Local permutations of products of Bell states and entanglement distillation

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We present different algorithms for mixed-state multicopy entanglement distillation for pairs of qubits. Our algorithms perform significantly better than the best-known algorithms. Better algorithms can be derived that are tuned for specific initial states. These algorithms are based on a characterization of the group of all locally realizable permutations of the 4^n possible tensor products of n Bell states.

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I. INTRODUCTION

We study mixed-state multicopy entanglement distillation protocols for pairs of qubits. We start from n identical copies of a Bell-diagonal state of two qubits and end up, after local operations and classical communication, with $m<n$ Bell diagonals (possibly statistically dependent) with higher joint fidelity than m copies of the original Bell-diagonal state. For non-Bell-diagonal initial states, one can first perform n separate optimal single-copy distillation protocols to make them Bell diagonal [1]. Our protocol can be used in a recurrence scheme followed by the hashing protocol as in Refs. [2,3]. We propose a protocol with $n=4$ and $m=1$, which does significantly better than the existing protocols. Our results can be used to find even better protocols for other values of n and m that are tuned for specific initial states.

We see three main reasons for studying entanglement distillation protocols. The first and most obvious reason is that entanglement distillation protocols are a means of obtaining states that are closer to maximally entangled pure states, as needed in typical applications such as teleportation, from mixed states that can be reached by sending one qubit of an entangled pair through a realistic channel. A second reason to study distillation protocols is that asymptotic protocols yield a lower bound for ''entanglement of distillation,'' an important measure of entanglement, that is in itself a lower bound for any sensible measure of entanglement [4]. In this context, we also mention the upper bounds on entanglement of distillation obtained in Ref. [5]. A third reason is that multicopy entanglement protocols can be considered as applications of entanglement, where more can be done in the presence of entangled pairs than without. We hope that studying these mechanisms will reveal some information on the important problem of how exactly the presence of entanglement enables one to do things that are impossible without.

Multicopy mixed-state entanglement distillation for qubit pairs was first studied in Refs. [2,3]. An improved variant of the two-copy protocol in that paper was described in Ref. [6] under the title of quantum privacy amplification. Other variants appeared in Refs. [7–9]. These protocols as well as ours start from n identical qubit pairs in a Bell-diagonal state, shared by Alice and Bob. A crucial ingredient of these protocols is a local unitary operation, performed by Bob and Alice on their n qubits, which results globally in a permutation of the 4^n possible tensor products of Bell states. The key ingredient of this paper is a characterization, by means of a binary matrix group, of all possible local permutations of the products of Bell states. This enables a search for the best protocol within this setting. In Sec. II, we study local permutations of products of Bell states. In Sec. III, we discuss the protocols. In Sec. IV, we discuss the combination of our protocols with a recurrence scheme and the hashing protocol and show the strength of our protocols by computer simulations.

II. LOCAL PERMUTATIONS OF PRODUCTS OF BELL STATES

In this section, we study the class of local unitary operations that can be performed by Alice and Bob locally and result in a permutation of the 4^n (tensor) products of Bell states, where n is the number of qubit pairs. These local permutations are the key ingredient of the different distillation protocols described in the following section.

We will code products of Bell states by binary vectors by assigning two-bit vectors to the Bell states as follows

$$
\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)=|B_{00}\rangle,\\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)=|B_{01}\rangle,\\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle-|11\rangle)=|B_{10}\rangle,\\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle-|10\rangle)=|B_{11}\rangle.
\end{aligned}
\tag{1}
$$

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A product of n Bell states is then described by a $2n$-bit vector, e.g., $|B_{001101}\rangle = |B_{00}\rangle|B_{11}\rangle|B_{01}\rangle = |\Phi^+\rangle|\Psi^-\rangle|\Psi^+\rangle$.

