Multipartite entangled states in coupled quantum dots and cavity QED

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(Received 16 July 2002; published 6 February 2003)

We investigate the generation of multipartite entangled state in a system of N quantum dots embedded in a microcavity and examine the emergence of genuine multipartite entanglement by three different characterizations of entanglement. At certain times of dynamical evolution one can generate multipartite entangled coherent exciton states or multiqubit W states by initially preparing the cavity field in a superposition of coherent states or the Fock state with one photon, respectively. Finally, we study environmental effects on multipartite entanglement generation and find that the decay rate for the entanglement is proportional to the number of excitons.

DOI: 10.1103/PhysRevA.67.022302 PACS number(s): 03.67.Lx, 03.65.Ud, 68.65.Hb

I. INTRODUCTION

Quantum-information processing offers important capabilities for quantum communications and quantum computation in a variety of physical systems, and the solid-state implementation is one of the most promising candidates. Semiconductor quantum dots (QDs) are appealing for the realization of quantum computer and quantum-information processing. Spatial confinement of excitons in three dimensions leads to discrete energy levels, and the main causes of phase decoherence, namely, scattering events, are highly suppressed [1]. In this respect, QDs are very promising compared to other semiconductor structures with band states. Moreover, a quantum computer scheme based on quantum dot arrays is scalable up to $\geqslant 100$ qubits.

Entanglement is a essential source for quantum computing and quantum-information processing. It is well known that a controlled-NOT gate can generate a maximally entangled state. Various schemes for the realization of controlled-NOT gates in QDs [2–4] are proposed with either the electron spin [3–5] or the discrete electronic charge degree of freedom as qubits [6–8]. For example, Imamoğlu *et al.* [4] proposed a scheme that realized a controlled-NOT gate C_{NOT}^{ij} between two distant quantum dots i and j via the cavity quantum electromagnetic dynamics (CQED) techniques. Here, the first superscript i in C_{NOT}^{ij} denotes the control and the second denotes the target. These controlled-NOT gates in turn can be used to generate a tripartite maximally entangled Greenberger-Horne-Zeilinger (GHZ) [9] state as $|0\rangle\otimes|0\rangle\otimes|0\rangle+|1\rangle\otimes|1\rangle\otimes|1\rangle = C_{\mathrm{NOT}}^{13}C_{\mathrm{NOT}}^{12}(|0\rangle+|1\rangle)\otimes|0\rangle\otimes|0\rangle$. Quiroga *et al.* showed [10] that an optically controlled exciton transfer process can generate maximally entangled Bell states [11] and GHZ states. To demonstrate that excitons in coupled quantum dots are ideal candidates for reliable preparation of entangled states in solid-state systems, Chen *et al.* [12] experimentally investigated how to optically induce entanglement of excitons in a single-gallium arsenide QD; the entanglement is identified by the spectrum of the phase-sensitive homodyne-detected coherent nonlinear optical response.

Although no experimental observation of entanglement of many-exciton systems has yet occurred, the theoretical studies are necessary if we desire to have a large-scale quantum computing device. In a recent investigation by Liu *et al.* [13] for generating bipartite entangled coherent exciton states in a system of two coupled quantum dots and CQED with dilute excitons, they found that the bipartite maximally entangled coherent exciton states can be generated when the initial cavity field is in an odd coherent state. The relation between entanglement of exciton and photon numbers in the cavity was also discussed.

It is interesting to extend to multipartite entanglement in such systems. Multipartite entanglement is not only of intrinsic interest itself but also of practical importance in quantum-information proposals such as quantum teleportation and quantum cryptography. One motivation to consider multipartite entanglement in many QDs is that around ten QDs can be embedded in a microdisk structure and coupled to a single cavity mode in the quantum information process scheme proposed by Imamoğlu *et al.* We will first extend the model of Ref. [11] to the case of many QDs, and then study how to generate multipartite entangled states and examine multipartite entanglement in such systems.

