parameter space around a degeneracy of resonances is given by Mondragón and Hernández [19,20].

Some examples of simple quantum-mechanical systems with double poles in the scattering matrix have been recently described. Vanroose *et al.* [22] examined the formation of resonant double poles of the S matrix in a two-channel model with square-well potentials. Hernández *et al.* [23] investigated a one-channel model with two spherical concentric cavities bounded by δ-function barriers, and showed that a double pole of the S matrix can be induced by tuning the parameters of the model; Vanroose [24] generalized this model to the case of two finite width barriers.

The formal theory of multiple-pole resonances and Gamow-Jordan resonant-states in the rigged Hilbert-space formulation of quantum mechanics was developed by Bohm *et al.* [25] and Antoniou *et al.* [26].

In the present paper, we deal with the problem of multiple poles of the scattering matrix and the generalized complex energy eigenfunctions associated with them in the framework of the theory of the analytic properties of the radial wave functions.

The plan of this paper is as follows. In Secs. II and III, we introduce some basic concepts and fix the notation by way of a short reminder of resonances and resonant-states in the theory of the analytic properties of the radial wave functions. In Sec. IV, bound- and resonant-state eigenfunctions are characterized as elements of a rigged Hilbert space. Sections V and VI are devoted to a short discussion of the no-crossing rule for bound states and its nonapplicability to resonant-states. In Sec. VII, we show that a double pole of the scattering wave function (double zero of the Jost function) is associated with a chain of length 2 of Gamow-Jordan generalized eigenfunctions and derive explicit expressions for these generalized eigenfunctions in terms of the outgoing wave Jost solution, the Jost function, and its derivatives evaluated at the double pole. We also show that the Gamow-Jordan generalized eigenfunctions in the Jordan chain are elements of a complete set of states containing the real (bound states) and complex (resonant-states) energy eigenfunctions plus a continuum of scattering wave functions of complex wave number. Section VIII, is devoted to the characterization of the Gamow-Jordan generalized eigenfunctions as solutions of a Jordan chain of differential equations. In Sec. IX we derive expansion theorems in the complex energy basis (spectral representations) for operators $f(H_r^{(\ell)})$, which are regular functions of the radial Hamiltonian $H_r^{(\ell)}$, and show that, in this basis, the operator $f(H_r^{(\ell)})$ is represented by a complex matrix, which is diagonal except for a Jordan block of rank 2 associated with the double zero of the Jost function and the corresponding Jordan chain of generalized Gamow-Jordan eigenfunctions. We give the normalization and orthogonality rules for the generalized eigenfunctions in the Jordan chain associated with the double pole of the Green's function in Sec. X. We end our paper with a summary of the results and some conclusions in Sec. XI.

II. REGULAR AND PHYSICAL SOLUTIONS OF THE RADIAL EQUATION

The nonrelativistic scattering of a spinless particle by a short-range potential $v(r)$ is described by the solution of a Schrödinger equation. When the potential is rotationally invariant, the wave function is expanded in partial waves, and one is left with the radial equation

$$\frac{d^2\phi_\ell(k,r)}{dr^2}+\left[k^2-\frac{\ell(\ell+1)}{r^2}-v(r)\right]\phi_\ell(k,r)=0. \quad (1)$$

As is usually done when discussing the analytic properties of the solutions of Eq. (1) as functions of k, rather than starting by defining the physical solutions $\psi_\ell^{(+)}(k,r)$, we define the regular and irregular solutions of Eq. (1) by boundary conditions that lead to simple properties as functions of k. The regular solution $\phi_\ell(k,r)$ is uniquely defined by the boundary condition [27]

$$\lim_{r\to 0}(2\ell+1)!!r^{-\ell-1}\phi_\ell(k,r)=1. \quad (2)$$

$\phi_\ell(k,r)$ may be expressed as a linear combination of two independent, irregular solutions of Eq. (1), which behave as outgoing and incoming waves at infinity:

$$\phi_\ell(k,r)=\tfrac{1}{2}ik^{-\ell-1}[f_\ell(-k)f_\ell(k,r)-(-1)^\ell \times f_\ell(k)f_\ell(-k,r)], \quad (3)$$

where $f_\ell(-k,r)$ is an outgoing wave at infinity defined by the boundary condition

$$\lim_{r\to\infty}\exp(-ikr)f_\ell(-k,r)=(+i)^\ell \quad (4)$$

and $f_\ell(k,r)$ is an incoming wave at infinity related to $f_\ell(-k,r)$ by

$$f_\ell(k,r)=(-1)^\ell f_\ell^*(-k,r) \quad (5)$$

for k real and nonvanishing.

The Jost function $f_\ell(-k)=f_\ell(-k,0)$ is given by

$$f_\ell(-k)=(-1)^\ell k^\ell W[f_\ell(-k,r),\phi_\ell(k,r)], \quad (6)$$

where $W[f,g]=fg'-f'g$ is the Wronskian. The Jost function $f_\ell(-k)$ has zeros (roots) on the imaginary axis and in the lower half of the complex k plane.

When the first and second absolute moments of the potential exist, and the potential decreases at infinity faster than any exponential [e.g., if $v(r)$ has a Gaussian tail or if it vanishes identically beyond a finite radius], the functions $f_\ell(-k)$, $\phi_\ell(k,r)$, and $k^\ell f_\ell(-k,r)$, for fixed $r>0$, are entire functions of k [27].

The scattering wave function $\psi_\ell^{(+)}(k,r)$ is the solution of Eq. (1), which vanishes at the origin and behaves at infinity as the sum of a free incoming spherical wave of unit flux plus a free outgoing spherical wave,

$$\psi_\ell^{(+)}(k,0)=0 \quad (7)$$

and

$$\lim_{r\to\infty}\{\psi_\ell^{(+)}(k,r)-[\hat{h}_\ell^{(-)}(k,r)-S_\ell(k)\hat{h}_\ell^{(+)}(k,r)]\}=0. \quad (8)$$

In this expression, $\hat{h}_\ell^{(-)}(k,r)$ and $\hat{h}_\ell^{(+)}(k,r)$ are Ricatti-Hankel functions that describe incoming and outgoing waves, respectively, $S_\ell(k)$ is the scattering matrix.

Hence, the scattering wave function $\psi_\ell^{(+)}(k,r)$ and the regular solution are related by

$$\psi_\ell^{(+)}(k,r)=\frac{k^{\ell+1}\phi_\ell(k,r)}{f_\ell(-k)},\tag{9}$$

and the scattering matrix is given by

$$S_\ell(k)=\frac{f_\ell(k)}{f_\ell(-k)}.\tag{10}$$

The complete Green's function for outgoing particles or resolvent of the radial equation may also be written in terms of the regular solution $\phi_\ell(k,r)$, and the irregular solution $f_\ell(-k,r)$ which behaves as an outgoing wave at infinity:

$$G_\ell^{(+)}(k;r,r')=(-1)^{\ell+1}k^\ell\frac{\phi_\ell(k,r_<)f_\ell(-k,r_>)}{f_\ell(-k)}.\tag{11}$$

III. BOUND- AND RESONANT-STATE EIGENFUNCTIONS

Bound- and resonant-state energy eigenfunctions are the solutions of Eq. (1) that vanish at the origin,

$$u_{n\ell}(k_n,0)=0,\tag{12}$$

and at infinity satisfy the boundary condition

$$\lim_{r\to\infty}\left[\frac{1}{u_{n\ell}(k_n,r)}\frac{du_{n\ell}(k_n,r)}{dr}-ik_n\right]=0,\tag{13}$$

where k_n is a zero of the Jost function,

$$f_\ell(-k_n)=0.\tag{14}$$

From Eqs. (1) and (3) we verify that all roots (zeros) of the Jost function are associated with energy eigenfunctions of the Schrödinger equation.

From Eqs. (3), (4), and (14), bound states and Gamow or resonance eigenfunctions are related to the regular solution $\phi_\ell(k,r)$ by

$$u_{n\ell}(k_n,r)=N_{n\ell}^{-1}\phi_\ell(k_n,r),\tag{15}$$

where $N_{n\ell}$ is a normalization constant. Due to the vanishing of $f_\ell(-k_n)$, $\phi_\ell(k_n,r)$ is now proportional to the outgoing wave solution $f_\ell(-k_n,r)$ of Eq. (1). Hence, bound-state and resonant eigenfunctions take the form

$$u_{n\ell}(k_n,r)=N_{n\ell}^{-1}\frac{i}{2}\frac{(-1)^{\ell+1}}{k^{\ell+1}}f_\ell(k_n)f_\ell(-k_n,r).\tag{16}$$

Bound-state eigenfunctions are associated with the zeros of $f_\ell(-k)$ that are on the positive imaginary axis, while resonant or Gamow state eigenfunctions belong to the zeros of the Jost function that are in the fourth quadrant of the complex k plane.

Equation (16) shows, in a very explicit way, that the Gamow eigenfunctions $u_{n\ell}(k_n,r)$ with $k_n=\kappa_n-i\gamma_n$ and $\kappa_n>\gamma_n>0$ are solutions of Eq. (1) that vanish at the origin and asymptotically behave as purely outgoing waves that oscillate between envelopes that increase exponentially with r, the corresponding energy eigenvalues $\mathcal{E}_n$ are complex with $\mathrm{Re}\mathcal{E}_n>\mathrm{Im}\mathcal{E}_n$.

The bound-state eigenfunctions $u_{s\ell}(k_s,r)$ are also solutions of Eq. (1) that vanish at the origin and satisfy the outgoing wave boundary condition (13), but, in this case the wave number is purely imaginary, with $k_s=i\kappa_s$ and $\kappa_s>0$. Hence, the outgoing wave solution $f_\ell(-k_s,r)$ and the bound-state eigenfunction $u_{s\ell}(k_s,r)$ as functions of r behave asymptotically as decreasing exponentials vanishing as r goes to infinity,

$$\lim_{r\to\infty}u_{s,\ell}(k_s,r)=0,\tag{17}$$

the corresponding energy eigenvalues $\mathcal{E}_s=-\hbar^2\kappa_s^2/2\mu$ are real and negative. Hence, the bound-state eigenfunctions are bounded for all values of r and, as functions of r, they are square integrable.