We will also exploit a correspondence between Bell states and Pauli matrices

$$|\Phi^+\rangle \rightarrow \frac{1}{\sqrt{2}}\sigma_{00} = \frac{1}{\sqrt{2}}\sigma_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$|\Psi^+\rangle \rightarrow \frac{1}{\sqrt{2}}\sigma_{01} = \frac{1}{\sqrt{2}}\sigma_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$|\Phi^-\rangle \rightarrow \frac{1}{\sqrt{2}}\sigma_{10} = \frac{1}{\sqrt{2}}\sigma_z = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$|\Psi^-\rangle \rightarrow \frac{1}{\sqrt{2}}\sigma_{11} = \frac{1}{\sqrt{2}}\sigma_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

A tensor product of n Bell states is then described by a tensor product of Pauli matrices, e.g., $|\Phi^+\rangle|\Psi^-\rangle|\Psi^+\rangle \rightarrow 1/\sqrt{8}\,\sigma_0 \otimes \sigma_y \otimes \sigma_x = 1/\sqrt{8}\,\sigma_{00} \otimes \sigma_{11} \otimes \sigma_{01}$. We will also use longer vector subscripts to denote such tensor products, e.g.,

$$\sigma_{001101} = \sigma_{00} \otimes \sigma_{11} \otimes \sigma_{01}. \tag{2}$$

In this representation of pure states of $2n$ qubits as $2^n \times 2^n$ matrices $\tilde{\Psi}$, local unitary operations $|\psi\rangle \rightarrow (U_A \otimes U_B)|\psi\rangle$, in which Alice acts on her n qubits (jointly) with an operation U_A and Bob on his with an operation U_B, are represented by

$$\tilde{\Psi} \rightarrow U_A \tilde{\Psi} U_B^T. \tag{3}$$

We are now in a position to state the main result of this section.

Theorem 1. (i) If a local unitary operation (3) results in a permutation of the 4^n tensor products of n Bell states, this permutation can be represented on the binary vector representations (1) as an affine operation

$$\phi: \mathbb{Z}_2^{2n} \rightarrow \mathbb{Z}_2^{2n}: x \rightarrow Ax + b,$$

$$\text{with } A \in \mathbb{Z}_2^{2n \times 2n}, \quad b \in \mathbb{Z}_2^{2n}$$

$$\text{and } A^T P A = P,$$

$$\text{where } P = \text{diag}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right). \tag{4}$$

(ii) Conversely, any such permutation can be realized by local unitary operations.

Note that all multiplication and addition should be done modulo 2. We call a matrix A satisfying $A^T P A = P$ "P orthogonal." The affine and linear transformations considered are invertible and, therefore, amount to a permutation of $\mathbb{Z}_2^{2n}$. In the sequel we sometimes directly refer to the linear transformations as permutations.

Proof. We first prove part (i). One can easily check that $\sigma_v \sigma_w \sim \sigma_{v+w}$, where v and w are binary vector indices as in Eq. (2), and the $-$ sign means equal up to a complex phase (in this case 1, i, -1 or $-i$). Such a phase is irrelevant as the Pauli matrices here are matrix representations of pure state vectors.

Assume now that U_A and U_B indeed result in a permutation $\pi: \mathbb{Z}_2^{2n} \rightarrow \mathbb{Z}_2^{2n}$, then the null vector is mapped to some vector $v = \pi(0)$. Accordingly, $U_A \sigma_0 U_B^T \sim \sigma_v$. Since σ_0 is the identity matrix, we have $U_B \sim \sigma_v U_A^*$ where * denotes complex conjugation. If we want to represent π by $x \rightarrow Ax + b$, we clearly have to choose $b = v$. Note that Eq. (3) now reads $\tilde{\Psi} \rightarrow U_A \tilde{\Psi} U_A^\dagger \sigma_b$.