The paper is organized as follows. In Sec. II, we present a model for coupled QDs in CQED and determine the exact state vector at any time t, which will be shown to be a multipartite entangled state in some time ranges. In Sec. III, we use three different methods to examine the multipartite entanglement in the state. Then in Sec. IV, we show that multipartite entangled coherent exciton states can be generated at certain times during the dynamical evolution. In Sec. V, we analyze the effects of environment on the generation of multipartite entanglement. The conclusions are given in Sec. VI.

II. MULTIPARTITE ENTANGLED STATES

We consider quantum dots embedded in a single-mode cavity. We assume that the QDs are sufficiently large and there are only a few electrons excited from valence band to conduction band [13]. In the assumption of low exciton excitation density, exciton operators can be approximated by

boson operators, and all nonlinear dynamics such as exciton-exciton interaction can be neglected. As the distance between any two excitons is also assumed to be large, the interaction between any two excitons can be safely neglected. There exists a resonant interaction between the excitons and the cavity field. The Hamiltonian under the rotating-wave approximation is given by ($\hbar=1$) [13,14]

$$H=\omega b_0^\dagger b_0+\omega\sum_{n=1}^{N} b_n^\dagger b_n+\sum_{n=1}^{N} g_n(b_n^\dagger b_0+b_0^\dagger b_n), \quad (1)$$

where $b_0^\dagger$ (b_0) is the creation (annihilation) operator of the cavity field with frequency ω, $b_n^\dagger(b_n)$ ($n=1,2,\ldots,N$) denote the creation (annihilation) operator of the nth exciton with the same frequency ω, g_n is the coupling constant between the cavity field and nth exciton, and N is the number of excitons.

It is convenient to write the coupling constants g_i in generalized spherical coordinates [15,16] as

$$\begin{aligned} g_1&=G\cos\phi_1,\\ g_2&=G\sin\phi_1\cos\phi_2,\\ g_3&=G\sin\phi_1\sin\phi_2\cos\phi_3,\\ &\vdots\\ g_{N-1}&=G\sin\phi_1\sin\phi_2\cdots\cos\phi_{N-1},\\ g_N&=G\sin\phi_1\sin\phi_2\cdots\sin\phi_{N-1}, \end{aligned}$$

where $G=\sqrt{\Sigma_{k=1}^{N} g_k{}^2}$. The evolution operator corresponding to the Hamiltonian is then obtained as

$$\begin{aligned} U(t)=&V_{N-1,N}^\dagger(\phi_{N-1})\cdots V_{2,3}^\dagger(\phi_2)V_{1,2}^\dagger(\phi_1)\\ &\times\tilde{V}_{0,1}(t)V_{1,2}(\phi_1)V_{2,3}(\phi_2)\cdots V_{N-1,N}(\phi_{N-1})U_0(t), \end{aligned} \quad (2)$$

where

$$\begin{aligned} V_{i,j}(\phi_i)&=\exp[\phi_i(b_i^\dagger b_j-b_j^\dagger b_i)] \quad (i\neq j),\\ \tilde{V}_{0,1}(t)&=\exp[-iGt(b_0^\dagger b_1+b_1^\dagger b_0)],\\ U_0(t)&=\exp\left[-i\omega t\left(b_0^\dagger b_0+\sum_{i=1}^{N} b_i^\dagger b_i\right)\right]. \end{aligned} \quad (3)$$

Both $V_{i,j}(\phi_i)$ and $\tilde{V}_{0,1}(t)$ are operators for the beam splitters. We see that the evolution operator can be written as a free evolution $U_0(t)$ followed by a series of beam splitter operators.