IV. ENERGY EIGENFUNCTIONS AS ELEMENTS OF A RIGGED HILBERT SPACE

Since bound-state radial eigenfunctions vanish at the origin and are square integrable, they are elements of the Hilbert space $\mathcal{H}$ of square-integrable functions of r,

$$\mathcal{H}=\mathcal{L}^2([0,\infty),dr).\tag{18}$$

Therefore, making an abstraction of the name of the eigenfunctions, we may refer to the formal differential expression

$$H_r^{(\ell)}\equiv\frac{\hbar^2}{2\mu}\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+v(r)\right],\tag{19}$$

occurring on the left-hand side of the radial Schödinger equation, Eq. (1), as a "formal Hamiltonian," which may be considered (defined) as an operator acting in a space of functions. If this space is the Hilbert space $\mathcal{H}$ of square-integrable functions that vanish at the origin, the Hamiltonian H_r^ℓ is bounded from below and essentially self-adjoint [25]. By this last statement we mean that, if $H_r^{(\ell)\dagger}$ is the adjoint of $H_r^{(\ell)}$ in $\mathcal{H}$, and $\overline{H}_r^{(\ell)}$ is the closure of $H_r^{(\ell)}$ in $\mathcal{H}$, then

$$H_r^{(\ell)\dagger}=\overline{H}_r^{(\ell)}=H_r^{(\ell)}.\tag{20}$$

The self-adjointness (Hermiticity) of $H_r^{(\ell)}$ in $\mathcal{H}$ implies that $H_r^{(\ell)}$ as an operator in $\mathcal{H}$ has only real eigenvalues. Hence, the Gamow state eigenfunctions cannot be elements of this space.

Indeed, due to their nondecreasing oscillating behavior at large values of r, the scattering wave functions $\psi_\ell^{(+)}(k,r)$ and the Gamow eigenfunctions $u_{n,\ell}(k_n,r)$ are not square-integrable functions of r, that is, they are not elements of the Hilbert space $\mathcal{H}$. In other words, the Hilbert space $\mathcal{H}$ is not ample enough to contain either the scattering wave functions associated with the real energies in the continuum spectrum of $H_r^{(\ell)}$ or the resonant eigenfunctions belonging to the complex eigenenergies $\mathcal{E}_n=\hbar^2k_n^2/2\mu$.

If we want to consider all the physical solutions of the radial Schrödinger equation (1) as elements of a space, then, a space larger than the Hilbert space $\mathcal{H}$ is required. This larger space is a rigged Hilbert space [28], which is a triplet of spaces,

$$\Phi \subset \mathcal{H} \subset \Phi^{\times}, \tag{21}$$

where $\mathcal{H}$ is the Hilbert space $\mathcal{L}^2([0,\infty),dr)$, Φ is the space of very well behaved functions in $\mathcal{H}$ (the Schwarz space of test functions, i.e., the subspace of $\mathcal{H}$ of all functions admitting derivatives at all orders and such that they and their derivatives go to zero faster than any exponential at infinity), and $\Phi^{\times}$ is the space of antilinear functionals defined over the space Φ.

The domain $\mathcal{D}(H_r^{(\ell)})$ of the radial Hamiltonian is the subspace of $\mathcal{H}$ with elements that are functions $f(r)$ such that they vanish at the origin and have a continuous derivative in $[0,\infty)$ and for which $f'(r)$ is not only continuous but also absolutely continuous over each compact subinterval of $[0,\infty)$ [29,30]. Thus the domain of $H_r^{(\ell)}$ is

$$\mathcal{D}(H_r^{(\ell)}) = \{ f \epsilon \mathcal{L}^2([0,\infty),dr) | f \epsilon AC^2[0,\infty), H_r^{(\ell)} f \epsilon \mathcal{L}^2 \times([0,\infty),dr), f(0)=0\}. \tag{22}$$

Therefore, the domain $\mathcal{D}(H_r^{(\ell)})$ of the Hamiltonian $H_r^{(\ell)}$ lies somewhere between Φ and $\mathcal{H}$,

$$\Phi \subset \mathcal{D}(H_r^{(\ell)}) \subset \mathcal{H}. \tag{23}$$

The action of the Hamiltonian $H_r^{(\ell)}$, which, as an operator, is, in principle, only defined on the elements of its domain in $\mathcal{H}$, can be extended to the elements of $\Phi^{\times}$ by defining the following extended operator $H_r^{(\ell)\times}$:

$$\langle \varphi | H_r^{(\ell)\times} F\rangle := \langle H_r^{(\ell)} \varphi | F\rangle \forall \varphi \epsilon \Phi,\ F \epsilon \Phi^{\times}, \tag{24}$$

where the notation means, as usual,

$$\langle H_r^{(\ell)} \varphi | F\rangle = \int_0^\infty [H_r^{(\ell)} \varphi(r)]^* F(r) dr. \tag{25}$$

We verify that Eq. (24) is satisfied as

$$\int_0^\infty \varphi^*(r)[H_r^{(\ell)\times} F(r)] dr = \int_0^\infty [H_r^{(\ell)} \varphi(r)]^* F(r) dr. \tag{26}$$

Finally, using the definition (24) and Eq. (26), and integrating by parts on the right-hand side of Eq. (26), when $F(r)$ is the Gamow eigenfunction $u_{n\ell}(k_n,r)$, we get

$$\int_0^\infty \varphi^*(r)[H_r^{(\ell)} u_n(k_n,r)] dr = k_n^2 \int_0^\infty \varphi^*(r) u_n(k_n,r) dr. \tag{27}$$

If the arbitrary test function $\varphi \epsilon \Phi$ is omitted in this last equation, we recover the differential equation

$$H_r^{(\ell)} u_{n\ell}(k_n,r) = k_n^2 u_{n\ell}(k_n,r), \tag{28}$$

where $H_r^{(\ell)}$ is the "formal" Hamiltonian given in Eq. (19). This last result shows that we may consider (define) the same formal Hamiltonian $H_r^{(\ell)}$ as an extended operator $H^{(\ell)\times}$, acting in the larger space $\Phi^{\times}$ which contains the Hilbert space $\mathcal{H}$ of square-integrable functions, the scattering wave functions, and the Gamow eigenfunctions. In such a space, the formal Hamiltonian $H_r^{(\ell)}$ can have eigenfunctions that are not square integrable and have complex energy eigenvalues. We also notice that when $H_r^{(\ell)\times}$ is expressed as a formal differential operator acting on the elements of $\Phi^{\times}$, it has exactly the same form as $H_r^{(\ell)}$ given in Eq. (19), which is essentially self-adjoint, that is, Hermitian, in the Hilbert space $\mathcal{H}$.

Considered as elements of a rigged Hilbert space, Gamow eigenfunctions and scattering wave functions may be characterized as energy eigenkets. But whereas Dirac kets describing scattering states are associated with a real value of the energy in the continuous Hilbert-space spectrum of the self-adjoint Hamiltonian $H_r^{(\ell)}$, the Gamow eigenkets are not, but have complex eigenvalues. The existence of these Gamow eigenfunctions (or eigenkets) allows us to interpret resonances as well-defined quantum states of physical systems labeled with a complete set of quantum numbers.

V. THE NO-CROSSING RULE FOR BOUND-STATES

In the case of bound-states, the normalization constant occurring in Eqs. (15) and (16) is related to the derivative of the Jost function evaluated at k_s and it may also be expressed as a normalization integral. The zero of the Jost function is on the positive imaginary axis, and the bound-state eigenfunction is quadratically integrable [for time reversal invariant forces $\phi_\ell(i\kappa_s,r)$ is real]. Newton gives the following expression [27]

$$N_{s\ell}^2 = \frac{1}{i4k_s^{2(\ell+1)}} \left(\frac{df_\ell(-k)}{dk} \right)_{k_s} f_\ell(k_s) = \int_0^\infty |\phi_\ell(k_s,r)|^2 dr. \tag{29}$$

Since the normalization integral is positive and the function $f_\ell(k)$ is regular at $k_s = i\kappa_s$, the derivative of the Jost function evaluated at $k_s = i\kappa_s$ cannot vanish. Therefore, the zero of $f_\ell(-k)$ at $k_s = i\kappa_s$ must be simple. The corresponding pole in $G_\ell^{(+)}(k;r,r')$, $\psi_\ell^{(+)}(k,r)$, and $S_\ell(k)$ must also be simple.

It follows that, in the absence of symmetry, the real negative-energy eigenvalues of the radial equation for a one-channel problem cannot be degenerate.

VI. CROSSING OF RESONANT STATES

In the case of a resonant-state, the zero of the Jost function $f_\ell(-k)$ lies in the fourth quadrant of the complex k plane,

$$k_n = \kappa_n - i\gamma_n, \tag{30}$$

with $\kappa_n > \gamma_n > 0$.

The resonant or Gamow eigenfunction $\phi_\ell(k_n,r)$ is an outgoing spherical wave of complex wave number k_n and angular momentum ℓ. Therefore, for large values of r, $\phi_\ell(k_n,r)$ oscillates between envelopes that grow exponentially with r. Hence, the integrals over r must be properly defined. This may be done by means of a Gaussian regulator and a limiting procedure [31]; Berggren [32,33] gives the following expression

$$\frac{1}{i4k_n^{2(\ell+1)}}\left(\frac{df_\ell(-k)}{dk}\right)_{k_n} f_\ell(k_n) = \lim_{\nu\to 0}\int_0^\infty \exp(-\nu r^2)\phi_\ell^2(k_n,r)dr. \qquad (31)$$

The integral on the right-hand side is a complex number and may vanish.

Since $f_\ell(k_n)$ has no zeros in the lower half of the complex k plane, the left-hand side of Eq. (31) vanishes only when $[df_\ell(-k)/dk]_{k_n}$ vanishes. Then, we have two possibilities,

(i) When $[df_\ell(-k)/dk]_{k_n}$ does not vanish, $f(-k)$ has a simple zero at $k=k_n$, the integral on the right-hand side of Eq. (31) does not vanish, and the normalization constant $N_{n\ell}^2$ occurring in Eq. (15) is given by Eq. (31).

(ii) When

$$\left(\frac{df_\ell(-k)}{dk}\right)_{k_n}=0, \qquad (32)$$

the integral on the right-hand side of Eq. (31) vanishes,

$$\lim_{\nu\to 0}\int_0^\infty \exp(-\nu r^2)\phi_\ell^2(k_n,r)dr=0, \qquad (33)$$

and the Jost function $f_\ell(-k)$ has a multiple zero at $k=k_n$. In this case, the Green's function $G_\ell^{(+)}(k;r,r')$, the scattering wave function $\psi_\ell^{(+)}(k,r)$ and the scattering matrix $S_\ell(k)$ have a multiple pole at $k=k_n$. The normalization constant of the Gamow eigenfunction is no longer given by Eq. (31).