We now have to show that the permutation $\pi': x \rightarrow \pi(x) + b$, which maps $\tilde{\Psi}$ to $U_A \tilde{\Psi} U_A^\dagger$, is a linear P-orthogonal map $\pi': x \rightarrow Ax$. Linearity of binary maps means that sums are mapped to sums $\pi'(v+w) = \pi'(v) + \pi'(w)$. This is clearly true since $U_A \sigma_{v+w} U_A^\dagger \sim U_A \sigma_v \sigma_w U_A^\dagger = U_A \sigma_v U_A^\dagger U_A \sigma_w U_A^\dagger$. Furthermore, it can be verified using the commutation and anticommutation laws for Pauli matrices that σ_v and σ_w are commutable operators if and only if $v^T P w = 0$. Since σ_v and σ_w are commutable if and only if $U_A \sigma_v U_A^\dagger$ and $U_A \sigma_w U_A^\dagger$ are commutable, it must hold that $v^T A^T P A w = v^T P w$ for all v and w, which proves $A^T P A = P$.

To prove part (ii), we will first consider $n = 2$ and show that all permutations $\pi: x \rightarrow Ax$ with $A^T P A = P$ can be generated with the operations $\phi_u: \tilde{\Psi} \rightarrow U_u \tilde{\Psi} U_u^\dagger$ with $U_u = e^{i(\pi/4)\sigma_u} = 1/\sqrt{2}(I + i\sigma_u)$ (with $u \in \mathbb{Z}_2^4$). (This is also true for $n > 2$, but generators affecting more than two qubits at a time will not be needed). Using $\sigma_v \sigma_w = (-1)^{v^T P w} \sigma_w \sigma_v$, it can be shown that ϕ_u translated into the binary language results in a permutation $\pi_u: x \rightarrow x + u(u^T P x) = (I + uu^T P)x$.

We will now show that the group of permutations generated by the permutations π_u is isomorphic to S_6, the group of all permutations of six elements. Next, we will show that the group of P-orthogonal 4×4 matrices contains $6! = 720$ elements, which proves that all P-orthogonal permutations are generated. Since no permutation (except the identity) is commutable with all the permutations, S_6 is isomorphic to the group of transformations $\chi_q: S_6 \rightarrow S_6: p \rightarrow qpq^{-1}$, where p and q are permutations of six elements. Such a transformation χ_q is completely determined by specifying the images of the 15 commutations $p_{i,j}$, permutations on $\{1,2,3,4,5,6\}$ that permute i and j. This holds because any permutation is a composition of such commutations and $\chi_q(p_1 p_2) = \chi_q(p_1)\chi_q(p_2)$. Note that the image under χ_q of a commutation is again a commutation. As a result, S_6 is isomorphic to the group of permutations of 15 elements obtained by restricting χ_q to the commutations. We will show that this is exactly the group of permutations generated by the generators π_u (which can be considered as permutations of 15 elements as 0000 can be left out, being always mapped to itself). To this end, we establish the following correspondence between nonzero four-bit vectors and commutations $\gamma: u \rightarrow p_{i,j}$:

$$0001\to p_{5,6},\quad 0010\to p_{4,6},\quad 0011\to p_{4,5},$$

$$0100\to p_{2,3},\quad 0101\to p_{1,4},\quad 0110\to p_{1,5},\quad 0111\to p_{1,6},$$

$$1000\to p_{1,3},\quad 1001\to p_{2,4},\quad 1010\to p_{2,5},\quad 1011\to p_{2,6},$$

$$1100\to p_{1,2},\quad 1101\to p_{3,4},\quad 1110\to p_{3,5},\quad 1111\to p_{3,6}.$$

It can be verified that $\gamma[\pi_u(x)]=\chi_{\gamma(u)}[\gamma(x)]$ for all u and x. So π_u and $\chi_{\gamma(u)}$ realize the same permutation of 15 elements. As a consequence, also products $\pi_{u_1}\times\ldots\times\pi_{u_k}$ realize the same permutations as products $\chi_{\gamma(u_1)}\times\ldots\times\chi_{\gamma(u_k)}$. This finally establishes the isomorphism between S_6 and the permutations generated by the π_u.