Let us assume that the initial state is $|\psi(0)\rangle=|\alpha\rangle_0\otimes|0\rangle_1\otimes\cdots\otimes|0\rangle_N$, i.e., the cavity field is in a coherent state $|\alpha\rangle_0$ with $\alpha\neq0$ and all the excitons are in vacuum states. After the action of the unitary operator $U(t)$, the state evolves into

$$|\psi(t)\rangle=|\alpha_0(t),\alpha_1(t),\ldots,\alpha_N(t)\rangle_{0,\ldots,N}, \quad (4)$$

where $|\alpha_0(t),\ldots,\alpha_N(t)\rangle_{0,\ldots,N}\equiv|\alpha_0(t)\rangle_0\otimes|\alpha_1(t)\rangle_1\otimes\cdots\otimes|\alpha_N(t)\rangle_N$ and

$$\begin{aligned} \alpha_0(t)&=\alpha\cos(Gt)e^{-i\omega t},\\ \alpha_n(t)&=-i\alpha\sin(Gt)g_n/Ge^{-i\omega t}, \quad n=1,2,\ldots,N. \end{aligned} \quad (5)$$

Thus, if the initial state of the cavity field is in a superposition of coherent states, the resulting state will be an entangled coherent state. Let us assume the initial state of the cavity field to be in a superposition of two coherent states $|\pm\alpha\rangle$ and the excitons to be in vacuum states, i.e.,

$$\begin{aligned} |\Psi(0)\rangle=&[2+2\cos\theta\exp(-2|\alpha|^2)]^{-1/2}(|\alpha\rangle_0+e^{i\theta}|-\alpha\rangle_0)\\ &\otimes|0\rangle_1\otimes\cdots\otimes|0\rangle_N. \end{aligned} \quad (6)$$

Specifically, for $\theta=0$, π, and $\pi/2$, the cavity superposition state reduces to even, odd [17], and Yurke-Stoler [18] coherent states, respectively. Then the state vector at time t is given by

$$\begin{aligned} |\Psi(t)\rangle=&\mathcal{N}(|\alpha_0(t),\alpha_1(t),\ldots,\alpha_N(t)\rangle_{0,\ldots,N}+e^{i\theta}|-\alpha_0(t),\\ &-\alpha_1(t),\ldots,-\alpha_N(t)\rangle_{0,\ldots,N}), \end{aligned} \quad (7)$$

where

$$\mathcal{N}=\left[2+2\cos\theta\prod_{k=0}^{N} p_k(t)\right]^{-1/2} \quad (8)$$

is the normalization constant, and $p_k(t)\equiv\exp(-2|\alpha_k(t)|^2)$ is the overlap of the two coherent states $|\pm\alpha_k(t)\rangle$. The resulting state $|\Psi(t)\rangle$ is a multipartite entangled coherent state [19,20].

We now choose the orthogonal basis [20]

$$|\mathbf{0}\rangle_k\equiv|\alpha_k(t)\rangle, \quad |\mathbf{1}\rangle_k\equiv(|-\alpha_k\rangle-p_k(t)|\mathbf{0}\rangle)/\mathcal{M}_k(t), \quad (9)$$

where $\mathcal{M}_k(t)=\sqrt{1-p_k(t)^2}$. It then follows that:

$$|-\alpha_k(t)\rangle=\mathcal{M}_k(t)|\mathbf{1}\rangle+p_k(t)|\mathbf{0}\rangle. \quad (10)$$

Using this basis the state vector at time t can be rewritten as

$$\begin{aligned} |\Psi(t)\rangle=&\mathcal{N}\{|\mathbf{0}\rangle_0\otimes\cdots\otimes|\mathbf{0}\rangle_N+e^{i\theta}[\mathcal{M}_0(t)|\mathbf{1}\rangle_0+p_0(t)|\mathbf{0}\rangle_0]\\ &\otimes\cdots\otimes[\mathcal{M}_N(t)|\mathbf{1}\rangle_N+p_N(t)|\mathbf{0}\rangle_N]\}. \end{aligned} \quad (11)$$

After the ''encoding'' this state is a multiqubit state. Then, we can fully exploit the sophisticated tools available for examining multipartite entanglement of qubits to study our state and determine if it is genuinely multipartite entangled.

III. EXAMINATION OF MULTIPARTITE ENTANGLEMENT

We study the multipartite entanglement by examining (i) Mermin-Klyshko inequality [21,22], (ii) state preparation fidelity [23,24], and (iii) the square of the multiqubit concurrence [25,26]. The first two, namely, the Mermin-Klyshko

inequality and the state preparation fidelity, are related to two sufficient conditions that distinguish between genuinely N-partite entangled states and those in which only M particle are entangled ($M<N$). The last one, the square of the multiqubit concurrence, is not by itself a measure of N-particle entanglement, but it appears to be related to some kind of multipartite entanglement. That is, we can gain some information about the degree of multipartite entanglement by calculating the square of the multiqubit concurrence.