Furthermore, it will be shown below that when $f_\ell(-k)$ has a multiple zero [a multiple resonant pole of rank r in $G_\ell^{(+)}(k;r,r')$, $\psi_\ell^{(+)}(k,r,)$, and $S_\ell(k)$] the corresponding complex energy eigenvalues are degenerate even in the absence of symmetry. That is, the no-crossing rule does not hold for resonant eigenstates.

VII. COMPLETENESS AND THE EXPANSION IN COMPLEX RESONANCE ENERGY EIGENFUNCTIONS

In this section, it will be shown that associated with a double zero of the Jost function [double pole of the scattering wave function $\psi_\ell^{(+)}(k,r)$, the Green's function $G_\ell^{(+)}(k;r,r')$, and the scattering matrix $S_\ell(k)$] there is a chain of generalized Gamow-Jordan eigenfunctions, which together with the bound-state and resonant-state eigenfunctions form a biorthonormal set that may be completed with a continuum of scattering wave functions of complex wave number.

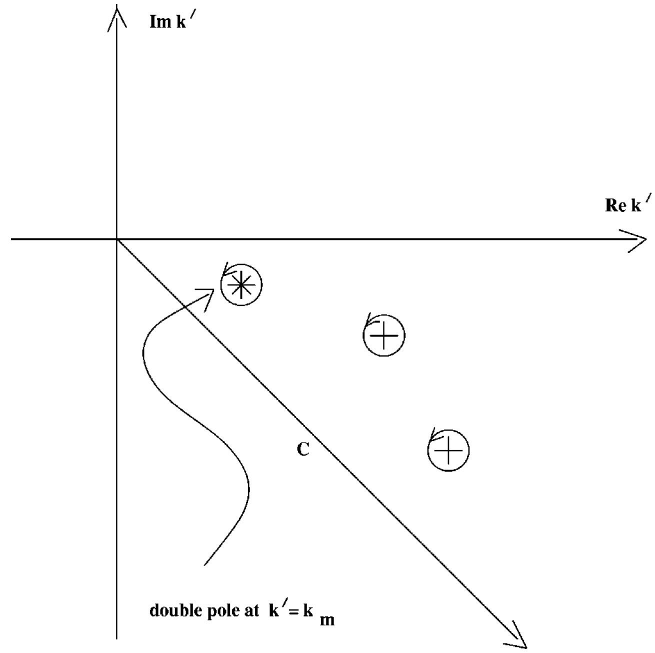

FIG. 1. Integration contour C in the complex k' plane.

Given two square-integrable and very well behaved functions $\Phi(r)$ and $\chi(r)$, which decrease at infinity faster than any exponential, the completeness of the orthonormal set of bound-state and scattering solutions of the radial Schrödinger equation [27] allows us to write

$$\langle\Phi|\chi\rangle = \sum_{s\ bound\ states}\langle\Phi|v_{s,\ell}\rangle\langle v_{s,\ell}|\chi\rangle + \frac{2}{\pi}\int_0^\infty \langle\Phi|\psi_\ell^{(+)}(k')\rangle\langle\psi_\ell^{(+)}(k')|\chi\rangle dk', \qquad (34)$$

where $\langle\Phi|\chi\rangle$ is the standard Dirac bracket notation

$$\langle\Phi|\chi\rangle = \int_0^\infty \Phi^*(r)\chi(r)dr. \qquad (35)$$

We shall assume that the Jost function $f_\ell(-k)$ has a double zero at $k=k_m$ in the fourth quadrant of the complex k' plane, all other zeros of $f_\ell(-k')$ in that quadrant being simple. Then the scattering function $\psi_\ell^{(+)}(k',r)$, as function of k' complex, has one double-resonance pole at $k'=k_m$ and simple resonance poles at $k=k_n$, $n=1,2,\ldots,m-1,m+1,\ldots$, all in the fourth quadrant of the complex k' plane. The function $\psi^{(+)*}(k',r)$ is regular and has no poles in the lower half of the k' plane.

In order to make explicit the contribution of the resonant-states to the expansion in eigenfunctions, the integration contour in the second term on the right-hand side of Eq. (34) is deformed as shown in Fig. 1.

When the deformed contour C crosses over resonant poles, the theorem of residues gives

$$\langle\Phi|\chi\rangle=\sum_{s\ \text{bound states}}\langle\Phi|v_{s,\ell}\rangle\langle v_{s,\ell}|\chi\rangle + \sum_{\text{resonance poles}} 2\pi i\ \text{Res}\left[\frac{2}{\pi}\langle\Phi|\psi_\ell^{(+)}(k')\rangle \times\langle\psi_\ell^{(+)}(k')|\chi\rangle\right]+\frac{2}{\pi}\int_C\langle\Phi|\psi_\ell^{(+)}(k')\rangle \times\langle\psi_\ell^{(+)}(k')|\chi\rangle dk'. \tag{36}$$

The residues may be readily computed from equations (9) and (36).

When $f_\ell(-k')$ has a simple zero at $k'=k_n$,

$$2\pi i\ \text{Res}\left[\frac{2}{\pi}\langle\Phi|\psi_\ell^{(+)}(k')\rangle\langle\psi_\ell^{(+)}(k')|\chi\rangle\right]_{k'=k_n} = 4i\ \text{Res}\left[\frac{\langle\Phi|\phi_\ell(k')\rangle\langle\phi_\ell(k')|\chi\rangle k'^{2(\ell+1)}}{(k'-k_n)\left(\frac{df_\ell(-k')}{dk'}\right)f_\ell(k')}\right]_{k'=k_n} = \frac{1}{\frac{f_\ell(k_n)}{4ik_n^{2(\ell+1)}}\left(\frac{df_\ell(-k')}{dk'}\right)_{k_n}}[\langle\Phi|\phi_\ell(k')\rangle]_{k'=k_n} \times[\langle\phi_\ell(k')|\chi\rangle]_{k'=k_n}, \tag{37}$$

where

$$[\langle\Phi|\phi_\ell(k')\rangle]_{k'=k_n}=\lim_{k'\to k_n}\int_0^\infty\Phi^*(r)\phi_\ell(k',r)dr, \tag{38}$$

and

$$[\langle\phi_\ell(k')|\chi\rangle]_{k'=k_n}=\lim_{k'\to k_n}\int_0^\infty\phi_\ell(k',r)\chi(r)dr, \tag{39}$$

since $\phi_\ell(k',r)$ is real and bounded for k' real, the integrals in Eqs. (38) and (39) exist.

Furthermore, since $\phi_\ell(k_n,r)$ is an outgoing wave that oscillates between envelopes that grow exponentially at infinity and $\Phi(r)$ and $\chi(r)$ are very well behaved functions of r that decrease at infinity faster than any exponential, the integrals of the products $\Phi^*(r)\phi_\ell(k_n,r)$ and $\phi_\ell(k_n,r)\chi(r)$ also exist, and we may take the limit indicated on the right-hand side of Eqs. (38) and (39) under the integration sign.

Therefore,

$$2\pi i\text{Res}\left[\frac{2}{\pi}\langle\Phi|\psi_\ell^{(+)}(k')\rangle\langle\psi_\ell^{(+)}(k')|\chi\rangle\right]_{k'=k_n} =\langle\Phi|u_{n\ell}(k_n))(u_{n\ell}(k_n)|\chi\rangle, \tag{40}$$

where the notation means

$$\langle\Phi|u_{n\ell}(k_n))=\int_0^\infty\Phi^*(r)u_{n\ell}(k_n,r)dr \tag{41}$$

and

$$(u_{n\ell}(k_n)|\chi\rangle=\int_0^\infty u_{n\ell}(k_n,r)\chi(r)dr, \tag{42}$$

The Gamow eigenfunction or normal mode $u_{n\ell}(k_n,r)$, is given by Eq. (16) and the normalization constant $N_{n\ell}$ is given by

$$N_{n\ell}^2=\frac{1}{i4k_n^{2(\ell+1)}}f_\ell(k_n)\left(\frac{df_\ell(-k')}{dk'}\right)_{k_n}, \tag{43}$$

in agreement with Berggren's result given in Eq. (31).

When the Jost function $f_\ell(-k')$ has a double zero at $k'=k_m$, $\psi_\ell^{(+)}(k',r)$ has a double pole at $k'=k_m$,

$$\psi_\ell^{(+)}(k',r)=\frac{\phi_\ell(k',r)k'^{(\ell+1)}}{(k'-k_m)^2g_{\ell m}(k')}. \tag{44}$$

The function $g_{\ell m}(k')$ is regular at $k'=k_m$ and may be expanded as

$$g_{\ell m}(k')=\frac{1}{2}\left(\frac{d^2f_\ell(-k')}{dk'^2}\right)_{k_m}+\frac{1}{6}(k'-k_m)\left(\frac{d^3f_\ell(-k')}{dk'^3}\right)_{k_m} +\cdots \tag{45}$$

with

$$\left(\frac{d^2f_\ell(-k')}{dk'^2}\right)_{k_m}\neq 0. \tag{46}$$

The function $\psi_\ell^{(+)*}(k',r')$ is regular at $k'=k_m$, since $f_\ell(k')$ has no zeros in the lower half of the complex k' plane,

$$\psi_\ell^{(+)*}(k',r')=\frac{\phi_\ell(k',r')k'^{(\ell+1)}}{f_\ell(k')}. \tag{47}$$

Thus, the residue of the term $(2/\pi)\langle\Phi|\psi_\ell^{(+)}(k')\rangle \times\langle\psi_\ell^{(+)}(k')|\chi\rangle$ at the double pole in $k'=k_m$ is obtained from the Cauchy integral formula as

$$2\pi i \,\mathrm{Res}\left[\frac{2}{\pi}\langle\Phi|\psi_\ell^{(+)}(k')\rangle\langle\psi_\ell^{(+)}(k')|\chi\rangle\right]_{k'=k_m}$$
$$=4i\,\mathrm{Res}\left[\frac{\langle\Phi|\phi_\ell(k')\rangle\langle\phi_\ell(k')|\chi\rangle k'^{2(\ell+1)}}{(k'-k_m)^2 g_{\ell m}(k')f_\ell(k')}\right]_{k'=k_m}$$
$$=4i\left[\frac{d}{dk'}\left(\frac{\langle\Phi|\phi_\ell(k')\rangle\langle\phi_\ell(k')|\chi\rangle k'^{2(\ell+1)}}{g_{\ell m}(k')f_\ell(k')}\right)\right]_{k'=k_m}. \tag{48}$$