It remains to be shown that there are 6! P-orthogonal 4×4 matrices. It follows from $A^TPA=P$ that A is P orthogonal if and only if all the pairs of columns of A represent commutable σ_{a_i} except for the first and second or the third and fourth columns. Therefore, to make an arbitrary P-orthogonal matrix, the first column a_1 can be chosen to be any nonzero four-bit vector (15 choices), the second column should satisfy $a_1^TPa_2=1$ (one linear condition yielding eight possible a_2), the third column should be commutable with a_1 and a_2 (two linear conditions yielding three choices after excluding 0000) and finally the fourth column should be commutable with a_1 and a_2 and noncommutable with a_3 (three linear conditions, yielding two possibilities). This results in $15\times8\times3\times2=720=6!$ possibilities. This ends the proof for $n=2$.

For $n>2$, we turn to the matrix picture and show that every P-orthogonal matrix A can be reduced to the identity matrix by two-qubit operations, i.e., 4×4 P-orthogonal matrices embedded in an identity matrix on rows and columns $2k+1,2k+2,2l+1,2l+2$ for some $k,l\in\{0,\ldots,n-1\}$. We concentrate on two columns of A at a time, first 1 and 2, then 3 and 4, and so on and transform them to the corresponding columns of the identity matrix with two-qubit operations. Assume, without loss of generality, that we are working on columns 1 and 2, then we name $K^{(k,l)}=A_{\{2k+1,2k+2,2l+1,2l+2\},\{1,2\}}$. If the two columns of $K^{(k,l)}$ are commutable, they can be thought of as the first and third columns of a 4×4 P-orthogonal matrix and can be reduced by a two-qubit operation to the first and third column of an identity matrix. If the two columns of $K^{(k,l)}$ are noncommutable, they can be reduced to the first and second columns of an identity matrix. One can see that by combining such two-qubit operations the first two columns of A can be reduced to the first two columns of an identity matrix. Due to the commutability relations between the columns of A, as a result, also the first two rows become the first rows of an identity matrix. One can now proceed in a similar way with the next pair of columns until the whole matrix is reduced to the identity matrix. The composition of the inverses of all two-qubit operations that were applied yields a decomposition of A into two-qubit operations that can be realized by local unitary operations as shown above. This ends the proof. □

In the proof, we saw that linear transformations ($b=0$) correspond to operations with $U_B=U_A^*$, i.e., $\tilde{\Psi}\to U_A\tilde{\Psi}U_A^\dagger$. The matrices U_A that under this action map tensor products of Pauli matrices to tensor products of Pauli matrices possibly with a minus sign are known to form the Clifford group, studied in Refs. [10,11] in the context of quantum error correction and quantum computation. The P-orthogonal matrices form a group that is isomorphic to a quotient group of the Clifford group. The Clifford group is known to be generated by controlled-NOT (CNOT) operations and one-qubit operations that map Pauli matrices to Pauli matrices. It is possible that this knowledge may be used to give other proofs for the theorem above. However, we think that our set of generators and the isomorphism between P-orthogonal 4×4 matrices and permutations of six elements are worthwhile results on their own. It also follows that the controlled-NOT operation should be decomposable in terms of our generators (at least realizing the same permutation of products of Pauli matrices up to signs, but the following formula gives the CNOT exactly). One can easily verify that $\mathrm{CNOT}=(1+i)/\sqrt{2}e^{-i(\pi/4)\sigma_{1000}}e^{i(\pi/4)\sigma_{1001}}e^{-i(\pi/4)\sigma_{0001}}$. Note that the first and last operations are actually one-qubit operations.

III. MIXED-STATE MULTICOPY ENTANGLEMENT DISTILLATION FROM PAIRS OF QUBITS

The distillation protocols presented in this paper can be summarized as follows.