A. Mermin-Klyshko inequality

Let us first use the Mermin-Klyshko inequality [21,22] to examine the N-partite entanglement. This inequality generalizes the Bell inequality [11] and Clauser-Horne-Shimony-Holt (CHSH) inequality [27], which not only tests the predictions of quantum mechanics against those of local hidden variable theory but also distinguishes entangled from nonentangled states [28,24]. The Mermin-Klyshko inequality is [21,22]

$$|\langle \mathcal{B}_N \rangle| \leqslant 2, \tag{12}$$

where $\mathcal{B}_N$ is the Bell operator defined recursively as

$$\mathcal{B}_N = \frac{1}{2}(A_N + A'_N) \otimes \mathcal{B}_{N-1} + \frac{1}{2}(A_N - A'_N) \otimes \mathcal{B}'_{N-1}, \tag{13}$$

$\mathcal{B}'_N$ is obtained from $\mathcal{B}_N$ by exchanging primed and unprimed terms,

$$\mathcal{B}'_N = \frac{1}{2}(A_N + A'_N) \otimes \mathcal{B}'_{N-1} - \frac{1}{2}(A_N - A'_N) \otimes \mathcal{B}_{N-1}, \tag{14}$$

$\mathcal{B}_1 = 2A_1$, and $\mathcal{B}'_1 = 2A'_1$. All A_i and A'_i are dichotomous observables. Let S_n denote the set of all N-particle states and S_N^{N-1} denote the subset of those states which are at most $(N-1)$-partite entangled. Then, from the results of Refs. [28,24], for a state ρ, we have

$$|\langle \mathcal{B}_N \rangle| \leqslant 2^{N/2}, \quad \forall \ \rho \in S_N^{N-1}, \tag{15}$$

$$|\langle \mathcal{B}_N \rangle| \leqslant 2^{(N+1)/2} \quad \forall \ \rho \in S_N, \tag{16}$$

which implies that a sufficient condition for N-partite entanglement is the violation of the inequality given by Eq. (15). Now we define a quantity B,

$$B(\rho) = \frac{|\langle \mathcal{B}_N \rangle| - 2^{N/2}}{2^{(N+1)/2} - 2^{N/2}}. \tag{17}$$

Then the state ρ is N-partite entangled when $B(\rho) > 0$ and maximally entangled when $B(\rho) = 1$.

Let $A_i = \sigma_x$ and $A'_i = \sigma_y$ for any i. Then the Bell operator $\mathcal{B}_N$ and $\mathcal{B}'_N$ become [29]

$$\begin{aligned} \mathcal{B}_N &= 2^{(N+1)/2}(e^{-i\beta_N}\sigma_+^{\otimes N} + e^{i\beta_N}\sigma_-^{\otimes N}), \\ \mathcal{B}'_N &= 2^{(N+1)/2}(-ie^{-i\beta_N}\sigma_+^{\otimes N} + ie^{i\beta_N}\sigma_-^{\otimes N}), \end{aligned} \tag{18}$$

where $\beta_N = \pi/4(N-1)$, $\sigma_+ = |\mathbf{0}\rangle\langle\mathbf{1}|$, and $\sigma_- = |\mathbf{1}\rangle\langle\mathbf{0}|$. For our state $|\Psi(t)\rangle$ (11), we have

$$\begin{aligned} \mathcal{B}(|\Psi(t)\rangle\langle\Psi(t)|) = \frac{2^{(N+1)/2}}{1 + \cos\theta e^{-2|\alpha|^2}} [\cos(\theta - \beta_N) \\ + \cos(\beta_N) e^{-2|\alpha|^2}] \prod_{k=0}^{N} \sqrt{1 - p_k^2}. \end{aligned} \tag{19}$$

We choose $\theta = \beta_N$. Then the above equation reduces to

$$\mathcal{B} = 2^{(N+1)/2} \prod_{k=0}^{N} \sqrt{1 - p_k^2}. \tag{20}$$

Therefore, this sufficient condition becomes

$$\prod_{k=0}^{N} \sqrt{1 - p_k^2} > 1/\sqrt{2}. \tag{21}$$

Numerical results are provided later. Next we discuss another method based on state preparation fidelity to examine multipartite entanglement.