After computing the derivative indicated in Eq. (48) and rearranging some terms, we obtain

$$2\pi i \,\mathrm{Res}\left[\frac{2}{\pi}\langle\Phi|\psi_\ell^{(+)}(k')\rangle\langle\psi_\ell^{(+)}(k')|\chi\rangle\right]_{k'=k_m}$$
$$=\frac{1}{\mathcal{N}_{m\ell}^2}[\langle\Phi|\hat{\phi}_\ell(k_m)\rangle\langle\phi_\ell(k_m)|\chi\rangle + \langle\Phi|\phi_\ell(k_m)\rangle\langle\hat{\phi}_\ell(k_m)|\chi\rangle], \tag{49}$$

where, according to Eq. (15), $\phi_\ell(k_m,r)$ is the non-normalized Gamow eigenfunction and $\hat{\phi}_\ell(k_m,r)$ is a generalized Gamow-Jordan eigenfunction or abnormal mode given by

$$\hat{\phi}_\ell(k_m,r)=\frac{d\phi_\ell(k_m,r)}{d\mathcal{E}_m}+C_\ell(k_m)\phi_\ell(k_m,r), \tag{50}$$

$\mathcal{E}_m$ is the complex energy eigenvalue, $\mathcal{E}_m=(\hbar^2/2\mu)k_m^2$ and the constant factor $C_\ell(k_m)$ multiplying $\phi_\ell(k_m,r)$ in Eq. (50) is

$$C_\ell(k_m)=\frac{2\mu}{\hbar^2}\frac{1}{2k_m}\left[\frac{\ell+1}{k_m}-\frac{1}{2}\frac{1}{f_\ell(k_m)}\frac{df_\ell(k_m)}{dk_m} - \frac{1}{6}\left(\frac{d^2f_\ell(-k')}{dk'^2}\right)_{k_m}^{-1}\left(\frac{d^3f_\ell(-k')}{dk'^3}\right)_{k_m}\right]. \tag{51}$$

The normalization constant $\mathcal{N}_{ml}^2$ is now

$$\mathcal{N}_{m\ell}^2=\left(\frac{2\mu}{\hbar^2}\right)\frac{1}{16ik_m^{2\ell+3}}f_\ell(k_m)\left(\frac{d^2f_\ell(-k')}{dk'^2}\right)_{k_m}. \tag{52}$$

The expression (49) suggests the following normalization rule for the chain of Gamow-Jordan generalized eigenfunctions belonging to a double zero of the Jost function:

$$u_{m\ell}(k_m,r)=\frac{1}{\mathcal{N}_{m\ell}}\phi_\ell(k_m,r) \tag{53}$$

and

$$\hat{u}_{m\ell}(k_m,r)=\frac{1}{\mathcal{N}_{m\ell}}\hat{\phi}_\ell(k_m,r). \tag{54}$$

Substitution of Eqs. (53) and (54) in Eq. (49) gives

$$2\pi i \,\mathrm{Res}\left[\frac{2}{\pi}\langle\Phi|\psi_\ell^{(+)}(k')\rangle\langle\psi_\ell^{(+)}(k')|\chi\rangle\right]_{k'=k_m}$$
$$=\langle\Phi|\hat{u}_{m\ell}(k_m))(u_{m\ell}(k_m)|\chi\rangle+\langle\Phi|u_{m\ell}(k_m))(\hat{u}_{m\ell}(k_m)|\chi\rangle, \tag{55}$$

where the notation means

$$\langle\Phi|\hat{u}_{m\ell}(k_m))=\int_0^\infty \Phi^*(r)\hat{u}_{m\ell}(k_m,r)dr \tag{56}$$

and

$$(\hat{u}_{m\ell}(k_m)|\chi\rangle=\int_0^\infty \hat{u}_{m,\ell}(k_m,r)\chi(r)dr; \tag{57}$$

$\hat{u}_{m\ell}(k_m,r)$ is defined in Eq. (54).

Finally, substitution of the expressions (40) and (55) in Eq. (36) gives the following expansion:

$$\langle\Phi|\chi\rangle=\sum_{s\ \mathrm{bound\ states}}\langle\Phi|v_{s\ell}\rangle\langle v_{s\ell}|\chi\rangle + \sum_{n\neq m\ \mathrm{resonances}}\langle\Phi|u_{n\ell})(u_{n\ell}|\chi\rangle + \langle\Phi|\hat{u}_{m\ell}(k_m))(u_{m\ell}(k_m)|\chi\rangle + \langle\Phi|u_{m\ell}(k_m))(\hat{u}_{m\ell}(k_m)|\chi\rangle + \frac{2}{\pi}\int_c\langle\Phi|\psi_\ell^{(+)}(k')\rangle\langle\psi_\ell^{(+)}(k')|\chi\rangle dk'. \tag{58}$$

This expression shows that, when the Jost function has many simple zeros and one double zero in the fourth quadrant of the complex k plane, the Gamow eigenfunctions $u_{n\ell}(k_m,r)$ associated with simple zeros of the Jost function, and the chain $\{u_{m\ell}(k_m,r),\hat{u}_{m\ell}(k_m,r)\}$ of Gamow-Jordan generalized eigenfunctions associated with the double zero of the Jost function are basis elements of an expansion in generalized bound- and resonant-state eigenfunctions plus a continuum of scattering functions of complex wave values k'.

Omitting the arbitrary function $\Phi(r)$ in Eq. (58), we obtain the complex basis expansion of an arbitrary square integrable and well-behaved function $\chi(r)$,

$$\chi(r)=\sum_{s\ \text{bound states}} v_{s\ell}(r)\langle v_{s\ell}|\chi\rangle+\sum_{n\neq m} u_{n\ell}(k_n,r)(u_{n\ell}|\chi\rangle +\hat{u}_{m\ell}(k_m,r)(u_{m\ell}|\chi\rangle+u_{m\ell}(k_m,r)(\hat{u}_{m\ell}|\chi\rangle +\frac{2}{\pi}\int_c \psi_\ell^{(+)}(k',r)\langle\psi_\ell^{(+)}(k')|\chi\rangle dk'. \tag{59}$$

In this expression, $u_{n\ell}(k_n,r)$ are the Gamow eigenfunctions representing decaying states associated with simple resonance poles of the scattering wave function $\psi_\ell^{(+)}(k,r)$, the matrix $S(k)$, and the Green's function $G^{(+)}(k;r,r')$. The set $\{u_{m\ell}(k_m,r),\hat{u}_{m\ell}(k_m,r)\}$ is a Jordan chain of length 2 of generalized Gamow-Jordan eigenfunctions associated with the double pole of the scattering matrix $S(k)$ and the Green's function $G^{(+)}(k;r,r')$ at $k=k_m$. The last term on the right-hand side of Eq. (58) and (59) is the background integral defined along the integration contour shown in Fig. 1.

VIII. CHAINS OF GAMOW-JORDAN GENERALIZED EIGENFUNCTIONS

The chain of Gamow-Jordan generalized eigenfunctions, introduced in the preceding section, may be characterized as solutions of a Jordan chain of differential equations with the same boundary conditions as those satisfied by the Gamow eigenfunctions.

According to the definition given in Eqs. (50) and (54), the Gamow-Jordan generalized eigenfunction $\hat{u}_{m,\ell}(k_m,r)$ is a linear combination of the Gamow eigenfunction and its derivative with respect to the complex energy eigenvalue

$$\hat{u}_{m,\ell}(k_m,r)=\frac{1}{\mathcal{N}_{m,\ell}}\left[\frac{\mu}{\hbar^2 k_m}\frac{\partial\phi_\ell(k_m,r)}{\partial k_m}+C_\ell(k_m)\phi_\ell(k_m,r)\right], \tag{60}$$

where we have written $d\mathcal{E}_m=(\hbar^2/\mu)k_m dk_m$, and $\mathcal{E}_m$ is the complex energy eigenvalue corresponding to the double zero of the Jost function at $k=k_m$. Consequently, in order to characterize $\hat{u}_{m,\ell}(k_m,r)$ as the solution of a differential equation with prescribed boundary conditions, it will be convenient to start by deriving the differential equation satisfied by $\partial\phi_\ell(k,r)/\partial k$.

The differential equations satisfied by the derivatives of the functions $\phi_\ell(k,r),f_\ell(k,r)$ and $f_\ell(-k,r)$ with respect to k are obtained from the radial Schrödinger equation, Eq. (1), taking derivatives with respect to k on both sides of the equation,

$$\frac{d^2\dot{\phi}_\ell(k,r)}{dr^2}+\left[k^2-\frac{\ell(\ell+1)}{r^2}-v(r)\right]\dot{\phi}_\ell(k,r)=-2k\phi_\ell(k,r), \tag{61}$$

in this equation we have used the notation $\dot{\phi}_\ell(k,r)=\partial\phi_\ell(k,r)/\partial k$. Similar expressions are valid for $\dot{f}_\ell(-k,r)$ and $\dot{f}_\ell(k,r)$.

Let us recall again that, when the first and second absolute moments of the potential $v(r)$ exist and the potential decreases at infinity faster than any exponential [e.g., if $v(r)$ has a Gaussian tail or if it vanishes identically beyond a finite radius], the functions $k^\ell f_\ell(-k,r),f_\ell(-k)$, and $\phi_\ell(k,r)$, for fixed $r>0$, are entire functions of k.

Therefore, the derivatives of these functions with respect to the wave number k exist and are entire functions of k for all values of k in the complex k plane.

If we take derivatives with respect to k on both sides of Eq. (3), the function $\dot{\phi}_\ell(k,r)$ may be written as a linear combination of the two linearly independent irregular solutions $f_\ell(k,r)$ and $f_\ell(-k,r)$ and their derivatives with respect to k,

$$\dot{\phi}_\ell(k,r)=-\frac{\ell+1}{k}\phi_\ell(k,r)+\frac{i}{2k^{\ell+1}}\{[\dot{f}_\ell(-k)f_\ell(k,r) +f_\ell(-k)\dot{f}_\ell(k,r)]-(-1)^\ell[\dot{f}_\ell(k)f_\ell(-k,r) +f_\ell(k)\dot{f}_\ell(-k,r)]\}. \tag{62}$$

From the boundary condition (2), which defines the regular solution $\phi_\ell(k,r)$, it may be readily shown that, for r close to zero [27],

$$\lim_{r\to 0}\frac{\partial\phi_\ell(k,r)}{\partial k}\approx\lim_{r\to 0}\frac{2(\ell+1)!!}{2\ell+3}r^{\ell+2}(kr)=0. \tag{63}$$

Similarly, from the boundary conditions satisfied by the irregular solutions, Eq. (4), it follows that, for large values of r,

$$f_\ell(\pm k,r)\approx e^{\pm ikr}. \tag{64}$$

Therefore,

$$\lim_{r\to\infty}\left\{\frac{1}{\dot{f}_\ell(\pm k,r)}\frac{d\dot{f}_\ell(\pm k,r)}{dr}\pm ik\right\}=0, \tag{65}$$

that is, at infinity, $\dot{f}_\ell(k,r)$ is an incoming wave and $\dot{f}_\ell(-k,r)$ is an outgoing wave.