(1) Start from n identical independent Bell-diagonal states with entanglement. This yields a mixture of 4^n tensor products of Bell states.

(2) Apply a local permutation of these 4^n products of Bell states as described in the preceding section. As a result, the n qubit pairs get statistically dependent.

(3) Check whether the last $n-m$ qubit pairs are $|\Phi\rangle$-states ($|\Phi^+\rangle$ or $|\Phi^-\rangle$). This can be accomplished locally by measuring both the qubits of each pair in the $|0\rangle,|1\rangle$ basis, and checking whether both measurements yield the same result.

(4) If all measured pairs were $|\Phi\rangle$ states, keep the first m pairs. This is a new mixture of 4^m products of Bell states.

This is a generalization of a protocol with $n=2$ and $m=1$, presented in Refs. [2,3]. In that protocol the applied local permutation consisted of a bilateral controlled-NOT operation by Alice and Bob. In our protocol, we will only consider linear permutations ($b=0$) as we expect that, in general, nothing can be gained by considering affine permutations. (For entangled states, the coefficient of $\sigma_{0,\ldots,0}$ dominates the other coefficients. Setting $b=0$ ensures that this coefficient will also contribute to the obtained entanglement after the protocol.)

In the following section, we discuss how to choose the local permutation so as to obtain a good protocol. The main result of this section is a formula for the resulting state of m pairs as a function of the permutation of Bell states performed in step (2) of the protocol

Theorem 2. If Alice and Bob apply the above protocol, starting from n-independent identical copies of a Bell-diagonal state $p_{00}|\Phi^+\rangle\langle\Phi^+|+p_{01}|\Psi^+\rangle\langle\Psi^+|+p_{10}|\Phi^-\rangle\langle\Phi^-|+p_{11}|\Psi^-\rangle\langle\Psi^-|$ *with* $p_{00}\geqslant p_{01}\geqslant p_{10}\geqslant p_{11}$, and with

entanglement, i.e., $p_{00}>\frac{1}{2}$, within step (2), a local operation by Alice and Bob that results in a permutation of products of Bell states $\pi: x \rightarrow Ax$ *with* $A^T P A = P$ as described in the preceding section, the resulting state of the remaining m qubit pairs is given by

$$2^{n-m} \sum_{y \in \mathbb{Z}_2^{2m}} \left(\frac{\sum_{x \in \mathcal{S}+PA^T P\bar{y}} p_x}{\sum_{x \in \mathcal{S}} s_x} \right) |B_y\rangle\langle B_y|, \tag{5}$$

where $\mathcal{S}$ is the subspace spanned by the rows of AP with indices $2m+2, 2m+4, \ldots, 2n$,

$$\begin{bmatrix} s_{00} \\ s_{01} \\ s_{10} \\ s_{11} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} p_{00} \\ p_{01} \\ p_{10} \\ p_{11} \end{bmatrix},$$

$\bar{y}$ *is* y extended with $2(n-m)$ zeros, and the long vector indices of p and s and B behave like the indices of σ in the preceding section, e.g., $p_{001101} = p_{00} p_{11} p_{01}$.

Proof. After the permutation, and before the measurement, the state of the n qubit pairs is given by $\Sigma_{x \in \mathbb{Z}_2^{2n}} p_x |B_{Ax}\rangle\langle B_{Ax}|$. The states $|B_{Ax}\rangle$ with $(Ax)_{2m+2}, (Ax)_{2m+4}, \ldots, (Ax)_{2n} = 0$ yield $|\Phi\rangle$ states and will be kept. These are the states $|B_{Ax}\rangle$, for which x is commutable with the rows $a_{2m+2}, \ldots, a_{2n}$ of AP. If we call the subspace of these vectors x, $\mathcal{R}$, the success rate (probability of keeping the first m pairs) is $\Sigma_{x \in \mathcal{R}} p_x$. Among the states that are kept, the ones with $(Ax)_j = y_j, j = 1, \ldots, 2m$ yield $|B_y\rangle$ states. Together with the conditions for being kept, these are $2m+(n-m)$ independent linear conditions, yielding a coset of an $(n-m)$-dimensional subspace of $\mathbb{Z}_2^{2n}$. This subspace must be $\mathcal{S}$ since the latter is $(n-m)$ dimensional and satisfies all homogeneous conditions (with $y=0$) by the P orthogonality of A. The right coset is obtained by adding $PA^T P\bar{y}$ (a combination of the first $2m$ rows of AP, determined by y). As a result, the state of the first m pairs after the measurement is