B. State preparation fidelity

The so-called state preparation fidelity $\mathcal{F}$ of an N-qubit state ρ is defined as

$$\mathcal{F}(\rho) = \langle \psi_{\mathrm{GHZ}} | \rho | \psi_{\mathrm{GHZ}} \rangle, \tag{22}$$

where

$$|\psi_{\mathrm{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|\mathbf{0}\rangle \otimes \cdots \otimes |\mathbf{0}\rangle + e^{i\gamma}|\mathbf{1}\rangle \otimes \cdots \otimes |\mathbf{1}\rangle) \tag{23}$$

is the GHZ state. A sufficient condition for N-partite entanglement is given by [23,24]

$$\mathcal{F}(\rho) > 1/2. \tag{24}$$

From Eqs. (22) and (11) the state preparation fidelity for the multipartite state $|\Psi(t)\rangle$ is obtained as

$$\begin{aligned} \mathcal{F}(|\Psi(t)\rangle\langle\Psi(t)|) = \frac{1}{4 + 4\cos\theta e^{-2|\alpha|^2}} \Bigg[1 + \prod_{k=0}^{N} (1 - p_k^2) \\ + e^{-4|\alpha|^2} + 2\cos\theta e^{-2|\alpha|^2} \\ + 2\cos(\theta - \gamma) \prod_{k=0}^{N} \sqrt{1 - p_k^2} \\ + 2\cos\gamma e^{-2|\alpha|^2} \prod_{k=0}^{N} \sqrt{1 - p_k^2} \Bigg]. \end{aligned} \tag{25}$$

Let $\theta = \gamma = \pi/2$. The above equation then reduces to

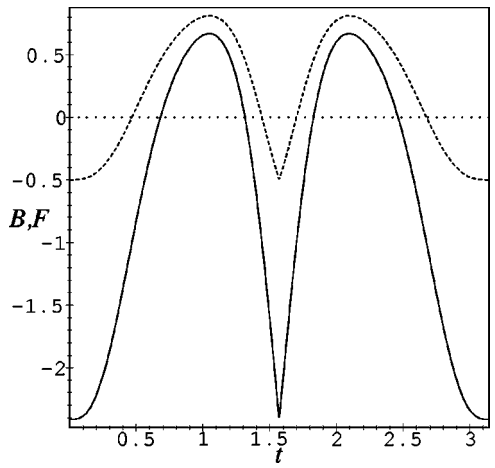

FIG. 1. The quantities B (solid line) and F (dashed line) against time. For B, we choose $\theta=\beta_N$. For F, we choose $\theta=\gamma=\pi/2$. The parameter $G=1$, $|\alpha|^2=3$, and $N=3$.

$$\mathcal{F}=\left(1+\prod_{k=0}^{N}\sqrt{1-p_k^2}\right)^2\Big/\,4+e^{-4|\alpha|^2}/4, \tag{26}$$

and the sufficient condition (24) becomes

$$\left(1+\prod_{k=0}^{N}\sqrt{1-p_k^2}\right)^2+e^{-4|\alpha|^2}>2. \tag{27}$$

For convenience, we define

$$F=2\mathcal{F}-1. \tag{28}$$

When $F>0$ the state is $(N+1)$-partite entangled.

In Fig. 1, we plot the quantity B and F against time t. Henceforth, we assume $g_1=g_2=\cdots=g_N$ and $G=1$. The period of B and F with respect to time is 2π. We can clearly see the two sufficient conditions can be satisfied in some time ranges which means that the state is a genuine multipartite entangled state. Moreover, we find the time range in which the sufficient condition is satisfied based on the state preparation fidelity, which is larger than that based on the Mermin-Klyshko inequality.