Therefore, from (3), (62), (63), and (65), in the general case of an arbitrary complex value of k, the function $\dot{\phi}_\ell(k,r)$, as a function of r, vanishes at the origin of the coordinates and asymptotically is a superposition of incoming and outgoing waves.

Now, let us consider the particular case when $k=k_n$ is a resonant zero of the Jost function, $f_\ell(-k_n)=0$, with k_n in the fourth quadrant of the complex k plane; then the coefficient of the incoming wave in $\phi_\ell(k,r)$ vanishes and $\phi_\ell(k_n,r)$ becomes proportional to the Gamow eigenfunction $u_{n\ell}(k_n,r)$, which vanishes at the origin of the coordinates and is an outgoing wave for large values of r,

$$\phi_\ell(k_n,r)=\frac{i}{2k_n^{\ell+1}}(-1)^{\ell+1}f_\ell(k_n)f_\ell(-k_n,r). \quad (66)$$

Then, Eq. (62) simplifies to

$$\left(\frac{\partial\phi_\ell(k,r)}{\partial k}\right)_{k_n}=\frac{\partial\phi_\ell(k_n,r)}{\partial k_n}+\frac{i}{2k_n^{\ell+1}}\left(\frac{\partial f_\ell(-k)}{\partial k}\right)_{k_n}f_\ell(k_n,r). \quad (67)$$

From this expression, we see that, when k_n is a simple zero of the Jost function, the coefficient of the incoming wave in $(\partial\phi_\ell(k,r)/\partial k)_{k_n}$ is nonvanishing, $[\partial f_\ell(-k)/\partial k]_{k_n}\neq 0$. Hence, the function $[\partial\phi_\ell(k,r)/\partial k]_{k_n}$, as a function of r, behaves at infinity as a superposition of an outgoing wave plus an incoming wave.

Therefore, when k_n is a simple zero of the Jost function, $\phi_\ell(k_n,r)$ is an unnormalized Gamow eigenfunction but $[\partial\phi_\ell(k,r)/\partial k]_{k_n}$ is not a generalized Gamow-Jordan eigenfunction.

When the Jost function has a double-resonance zero at $k=k_m$, $[\partial f_\ell(-k)/\partial k]_{k_m}$ vanishes and

$$\left(\frac{\partial\phi_\ell(k,r)}{\partial k}\right)_{k_m}=\frac{\partial\phi_\ell(k_m,r)}{\partial k_m}. \quad (68)$$

In this case, both functions $\phi_\ell(k_m,r)$ and $\partial\phi_\ell(k_m,r)/\partial k_m$, as functions of r, vanish at the origin of the coordinates and at infinity they behave as outgoing waves. Any linear combination of $\phi_\ell(k_m,r)$ and $\partial\phi_\ell(k_m,r)/\partial k_m$ also vanishes at the origin and at infinity behaves as an outgoing wave.

Therefore, when the Jost function has a double-resonance zero at $k=k_m$, there is a set of two generalized eigenfunctions of the radial Schrödinger equation, $u_m(k_m,r)$ and $\hat{u}_{m,\ell}(k_m,r)$, such that

$$H_r^{(\ell)}u_{m\ell}(k_m,r)=\mathcal{E}_m u_{m\ell}(k_m,r), \quad (69)$$

$$H_r^{(\ell)}\hat{u}_{m\ell}(k_m,r)=\mathcal{E}_m\hat{u}_{m\ell}(k_m,r)+u_{m\ell}(k_\ell,r), \quad (70)$$

which satisfy the same boundary conditions, namely, they vanish at the origin and at infinity they behave as outgoing waves.

The chain of Gamow-Jordan differential equations may also be seen as being associated with the Gamow generalized eigenfunction $u_{m\ell}(k_m,r)$. From this point of view, an eigenfunction $u_{m\ell}(k_m,r)$ is selected and the generalized Gamow-Jordan eigenfunctions are generated by successively solving the equations

$$\begin{aligned}
H_r^{(\ell)}u_{m_\ell}(k_m,r)&=\mathcal{E}_m u_{m_\ell}(k_m,r),\\
H_r^{(\ell)}\hat{u}_{m_\ell}^{(1)}(k_m,r)&=\mathcal{E}_m\hat{u}_{m_\ell}^{(1)}(k_m,r)+(u_m,r),\\
&\vdots \qquad (71)\\
H_r^{(\ell)}\hat{u}_{m_\ell}^{(s)}(k_m,r)&=\mathcal{E}\hat{u}_{m_\ell}^{(s)}(k_{m,r})+\hat{u}_{m_\ell}^{(s-1)}(k_m,r)
\end{aligned}$$

for as long as there exists a solution to the inhomogeneous equation

$$(H_r^{(\ell)}-\mathcal{E}_m)\hat{u}_{m,\ell}^{(j)}(k_m,r)=\hat{u}_{m,\ell}^{(j-1)}(k_m,r), \quad (72)$$

for $j=1,2,3,\ldots,s$ such that $\hat{u}_{m\ell}^{(j)}(k_m,r)$ vanish at the origin of coordinates and at infinity they are outgoing waves.

From our previous discussion, it is fairly obvious that these solutions exist as long as $[\partial^{(j)}\phi_\ell(k,r)/\partial k^{(j)}]_{k_m}$ behaves as an outgoing wave when r goes to infinity for $1\leqslant j\leqslant s$. But, this condition implies that all the coefficients of the incoming waves in $\partial^{(j)}\phi_\ell(k,r)/\partial k^j$, which are the derivatives of the Jost function $[\partial^{(i)}f_\ell(-k)/\partial k^{(i)}]$ for $1\leqslant i\leqslant j\leqslant s$, vanish at $k=k_m$. It follows that a necessary and sufficient condition for having a Gamow-Jordan chain of generalized resonance eigenfunctions of length $s+1$ associated with the resonance energy eigenvalue $\mathcal{E}_m=\hbar^2k_m^2/2\mu$ is that the Jost function has a zero of rank $s+1$ at $k=k_m$, with k_m in the fourth quadrant of the complex k plane.

Obviously, the length of any Gamow-Jordan chain of $H_r^{(\ell)}$ is finite, since $[\partial^{(s+1)}f_\ell(-k)/\partial k^{(s+1)}]_{k_m}\neq 0$, for some finite value of s, otherwise $f_\ell(-k)$ would be a constant.

Furthermore, the members of a Gamow-Jordan chain are linearly independent, as can be seen from the following reasoning.

Let

$$\sum_{i=0}^{s}\alpha_i\hat{u}_{m\ell}^{(i)}(k_m,r)=0, \quad (73)$$

with $u_{m\ell}(k_m,r)=u_{m\ell}^{(0)}(k_m,r),\hat{u}_{m\ell}(k_m,r)=\hat{u}_{m\ell}^{(1)}(k_m,r)$, etc. Applying the operator $(H_r^{(\ell)}-\mathcal{E}_m)^s$ from the left to both sides of Eq. (73), and noting that

$$(H_r^{(\ell)}-\mathcal{E}_m)^s\hat{u}_{m\ell}^{(j)}(k_m,r)=0, \tag{74}$$

for $j=1,\ldots,s-1$, we obtain

$$\alpha_s(H_r^{(\ell)}-\mathcal{E}_m)^s\hat{u}_{m,\ell}^{(s)}(k_m,r)=0, \tag{75}$$

but $(H_r^{(\ell)}-\mathcal{E}_m)^s\hat{u}_{m\ell}^{(s)}(k_m,r)$ is equal to the Gamow eigenfunction $u_{m\ell}(k_m,r)$ and is, hence, nonzero. Thus,

$$\alpha_s=0. \tag{76}$$

Applying the operator $(H_r^{(\ell)}-\mathcal{E}_m)^{s-1}$ to the equation

$$\sum_{i=1}^{s-1}\alpha_i\hat{u}_{m\ell}^{(i)}(k_m,r)=0, \tag{77}$$

we, similarly, derive

$$\alpha_{s-1}=0. \tag{78}$$

Repeating this procedure, the linear independence of the set of generalized eigenfunctions of the Gamow-Jordan chain is established.

IX. JORDAN BLOCKS IN THE COMPLEX ENERGY BASIS

Once it has been established that the Gamow eigenfunctions $u_{n\ell}(k_n,r)$ and the Jordan chain $\{u_{m\ell}(k_m,r),\hat{u}_{m\ell}(k_m,r)\}$ of generalized Gamow-Jordan eigenfunctions are linearly independent elements of the basis set of eigenfunctions in the expansions (58) and (59), we may represent any operator $f(H_r^{(\ell)})$, which is a regular function of the Hamiltonian $H_r^{(\ell)}$, in terms of its matrix elements in this basis.

Let us start by deriving an expression for the action of $f(H_r^{(\ell)})$ on the generalized Gamow-Jordan eigenfunction $\hat{u}_{m\ell}(k_m,r)$. With this purpose in mind, let us write the eigenvalue equation satisfied by $u_{m\ell}(k_m,r)$ as

$$H_r^{(\ell)}u_{m\ell}(k_m,r)=\mathcal{E}_m u_{m\ell}(k_m,r), \tag{79}$$

where

$$H_r^{(\ell)}=-\frac{\hbar^2}{2\mu}\left[\frac{d^2}{dr^2}-v(r)-\frac{\ell(\ell+1)}{r^2}\right], \tag{80}$$

$v(r)$ is a well-behaved short-ranged potential that satisfies the conditions stated in Sec. I. Now, let us consider a holomorphic function $f(\mathcal{E})$ of the complex variable $\mathcal{E}$, such that

$$f(\mathcal{E})=\sum_{j=0}^{\infty}a_j\mathcal{E}^j, \tag{81}$$

the coefficients a_j are independent of $\mathcal{E}$.