$$\sum_{y \in \mathbf{Z}_2^{2m}} \left(\frac{\sum_{x \in \mathcal{S}+PA^T P\bar{y}} p_x}{\sum_{x \in \mathcal{R}} p_x} \right) |B_y\rangle\langle B_y|$$

If all coefficients (for all y) are calculated, the denominator $\Sigma_{x \in \mathcal{R}} p_x$ can be calculated as the sum of the 2^{2m} numerators. If only one coefficient is needed (for instance, if only the fidelity of the end state is needed), the denominator can be calculated in a more efficient way as $\Sigma_{x \in \mathcal{R}} p_x = 2^{-(n-m)} \Sigma_{x \in \mathcal{S}} s_x$. One can easily verify that $s_v = \Sigma_{x \in \mathbf{Z}_2^{2n}} (-1)^{v^T P x} p_x$ (first verify for two bits and then extend). Therefore, $\Sigma_{v \in \mathcal{S}} s_v = \Sigma_{x \in \mathbf{Z}_2^{2n}} [\Sigma_{v \in \mathcal{S}} (-1)^{v^T P x}] p_x$. If x commutes with all $v \in \mathcal{S}$, $(-1)^{v^T P x} = 1$ for all $v \in \mathcal{S}$ and $\Sigma_{v \in \mathcal{S}} (-1)^{v^T P x} = 2^{n-m}$. If $v \notin \mathcal{S}$, one can easily show that half the coefficients $(-1)^{v^T P x}$ are one and half are -1. Now the states $x \in \mathcal{R}$ are exactly the ones, for which x is commutable with all elements of $\mathcal{S}$. Therefore, $\Sigma_{x \in \mathcal{S}} s_x = 2^{n-m} \Sigma_{x \in \mathcal{R}} p_x$. This concludes the proof. ■

IV. RECURRENCE SCHEMES

With the above formula for the end state of the protocol (Theorem 2), it is possible to derive good protocols by searching over all possible values for the relevant rows of the P-orthogonal matrix A and optimizing some quality measure. Typically this measure will depend on the fidelity of the end state and the success rate of the protocol (the probability of having $|\Phi\rangle$ states in the measured pairs). In that case, one only needs the first coefficient (the fidelity) and the denominator (the success rate) in Eq. (5), which both only depend on $\mathcal{S}$, a space spanned by only $n-m$ rows of A. Although this drastically limits the search space, it still grows exponentially with growing n.

Therefore, to come up with schemes for large n, one needs to use the recurrence scheme, as was proposed for $n=2$ and $m=1$ in Refs. [2,3]. If $m=1$, this scheme means that the above protocol is performed n times (with the same local permutation) and the n identical end states are taken as the input for a new step. Of course, more than two steps are possible too. One could also envision recurrence schemes with $m \neq 1$, for instance, combining two end states of an $n=4$, $m=2$ protocol to yield the input for a second step with $n=4$. In that case, however, the input for the second step would no longer consist of n independent pairs. Although, this only requires a minor modification of the above results (p_x and s_x can no longer be interpreted as products of $p_{00}, \ldots, p_{11}$), we will not consider this case in this paper.