C. Square of the multiqubit concurrence

Recently Coffman *et al.* [25] used concurrence [30] to examine three-qubit systems, and quantified the amount of tripartite entanglement in three-qubit systems by the quantity $\tau_{0,1,2}$ [25],

$$\tau_{0,1,2}=C_{0(12)}^2-C_{01}^2-C_{02}^2, \tag{29}$$

where $C_{0(12)}$ denotes the concurrence between qubit 1 and qubits 2 and 3. Applying the general result for concurrence of bipartite nonorthogonal pure states [31] to the state (11) for $N=2$ yields

$$C_{0(12)}=\frac{\sqrt{(1-p_0^2)(1-p_1^2p_2^2)}}{1+p_0p_1p_2\cos\theta}. \tag{30}$$

By the standard method for calculating the concurrence [30,31], we find

$$C_{01}=\frac{p_2\sqrt{(1-p_0^2)(1-p_1^2)}}{1+\cos\theta p_0p_1p_2},$$

$$C_{02}=\frac{p_1\sqrt{(1-p_0^2)(1-p_2^2)}}{1+\cos\theta p_0p_1p_2}. \tag{31}$$

The concurrence C_{02} is obtained by a transformation $1\leftrightarrow 2$ in the expression of C_{01}. From Eqs. (29), (30), and (31), we obtain

$$\tau_{0,1,2}=\frac{\prod_{n=0}^{2}(1-p_n^2)}{(1+\cos\theta e^{-2|\alpha|^2})^2}. \tag{32}$$

Now consider the case of odd N excitons ($N>2$). Wong and Christensen [26] proposed the square of the multiqubit concurrence as a *potential measure* of the multipartite entanglement for an even-number pure qubit state. The $(N+1)$-qubit concurrence for a pure state $|\psi\rangle$ is defined as [26]

$$C_{0,1,\ldots,N}\equiv|\langle\psi|\sigma_y^{\otimes N+1}|\psi^*\rangle|, \tag{33}$$

where $\sigma_y=-i(|\mathbf{0}\rangle\langle\mathbf{1}|-|\mathbf{1}\rangle\langle\mathbf{0}|)$ is a Pauli matrix.

Then applying Eq. (33) to the state (11) leads to the square of the multiqubit concurrence

$$\tau_{0,1,\ldots,N}=C_{0,1,\ldots,N}^2=\frac{\prod_{n=0}^{N}(1-p_n^2)}{(1+\cos\theta e^{-2|\alpha|^2})^2}. \tag{34}$$

From the expression of the square of multiqubit concurrence we can see that the multiqubit concurrence is unchanged by permutation of qubits, which implies that it really represents $(N+1)$-partite entanglement. We note that although the above formula is obtained for odd N, by comparing Eqs. (34) and (32), it is also applicable to $N=2$. We also note that Eqs. (34) and (32) are applicable to the more general state $(|\psi_0\rangle\otimes|\psi_1\rangle\otimes\cdots\otimes|\psi_N\rangle+e^{i\theta}|\phi_0\rangle\otimes|\phi_1\rangle\otimes\cdots\otimes|\phi_N\rangle)$ with real overlap $\langle\psi_i|\phi_i\rangle(i=0,1,\ldots,N)$ up to a normalization constant.

In Fig. 2, we plot the square of the multiqubit concurrence against time for $N=2$, $N=3$, and $N=5$. We observe that the entanglement periodically reaches its maximum twice per period, and the multipartite entanglement is suppressed with the increase of the number of excitons. One way to overcome this suppression is to increase the parameter $|\alpha|^2$. From Eq. (34), we know that the larger the parameter $|\alpha|^2$, the larger the multipartite entanglement.