Then, from Eqs. (79) and (81),

$$f(H_r^{(\ell)})u_{m\ell}(k_m,r)=f(\mathcal{E}_m)u_{m\ell}(k_m,r). \tag{82}$$

Taking derivatives with respect to the eigenvalue $\mathcal{E}_m$ on both sides of Eq. (82), we obtain,

$$f(H_r^{(\ell)})\frac{\partial u_{m\ell}(k_m,r)}{\partial\mathcal{E}_m}=f(\mathcal{E}_m)\frac{\partial u_{m\ell}(k_m,r)}{\partial\mathcal{E}_m}+\frac{\partial f(\mathcal{E}_m)}{\partial\mathcal{E}_m}u_{m\ell}(k_m,r). \tag{83}$$

From this equation and the definition, Eqs. (50), (51), and (54), of $\hat{u}_{m\ell}(k_m,r)$, it follows immediately that

$$f(H_r^{(\ell)})\hat{u}_{m\ell}(k_m,r)=f(\mathcal{E}_m)\hat{u}_{m\ell}(k_m,r)+\frac{\partial f(\mathcal{E}_m)}{\partial\mathcal{E}_m}u_m(k_m,r). \tag{84}$$

Notice that a necessary and sufficient condition for the existence of $\partial u_{m\ell}(k_m,r)/\partial\mathcal{E}_m$ is the vanishing of $[df(-k)/dk]_{k_m}$.

The rule stated in Eq. (84) permits us to calculate the action of $f(H_r^{(\ell)})$ on the generalized Gamow-Jordan vectors occurring in the complex basis expansions (58) and (59).

Now, we can write the operator $f(H_r^{(\ell)})$ in terms of its matrix elements in the complex energy basis. This may be done by operating $f(H_r^{(\ell)})$ from the left in both sides of Eq. (59),

$$\begin{aligned}f(H_r^{(\ell)})\chi(r)=&\sum_s f(\mathcal{E}_s)v_{s\ell}(r)\langle v_{s\ell}|\chi\rangle\\&+\sum_{n\neq m}f(\mathcal{E}_n)u_{n\ell}(k_n,r)(u_{n\ell}|\chi\rangle\\&+\left(f(\mathcal{E}_m)\hat{u}_{m\ell}(k_m,r)+\frac{\partial f(\mathcal{E}_m)}{\partial\mathcal{E}_m}u_{m\ell}(k_m,r)\right)\\&\times(u_{m\ell}|\chi\rangle+f(\mathcal{E}_m)u_{m\ell}(k_m,r)(\hat{u}_{m\ell}|\chi\rangle\\&+\frac{2}{\pi}\int_c f(\mathcal{E}')\phi_\ell^{(+)}(k',r)\langle\phi_\ell^{(+)}(k')|\chi\rangle dk'.\end{aligned} \tag{85}$$

Multiplying both sides of Eq. (85) by $\Phi^*(r)$ and integrating over r, we get

$$\langle\Phi|f(H_r^{(\ell)})|\chi\rangle=\sum_s \langle\Phi|v_{s\ell})f(\mathcal{E}_s)\langle v_{s\ell}|\chi\rangle+\sum_{n\neq m}\langle\Phi|u_{n\ell})f(\mathcal{E}_n)(u_{n\ell}|\chi\rangle+\langle\Phi|\hat{u}_{m\ell})f(\mathcal{E}_m)(u_{m\ell}|\chi\rangle$$
$$+\langle\Phi|u_{m\ell})\left(f(\mathcal{E}_m)(\hat{u}_{m\ell}|\chi\rangle+\frac{\partial f(\mathcal{E}_m)}{\partial\mathcal{E}_m}(u_{m\ell}|\chi\rangle\right)+\frac{2}{\pi}\int_c\langle\Phi|\psi_\ell^{(+)}(k')\rangle f(\mathcal{E}')\langle\psi_\ell^{(+)}(k')|\chi\rangle dk'. \qquad (86)$$

To simplify the notation, suppose that the system has no bound states, only resonances, and that the first two resonances are degenerate. Rearranging Eq. (86) in matrix form, we get

$$\langle\Phi|f(H_r^{(\ell)})|\chi\rangle=(\langle\Phi|u_{1\ell}),\langle\Phi|\hat{u}_{1\ell}),\langle\Phi|u_{3\ell}),\ldots)\times\begin{pmatrix} f(\mathcal{E}_1) & \frac{\partial f(\mathcal{E}_1)}{\partial\mathcal{E}_1} & 0 & 0 & . \\ 0 & f(\mathcal{E}_1) & 0 & 0 & . \\ 0 & 0 & f(\mathcal{E}_3) & 0 & . \\ 0 & 0 & 0 & f(\mathcal{E}_4) & . \\ . & . & . & . & . \end{pmatrix}\times\begin{pmatrix}(\hat{u}_{1\ell}|\chi\rangle \\ (u_{1\ell}|\chi\rangle \\ (u_{3\ell}|\chi\rangle \\ . \\ . \end{pmatrix}+\frac{2}{\pi}\int_c\langle\Phi|\psi_\ell^{(+)}(k')\rangle f(\mathcal{E}')\times\langle\psi_\ell^{(+)}(k')|\chi\rangle dk'. \qquad (87)$$

In this matrix representation of $f(H_r^{(\ell)})$ [37], the upper left 2×2 submatrix is a Jordan block of rank 2 [34–36] associated with the chain of Gamow-Jordan generalized eigenfunctions $\{\hat{u}_{1\ell}(k_1,r),u_{1\ell}(k_1,r)\}$ belonging to the double zero of the Jost function $f_\ell(-k)$ (double pole of the scattering matrix and the Green's function). Except for this 2×2 block, this matrix is diagonal with the eigenvalues $f(\mathcal{E}_n)$ in the diagonal entries. Simple zeros of the Jost function correspond to simple (nonrepeated) eigenvalues of $f(H_r^{(\ell)})$, while the double zero of $f_\ell(-k)$ corresponds to the twice repeated (degenerate) eigenvalue $f(\mathcal{E}_1)$ occurring in the Jordan block. The off-diagonal nonvanishing element in this block is $\partial f(\mathcal{E}_1)/\partial\mathcal{E}_1$.

The difference in physical dimensions of the off-diagonal and the diagonal entries in the 2×2 Jordan block is compensated by the difference in normalization of the Gamow-Jordan chain $\{\hat{u}_{1\ell}(k_1,r),u_{1\ell}(k_1,r)\}$ and the Gamow eigenfunctions $u_{n\ell}(k_n,r)$ $(n=3,4,\ldots)$, which are normalized according to Eqs. (52), (53) and (54), and Eqs. (15), (29), respectively.

It will be instructive to consider some simple examples.

We first choose $f(H_r^{(\ell)})=H_r^{(\ell)}$. Then, from Eq. (87) we obtain

$$\langle\Phi|f(H_r^{(\ell)})|\chi\rangle=(\langle\Phi|u_{1\ell}),\langle\Phi|\hat{u}_{1\ell}),\langle\Phi|u_{3\ell}),\ldots)\times\begin{pmatrix}\mathcal{E}_1 & 1 & 0 & 0 & . \\ 0 & \mathcal{E}_1 & 0 & 0 & . \\ 0 & 0 & \mathcal{E}_3 & 0 & . \\ 0 & 0 & 0 & \mathcal{E}_4 & . \\ . & . & . & . & . \end{pmatrix}\begin{pmatrix}(\hat{u}_{1\ell}|\chi\rangle \\ (u_{1\ell}|\chi\rangle \\ (u_{3\ell}|\chi\rangle \\ . \\ . \end{pmatrix}+\frac{2}{\pi}\int_c\langle\Phi|\psi_\ell^{(+)}(k')\rangle\mathcal{E}'\langle\psi_\ell^{(+)}(k')|\chi\rangle dk'. \qquad (88)$$

From this example, it is evident that in a degeneracy of two resonances in the absence of symmetry, the degenerate complex eigenvalue $\mathcal{E}_1$ occurs twice in the spectral representation of the radial Hamiltonian $H_r^{(\ell)}$ given in Eq. (88), while there is only one Gamow eigenvector or normal mode, $u_{1\ell}(k_1,r)$, associated with the degeneracy. This is so because the Gamow-Jordan generalized eigenfunction or abnormal mode, $\hat{u}_{1\ell}(k_1,r)$, is not an eigenfunction of the radial Hamiltonian $H_r^{(\ell)}$. This is a generic property of this kind of degeneracy, which may be stated in slightly more formal terms as follows: In a degeneracy of resonances in the absence of symmetry, the algebraic multiplicity is always larger than the geometric multiplicity. Here, we mean by algebraic multiplicity of a degeneracy, μ_a, the number of times the degenerate complex eigenvalue is repeated, and by geometric multiplicity of the degeneracy, μ_g, the dimensionality of the subspace spanned by the eigenvectors associated with the degenerate eigenvalue [34–36].

Then,

$$\mu_a>\mu_g. \qquad (89)$$

Let us consider now, the complex energy representation of the resolvent operator. In this case $f(H_r^{(\ell)})=1/(E-H_r^{(\ell)})$. Then, from Eq. (87), we obtain,

$$\langle\Phi|\frac{1}{E-H_r^{(\ell)}}|\chi\rangle=(\langle\Phi|u_{1\ell}),\langle\Phi|\hat{u}_{1\ell}),\langle\Phi|u_{3\ell}),\ldots)\begin{pmatrix}\frac{1}{E-\mathcal{E}_1} & \frac{1}{(E-\mathcal{E}_1)^2} & 0 & 0 & . \\ 0 & \frac{1}{E-\mathcal{E}_1} & 0 & 0 & . \\ 0 & 0 & \frac{1}{E-\mathcal{E}_3} & 0 & . \\ 0 & 0 & 0 & \frac{1}{E-\mathcal{E}_4} & . \\ . & . & . & . & .\end{pmatrix}\begin{pmatrix}(\hat{u}_{1\ell}|\chi\rangle \\ (u_{1\ell}|\chi\rangle \\ (u_{3\ell}|\chi\rangle \\ . \\ .\end{pmatrix}$$

$$+\frac{2}{\pi}\int_c\langle\Phi|\psi_\ell^{(+)}(k')\rangle\frac{1}{(E-\mathcal{E}')}\langle\psi_\ell^{(+)}(k')|\chi\rangle dk'. \tag{90}$$

It may easily be verified that, when we delete the arbitrary functions $\Phi(r)$ and $\chi(r)$ in this expression, the resulting expansion for $\langle r|[1/(E-H_r^{(\ell)})]|r'\rangle$ is just the expansion in resonance eigenfunctions of the complete Green's function

$$G_\ell^{(+)}(k;r,r')=\frac{\hbar^2}{2\mu}\Bigg[\sum_{\substack{s\ \text{bound}\\ \text{states}}}\frac{v_{s\ell}(k,r)v_{s\ell}^*(k,r')}{E+|E_s|}+\sum_{\substack{n\neq m\\ \text{resonant}\\ \text{states}}}\frac{u_{n\ell}(k_n,r)u_{n\ell}(k_n,r')}{E-\mathcal{E}_n}+\frac{u_{m\ell}(k_m,r)u_{m\ell}(k_m,r')}{(E-\mathcal{E}_m)^2}$$

$$+\frac{u_{m\ell}(k_m,r)\hat{u}_{m\ell}(k_m,r')+\hat{u}_{m\ell}(k_m,r)u_{m\ell}(k_m,r')}{(E-\mathcal{E}_m)}\Bigg]+\frac{2}{\pi}\int_C\frac{\psi_\ell^{(+)}(k',r)\psi_\ell^{(+)*}(k',r')}{(k^2-k'^2)}dk'. \tag{91}$$

The occurrence of the double pole in $G_\ell^{(+)}(k;r,r')$, as a function of the complex energy, is thus associated with the occurrence of a Jordan block of rank 2 in the complex basis representation of the resolvent operator and a Jordan chain of Gamow-Jordan generalized eigenfunctions $\{\hat{u}_{1\ell}(k_1,r),u_{1\ell}(k_1,r)\}$ associated with the double zero of the Jost function.