To end up with almost pure Bell states, the recurrence scheme can best be combined with the hashing protocol as in Refs. [2,3]. The hashing protocol is the best-known asymptotic protocol (for $n \rightarrow \infty$) but can only be applied to Bell-diagonal states with high-enough fidelity. The combined protocol then consists of first applying a few recurrence steps and then switching to the hashing protocol.

The best-known $n=2$, $m=1$ recurrence scheme is the one of Ref. [6]. In our language it amounts to a scheme with a 4×4 P-orthogonal matrix whose last line is 11 11. It can be proven that this scheme yields the best achievable fidelity after one step (though not achieved with the best success rate) for initial probabilities that are ordered $p_{00} > p_{01} \geqslant p_{10} \geqslant p_{11}$. For this reason, it is also best to apply a pair-per-pair transformation after each recurrence step, which reorders the probabilities of the end state if they are not ordered. (One can easily find such one-pair transformations using the theory of Sec. II or equivalently using the local operations of Refs. [2,3]. This reordering scheme was also introduced in another mathematical setting in Ref. [9].)

Although, it is probably best to search for a new protocol for every given initial state, we propose below a protocol which we think is good if one does not have the time for such a search. We show by computer simulations that it performs better than the $n=2$ scheme.

Our scheme is an $n=4$, $m=1$ recurrence scheme com-

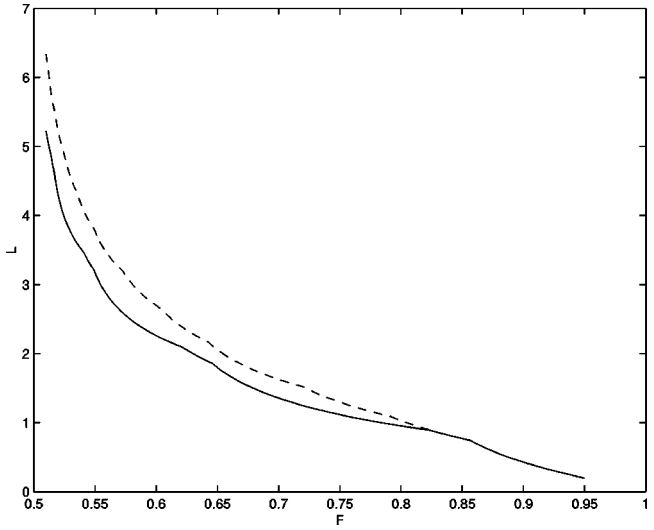

FIG. 1. Comparison of 10-logarithm of inverse asymptotic yield L for input Werner states with fidelity F for proposed protocol (full line) and existing recurrence-hashing protocol (dashed line).

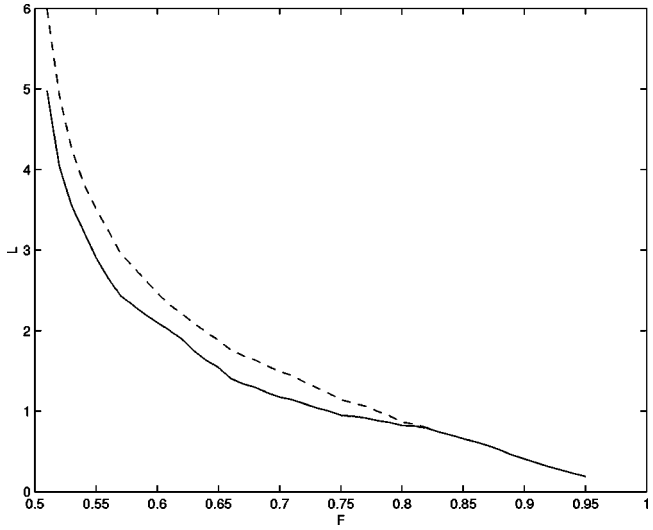

FIG. 2. Comparison of 10-logarithm of inverse asymptotic yield L averaged over 100 random non-Werner input states with fidelity F for proposed protocol (full line) and existing recurrence or hashing protocol (dashed line).