From Eq. (34), we know that the square of the $(N+1)$-qubit concurrence reaches a maximum value at $\theta=\pi$ when other parameters are fixed. This implies that the best input state is the odd coherent state in order to generate multipartite entanglement. On the other hand, we know that the average photon number of the cavity field in the initial state $|\Psi(0)\rangle$ is

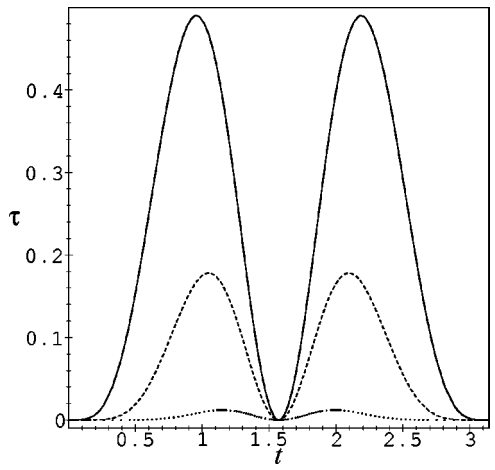

FIG. 2. The square of the multiqubit concurrence against time. The parameter $g=1$, $|\alpha|^2=0.9$, $\theta=\pi$, and all g_i are equal. The solid, dashed, and dotted lines correspond to $N=2$, 3, and 5, respectively.

$$\langle b_0^\dagger b_0\rangle=\frac{1-\cos\theta e^{-2|\alpha|^2}}{1+\cos\theta e^{-2|\alpha|^2}}|\alpha|^2. \tag{35}$$

We see that the mean photon number of the cavity field reaches its maximum when the field is in the odd coherent state ($\theta=\pi$) when other parameters are fixed. The mean photon number represents the energy of the system. Therefore, it turns out that the more energy contained in the initial cavity field, the larger the multipartite entanglement.

By three different methods, we have examined multipartite entanglement and found that the state (11) is a genuine multipartite entangled state over a large range of parameters by each of the indicators.

IV. MULTIPARTITE ENTANGLED COHERENT EXCITON STATES

It is clear that the cavity field and excitons are decoupled when $t=n\pi/(2g)$ for integer n. Without loss of generality, we examine the state vector $|\Psi(t)\rangle$ at times π and $\pi/2$ in one period. From the discussion of the last section, at these times there is no global multipartite entanglement. At time $t=\pi$, the state vector returns to the initial state (6) with $\alpha\rightarrow-\alpha e^{-i\omega\pi}$, and the excitons are in the vacuum state. It is interesting to see that at time $t=\pi/2$ the cavity field is decoupled from the excitons; however, the excitons are left in a multipartite entangled coherent exciton states given by

$$|\Psi\rangle_{\text{excitons}}=\mathcal{N}'(|\beta,\beta,\ldots,\beta\rangle_{1,\ldots,N}+e^{i\theta}|-\beta,-\beta,\ldots,-\beta\rangle_{1,\ldots,N}), \tag{36}$$

where $\beta=-i(\alpha/\sqrt{N})e^{-i\omega\pi/2}$, and the normalization constant $\mathcal{N}'=[2+2\cos\theta\exp(-2|\beta|^2)]^{-1/2}$. Now the cavity field is in a vacuum state, and all its energy has been transferred to the excitons.

Multipartite entanglement for the entangled coherent exciton state $|\Psi\rangle_{\text{excitons}}$ can be studied using the methods of the last section. From Eqs. (20), (26), and (34), we obtain the quantities $\mathcal{B}$, $\mathcal{F}$, and the square of the multiqubit concurrence for our state $|\Psi\rangle_{\text{excitons}}$ as

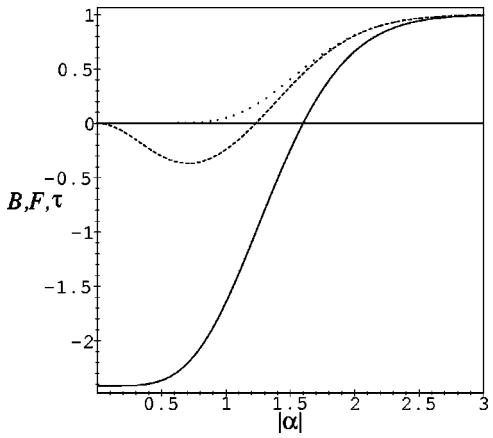

FIG. 3. The quantity B (solid line), F (dashed line), and square of the multiqubit concurrence (dotted line) against $|\alpha|$. The parameter $N=5$.