Finally, let us consider the time evolution operator $\exp(-iHt)$. For each fixed value of the angular momentum, it will be enough to consider the operator $f(H_r^{(\ell)})=\exp(-iH_r^{(\ell)}t)$. In this case, from Eq. (87),

$$\langle\Phi|\exp(-iH_r^{(\ell)}t)|\chi\rangle=(\langle\Phi|u_{1\ell}),\langle\Phi|\hat{u}_{1\ell}),\langle\Phi|u_{3\ell}),\ldots)\begin{pmatrix}\exp(-i\mathcal{E}_1t) & -it\exp(-i\mathcal{E}_1t) & 0 & 0 & . \\ 0 & \exp(-i\mathcal{E}_1t) & 0 & 0 & . \\ 0 & 0 & \exp(-i\mathcal{E}_3t) & 0 & . \\ 0 & 0 & 0 & \exp(-i\mathcal{E}_4t) & . \\ . & . & . & . & .\end{pmatrix}$$

$$\times\begin{pmatrix}(\hat{u}_{1\ell}|\chi\rangle \\ (u_{1\ell}|\chi\rangle \\ (u_{3\ell}|\chi\rangle \\ . \\ .\end{pmatrix}+\frac{2}{\pi}\int_c\langle\Phi|\psi_\ell^{(+)}(k')\rangle\exp(-i\mathcal{E}'t)\langle\psi_\ell^{(+)}(k')|\chi\rangle dk'. \tag{92}$$

As in the previous examples, the time evolution operator is nondiagonal in the complex energy basis representation. The time evolution of the Jordan chain of Gamow-Jordan generalized eigenfunctions $\{\hat{u}_{1\ell}(k_1,r),u_{1\ell}(k_1,r)\}$ is given by a Jordan block of 2×2 with an exponential time dependence in the diagonal entries and a first-order polynomial times an exponential in the off-diagonal entry. Hence, the time evolution of the Gamow-Jordan generalized eigenfunction or abnormal mode is a superposition of the abnormal mode $\hat{u}_{1\ell}(k_1,r)$ evolving exponentially in time plus the normal mode $u_{1\ell}(k,r)$ evolving according to the product of a first-order polynomial times an exponential time evolution factor. The time evolution of the normal mode $u_{1\ell}(k_1,r)$ in the Gamow-Jordan chain $\{\hat{u}_{1\ell}(k_1,r),u_{1\ell}(k_1,r)\}$, as well as the time evolution of all other normal modes $u_{n\ell}(k_n,r)$ associated with the simple zeros of the Jost function (simple poles of the scattering matrix) are purely exponential.

An alternative derivation of the main results in this section in the rigged Hilbert-space formulation of quantum mechanics may be found in the papers by Bohm *et al.* [25] and Antoniou, Gadella, and Pronko [26].

X. ORTHOGONALITY AND NORMALIZATION INTEGRALS FOR GAMOW-JORDAN EIGENFUNCTIONS

As in the case of bound- and resonant-state eigenfunctions associated with simple poles of the Green's function, we may derive orthogonality and normalization rules for the Gamow-Jordan eigenstates in terms of regularized integrals of the generalized Gamow-Jordan eigenfunctions. Following the same procedure as by Berggren [32,33], it may be shown that, when $f_\ell(-k')$ has a double zero at $k'=k_m$, the following relations are valid:

$$\frac{1}{i8k_m^{2(\ell+1)}}f_\ell(k_m)\left(\frac{d^2f_\ell(-k')}{dk'^2}\right)_{k'=k_m} = \lim_{\nu\to 0}\int_0^\infty e^{-\nu r^2}\frac{d\phi_\ell(k_m,r)}{dk_m}\phi_\ell(k_m,r)dr \quad (93)$$

and

$$\frac{1}{i8k_m^{2(\ell+1)}}f_\ell(k_m)\left[\frac{1}{3}\left(\frac{d^3f_\ell(-k')}{dk'^3}\right)_{k'=k_m} - \left(\frac{d^2f_\ell(-k')}{dk'^2}\right)_{k'=k_m}\left(\frac{2(\ell+1)}{k_m}-\frac{1}{f_\ell(k_m)}\frac{df_\ell(k_m)}{dk_m}\right)\right] = \lim_{\nu\to 0}\int_0^\infty e^{-\nu r^2}\left(\frac{d\phi_\ell(k_m,r)}{dk_m}\right)^2 dr. \quad (94)$$

From the expression (51) for $C_\ell(k_m)$ and Eqs. (93) and (94), it follows that

$$\lim_{\nu\to 0}\int_0^\infty e^{-\nu r^2}\left(\frac{d\phi_\ell(k_m,r)}{dk_m}\right)^2 dr + 2C_\ell(k_m)\frac{\hbar^2k_m}{\mu}\times\left(\lim_{\nu\to 0}\int_0^\infty e^{-\nu r^2}\frac{d\phi_\ell(k_m,r)}{dk_m}\phi_\ell(k_m,r)dr\right)=0, \quad (95)$$

which may be rewritten as

$$\lim_{\nu\to 0}\int_0^\infty e^{-\nu r^2}\left[\frac{d\phi_\ell(k_m,r)}{d\mathcal{E}_m}+C_\ell(k_m)\phi_\ell(k_m,r)\right]^2 dr = C_\ell^2(k_m)\lim_{\nu\to 0}\int_0^\infty e^{-\nu r^2}\phi_\ell^2(k_m,r)dr, \quad (96)$$

but, according to Eq. (32) and (33), when $f_\ell(-k)$ has a double zero at $k=k_m$, the integral on the right-hand side of Eq. (96) vanishes. Therefore, the integrand on the left-hand side of Eq. (96) is the square of the generalized Jordan-Gamow eigenfunction, and the relation (94) translates into

$$\lim_{\nu\to 0}\int_0^\infty e^{-\nu r^2}\hat{\phi}_\ell^2(k_m,r)dr=0, \quad (97)$$

which shows that also the regularized integral of the square of the generalized Gamow-Jordan eigenfunction vanishes.

An expression for the normalization constant $\mathcal{N}_{m\ell}^2$ in terms of a normalization integral may be obtained from Eq. (93),

$$\mathcal{N}_{m\ell}^2=\lim_{\nu\to 0}\int_0^\infty e^{-\nu r^2}\frac{d\phi_\ell(k_m,r)}{d\mathcal{E}_m}\phi_\ell(k_m,r)dr. \quad (98)$$

Writing $d\phi_\ell/d\mathcal{E}_m$ in terms of $\hat{\phi}_\ell(k_m,r)$ and recalling that the integral of $\phi_\ell^2(k_m,r)$ vanishes, we get

$$\mathcal{N}_{m\ell}^2=\lim_{\nu\to 0}\int_0^\infty e^{-\nu r^2}\hat{\phi}_\ell(k_m,r)\phi_\ell(k_m,r)dr, \quad (99)$$

which shows that the right-hand side of Eq. (99) is the normalization integral for the Gamow-Jordan generalized eigenfunctions associated with a double-pole degeneracy of resonances with $\mathcal{N}_{m\ell}^2$ as given in Eq. (52). However, it is convenient to note that this expression does not fix the normalization rule for $\phi_\ell(k_m,r)$ and $\hat{\phi}_\ell(k_m,r)$ in a unique way. Since $\phi_\ell(k_m,r)$ and $\hat{\phi}_\ell(k_m,r)$ are linearly independent, they have different dimensions and their product has no obvious interpretation in terms of observable quantities, therefore, there is no *a priori* reason to normalize both functions with the same normalization constant. Thus, we still have the freedom to write Eq. (99) as

$$\lim_{\nu\to 0}\int_0^\infty e^{-\nu r^2}\left(\frac{X_m}{\mathcal{N}_{m\ell}}\hat{\phi}_\ell(k_m,r)\right)\left(\frac{1}{X_m\mathcal{N}_{m\ell}}\phi_\ell(k_m,r)\right)dr=1, \quad (100)$$

where $\mathcal{N}_{m\ell}^2$ is given in Eq. (52) and X_m is a nonvanishing real or complex number that we associate with the double-pole singularity of $G_\ell^{(+)}(k;r,r')$ at $k=k_m$. Therefore, a more general normalization rule for the Gamow and Gamow-Jordan generalized eigenfunction than that proposed in Eqs. (52)–(54) would be

$$u_{m\ell}(k_m,r)=\frac{1}{X_m\mathcal{N}_{m\ell}}\phi_\ell(k_m,r) \quad (101)$$

and

$$\hat{u}_{m\ell}(k_m,r)=\frac{X_m}{\mathcal{N}_{m\ell}}\hat{\phi}_\ell(k_m,r). \tag{102}$$

With this normalization, the orthogonality and normalization integrals for the generalized Gamow-Jordan eigenfunction associated with a double pole of the Green's function, Eqs. (33), (97), and (99), take the form

$$\lim_{\nu\to 0}\int_0^\infty e^{-\nu r^2}u_{m\ell}^2(k_m,r)dr=0, \tag{103}$$

$$\lim_{\nu\to 0}\int_0^\infty e^{-\nu r^2}\hat{u}_{m\ell}^2(k_m,r)dr=0, \tag{104}$$

and

$$\lim_{\nu\to 0}\int_0^\infty e^{-\nu r^2}u_{m\ell}(k_m,r)\hat{u}_{m\ell}(k,r)dr=1. \tag{105}$$