bined with hashing, and with as the last step possibly an $n=2$, $m=1$ step if this can lead to better performance. For the local permutation (determined by the P-orthogonal matrix A), we choose a permutation that is found experimentally to often lead to the best fidelity after one step, when starting with ordered probabilities. For this reason, we also apply a reordering in between recurrence steps as discussed for the $n=2$ protocol above. The chosen local permutation corresponds to an 8×8 P-orthogonal matrix A whose fourth, sixth, and eighth rows span the space spanned by $\{10\,11\,11\,10, 01\,10\,11\,00, 11\,10\,10\,11\}$. This can be achieved by the operations

$$U_A = U_B^* = e^{i\pi/4\sigma_{10\,01\,00\,00}} e^{i\pi/4\sigma_{01\,00\,00\,01}} \times e^{i\pi/4\sigma_{10\,00\,11\,00}} e^{i\pi/4\sigma_{00\,01\,10\,00}}. \quad (6)$$

In this realization, the first and second rows of the P-orthogonal matrix A are $01\,10\,00\,10$ and $10\,10\,10\,10$. These rows are needed to compute the reordering operations between the steps, for although the three values of p'_{01}, p'_{10}, and p'_{11} after one step of the protocol are fixed, their order is not. (The three cosets of $\mathcal{S}$ in $\mathcal{R}$ in Eq. (5) are fixed but not their order.)

This realization was found by exhaustive search over all operations that can be realized by four consecutive elementary two-qubit operations. If, for protocols with larger n for instance, no such simple realization can be found in a reasonable amount of time, one can always find a realization using the theory of Sec. II but this can increase the total amount of work for executing the distillation protocol. This was also one of the reasons for choosing $n=4$ in the proposed protocol.

As a performance measure, we have chosen the expected number of input pairs needed per output Bell state in an asymptotic protocol (the inverse of the asymptotic yield). The number of recurrence steps was also chosen as to optimize this measure. Figures 1 and 2 show the performance for our method ($n=4$, $m=1$ recurrence with the local permutation realized by U_A as in Eq. (6), with reordering between the steps, possibly one last $n=2$, $m=1$ step, and optimal switching to hashing protocol) and the method of Ref. [6] with reordering between the steps and optimal switching to the Hashing protocol. Figure 1 shows the results for Werner states (with $p_{00}=F>\frac{1}{2}$ and $p_{01}=p_{10}=p_{11}=c(1-F)/3$). Figure 2 shows the average performance of (for each value of the input fidelity) 100 random non-Werner states.

To do better than this protocol for a specific initial Bell-diagonal state, one can do several things depending on the amount of computing time available. One can try recurrence schemes with higher n and even higher m, but the amount of time needed increases fast with increasing n. There is, of course, no obligation to take the same local permutation in consecutive recurrence steps. One can also consider distilling more than one end state at once. Making two states with two $n=4$, $m=1$ protocols is just a special case of a nonoptimal $n=8$, $m=2$ protocol. One can, of course, search for better ones if one has the time. In this case, the two obtained Bell states will not be independent but as the fidelity goes to 1, their dependence will vanish. Also two consecutive recurrence steps, say two $n=2$, $m=1$ steps, can be considered as one bigger nonoptimal step, in this case with $n=4$, $m=1$. So if one has the time, he can in theory always go for a one shot protocol (no recurrence), but if one combines with the recurrence scheme, he can always afford lower initial entanglement with the same amount of computing time.

V. CONCLUSION

We have derived different protocols for distillation of entanglement from mixed states of two qubits. The protocols were based on a characterization of the group of all locally realizable permutations of the 4^n possible tensor products of n Bell states. Our protocols perform significantly better than

existing protocols as was shown by computer simulation. We also indicated how to derive even better protocols for specific initial states.

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