The form of these orthogonality and normalization conditions is independent of the value of the constant X_m. However, if the Gamow-Jordan generalized eigenfunctions are normalized according to Eqs. (101) and (102), the expression for the residue at the double pole of $G_\ell^{(+)}(k;r,r')$ would be explicitly dependent on X_m, since a factor X_m^2 will appear, multiplying the term $u_{m\ell}(k_m,r)u_{m\ell}(k_m,r')$ in the expression for the residue at the double pole of $G_\ell^{(+)}(k;r,r')$ given in Eq. (91).

$$\frac{X_m^2 u_{m\ell}(k_m,r)u_{m\ell}(k_m,r')}{(E-\mathcal{E}_m)^2} + \frac{u_{m\ell}(k_m,r)\hat{u}_{m\ell}(k_m,r')+\hat{u}_{m\ell}(k_m,r)u_{m\ell}(k_m,r')}{(E-\mathcal{E}_m)}. \tag{106}$$

As is evident from the definition (50), the generalized eigenfunctions $\phi_{n\ell}(k_m,r)$ and $\hat{\phi}_{n\ell}(k_m,r)$ have different dimensions, if one takes X_m of dimension (energy)$^{1/2}$, the normalized eigenfunctions $u_{n\ell}(k_n,r)$ and $\hat{u}_{n\ell}(k_n,r)$ have the same dimensions, namely, (energy)$^{-1/2}$ so that when $[X_m] =$(energy)$^{1/2}$ the higher-order Gamow-Jordan vectors become Jordan vectors with the same dimensions as the Gamow vectors.

This freedom in the normalization rules could be used to define normalized Gamow-Jordan eigenfunctions with the same dimensions as those of the Gamow eigenfunctions associated with simple poles of $G_\ell^{(+)}(k;r,r')$.

XI. SUMMARY AND CONCLUSIONS

In the theory of the scattering of a beam of particles by a short-ranged potential, resonances are associated with the occurrence of poles of the scattering matrix $S_\ell(k)$, the Green's function $G_\ell^{(+)}(k;r,r')$, and the scattering wave function $\psi_{n\ell}(k,r)$. These resonance poles are caused by zeros of the Jost function lying in the fourth quadrant of the complex k plane. Accordingly, a degeneracy of resonances, that is, the exact coincidence of two (or more) simple resonance poles of the scattering matrix, results from the exact coincidence of two (or more) simple resonance zeros of the Jost function, which merge into one double (or higher rank) zero lying in the fourth quadrant of the complex k plane.

We found that, associated with a double-resonance zero of the Jost function, there is a Jordan chain of length 2 [35,36] of generalized Gamow-Jordan eigenfunctions $\{\hat{u}_{m\ell}(k_m,r),u_{m\ell}(k_m,r)\}$ belonging to the same degenerate complex energy eigenvalue $\mathcal{E}_m$. Hence, the corresponding second-rank pole occurring in the scattering matrix, $S_\ell(k)$, the Green's function $G_\ell^{(+)}(k;r,r')$ and the scattering wave function $\psi_\ell^{(+)}(k,r)$ is also associated with this Jordan chain of Gamow-Jordan generalized resonance eigenfunctions.

As the two simple zeros of the Jost function merge into one double zero, the two Gamow eigenfunctions corresponding to the two resonances that become degenerate merge into one Gamow eigenfunction or normal mode belonging to the double zero of the Jost function. The other element in the Jordan chain, namely, the Gamow-Jordan generalized eigenfunction or abnormal mode is not a proper eigenfunction of the radial Hamiltonian. Hence, at a degeneracy of resonances, one resonance eigenfunction or normal mode is lost, and a new kind of generalized resonance eigenfunction or abnormal mode is generated. Therefore, the dimensionality of the subspace of eigenfunctions associated with a degeneracy of two resonances or geometric multiplicity μ_g of the degeneracy is 1, yet, the number of times the degenerate complex energy eigenvalue is repeated in the spectral representation of $H_r^{(\ell)}$ or algebraic multiplicity of the degeneracy μ_a is 2. It follows that the algebraic multiplicity is larger than the geometric multiplicity of a degeneracy of resonances.

Explicit expressions for the normalized Gamow and Gamow-Jordan generalized eigenfunctions in the Jordan chain, written in terms of the outgoing wave Jost solution, the Jost function, and its derivatives evaluated at the double zero, are obtained from the computation of the residue of the scattering wave $\psi_\ell^{(+)}(k,r)$ function at the double pole. The chain of Gamow-Jordan generalized eigenfunctions are solutions of a Jordan chain of differential equations with the same boundary conditions as those satisfied by the Gamow eigenfunctions.

We also showed that the Jordan chain of generalized eigenfunctions are elements of the complex biorthonormal basis formed by the real (bound states) and complex (resonance states) energy eigenfunctions, which can be completed by means of a continuum of scattering wave functions of complex wave number. This set is a complete basis of a rigged Hilbert space. With the help of this result, we derived expansion theorems (spectral representations) for operators $f(H_r^{(\ell)})$, which are regular functions of the radial Hamiltonian $H_r^{(\ell)}$. In this basis, the operator $f(H_r^{(\ell)})$ is represented by a complex matrix which is diagonal except for one Jordan block of rank 2 [35,36] associated with the double zero of the Jost function and the corresponding chain of generalized eigenvectors. The diagonal entries in this matrix are the eigenvalues $f(\mathcal{E}_n)$, simple zeros of the Jost function correspond to nondegenerate eigenvalues of $f(H_r^{(\ell)})$, while the double zero of the Jost function corresponds to the twice repeated (degenerate) eigenvalue $f(\mathcal{E}_m)$ in the diagonal entries of the Jordan block. The off-diagonal, nonvanishing element in this block is $\partial f(\mathcal{E}_m)/\partial\mathcal{E}_n$. In particular, the occurrence of a double pole in the Green's function, as a function of the complex energy, is thus associated with the occurrence of a Jordan block of rank 2 in the complex basis representation of the resolvent operator and the corresponding Jordan chain of Gamow-Jordan generalized eigenfunctions.

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[1] A. Bohm, *Quantum Mechanics*, 3rd ed. (Springer-Verlag, Berlin, 1993).
[2] M.L. Goldberger and K.M. Watson, Phys. Rev. **136**, B1472 (1964).
[3] K.M. McVoy, in *Fundamentals in Nuclear Theory*, edited by A. de Shalit and C. Villi (IEAE, Vienna, 1967).
[4] L. Stodolsky, in *Experimental Meson Spectroscopy*, edited by C. Balthay and A.H. Rosenfeld (University Press, New York, 1970), p. 395.
[5] J. Lukierski, Bull. Acad. Pol. Sci., Ser. Sci., Math., Astron. Phys. **XV**, 217 (1967); **XV**, 223 (1967).
[6] F. Hinterberger *et al.*, Nucl. Phys. A **299**, 397 (1978).
[7] P. von Brentano, Phys. Rep. **264**, 57 (1996).
[8] E. Hernández and A. Mondragón, Phys. Lett. B **326**, 1 (1994).
[9] L.M. Baskov *et al.*, Nucl. Phys. B **256**, 365 (1985), and references therein.
[10] P. von Brentano, Z. Phys. A **348**, 41 (1994).
[11] P. von Brentano, R.V. Jolos, and H.A. Weidenmüller, Phys. Lett. B **534**, 63 (2002).
[12] P. von Brentano, Rev. Mex. Fis. **48**, 1 (2002).
[13] K.E. Lassila and V. Ruuskanen, Phys. Rev. Lett. **17**, 490 (1966).
[14] P.L. Knight, Phys. Lett. **72A**, 309 (1979).
[15] M. Pont, R.M. Potvliege, R. Shakeshaft, and P.H.G. Smith, Phys. Rev. A **46**, 555 (1992).
[16] O. Latinne *et al.*, Phys. Rev. Lett. **74**, 46 (1995).
[17] N.J. Kylstra and C.J. Joachain, Phys. Rev. A **57**, 412 (1998).
[18] E. Hernández, A. Jáuregui and A. Mondragón, Rev. Mex. Fis. **38**, 128 (1992).
[19] A. Mondragón and E. Hernández, J. Phys. A **29**, 2567 (1996).
[20] A. Mondragón and E. Hernández, in *Irreversibility and Causality: Semigroups and Rigged Hilbert Space*, edited by A. Bohm, D.-H. Doebner, and P. Kielanowski, Lecture Notes in Physics, Vol. 504 (Springer-Verlag, Berlin, 1998) p. 257.
[21] C. Dembowski *et al.*, Phys. Rev. Lett. **86**, 787 (2001)
[22] W. Vanroose *et al.*, J. Phys. A **30**, 5543 (1997).
[23] E. Hernández, A. Jáuregui, and A. Mondragón, J. Phys. A **33**, 4507 (2000).
[24] W. Vanroose, Phys. Rev. A **64**, 062708 (2001).
[25] A. Bohm *et al.*, J. Math. Phys. **38**, 6072 (1997).
[26] I. Antoniou, M. Gadella, and G. Pronko, J. Math. Phys. **39**, 2459 (1998).
[27] R.G. Newton, *Scattering Theory of Waves and Particles*, 2nd ed. (Springer-Verlag, New York, 1982) Chap. 12.
[28] A. Bohm and M. Gadella, *Dirac Kets, Gamow Vectors and Gel'fand Triplets*, Lecture Notes in Physics, Vol. 348 (Springer-Verlag, Berlin 1989).
[29] R. de la Madrid, J. Phys. A **35**, 319 (2002)
[30] N. Dunford and J Schwarz, *Linear Operators* (Interscience, New York, 1963), Vol. 2, p. 1306.
[31] Ya.B. Zel'dovich, Zh. Eksp. Teor. Fiz. **39**, 776 (1960) [Sov. Phys. JETP **12**, 542 (1961)].
[32] T. Berggren, Nucl. Phys. A **109**, 265 (1968).
[33] T. Berggren, Phys. Lett. B **373**, 1 (1996).
[34] Tosio Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, Berlin, 1980).
[35] Nathan Jacobson, *Linear Algebra*, *Lecture Notes in Abstract Algebra*, Vol. II (Van Nostrand, New York, 1953), Chap. III.
[36] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, 2nd ed. (Academic Press, New York, 1985).
[37] From the way it was derived, it is evident that the matrix in Eq. (87) represents the action of $f(H_r^{(\ell)})$ as an operator on the space of continuous antilinear functionals on the Schwarz space of very well behaved test functions.