$$\mathcal{B}=2^{N/2}(1-e^{-4|\alpha|^2/N})^{N/2}, \tag{37}$$

$$\mathcal{F}=[1+(1-e^{-4|\alpha|^2/N})^{N/2}]^2/4+e^{-4|\alpha|^2}/4, \tag{38}$$

$$\tau_{1,\ldots,N}=\frac{(1-e^{-4|\alpha|^2/N})^N}{(1+\cos\theta e^{-2|\alpha|^2})^2}. \tag{39}$$

In Fig. 3, we plot the quantity B, F, and τ against $|\alpha|$. We observe that the state is multipartite entangled when $|\alpha|$ is large enough. Quantitively, the corresponding Bell inequality is violated when $|\alpha|>1.601$, and the state preparation fidelity $\mathcal{F}$ is larger than 1/2 when $|\alpha|>1.228$. We also observe that the square of the $(N+1)$-qubit concurrence is significantly larger than zero only if $|\alpha|$ is large enough. For fixed N and very large $|\alpha|$ the square of the $(N+1)$-qubit concurrence $\tau_{1,\ldots,N}\approx 1$, which implies that the entangled coherent exciton states becomes a GHZ-like state. On the other hand $\tau_{1,\ldots,N}\approx 0$ for fixed N and small enough $|\alpha|^2$. As discussed in Ref. [20] the state $|\Psi\rangle_{\text{excitons}}$ with $\theta=\pi$ reduces to the multiqubit W state [32] in the limit of $|\alpha|^2\rightarrow 0$. It means that we can also prepare the W state in our system at time $t=\pi/2$ with the initial cavity field in a Fock state with one photon and all excitons initially in the vacuum states. For the case of only two excitons the W state is just the maximally entangled state (one Bell state) as discussed by Liu *et al.* [13].

V. EFFECTS OF ENVIRONMENT ON MULTIPARTITE ENTANGLEMENT

Environmental losses and decoherence are important effects in quantum-information processing [33]. The lifetime of both cavity photon and exciton is generally considered to be of the order of picoseconds. Whereas, if we assume $\hbar g=0.5$ meV [13], the time we need to get maximal entanglement in our model is also of the order of picoseconds. How-

ever, dynamical evolution suffers from decay of photons and excitons. Recent experiments show that the lifetime of photons and excitons can be prolonged [34–36]. In particular, for cavity decay, recent experiments display an elongated decay time of photons in the microwave domain [36]. Thus, we only consider the decay of excitons in the following discussions.

We follow the method of Ref. [13], and assume that the environment is at zero temperature and the system dissipates by interaction between excitons and a multimode electromagnetic field. Under the rotating-wave approximation, we write the Hamitonian as

$$H=\omega b_0^\dagger b_0+\omega\sum_{n=1}^{N} b_n^\dagger b_n+\sum_k \omega_k a_k^\dagger a_k + g\sum_{n=1}^{N}(b_n^\dagger b_0+b_0^\dagger b_n)+\sum_{n=1}^{N}\sum_k \lambda_k(b_n^\dagger a_k+b_n a_k^\dagger), \tag{40}$$

where $a_k^\dagger$ (a_k) denotes the creation (annihilation) operator of the multimode magnetic field with frequency ω_k. We assume $g_1=g_2=\cdots=g$ implying that the excitons are equally coupled to the cavity mode, and n independence of the λ_k implying that the excitons are also equally coupled to the environment.

Now we use the Heisenberg picture to study the problem. The Heisenberg equations for related operators are obtained as follows:

$$\partial b_0/\partial t=i\omega b_0+ig\sum_{n=1}^{N} b_n,$$

$$\partial b_n/\partial t=i\omega b_n+igb_0+i\sum_k \lambda_k a_k,$$

$$\partial a_k/\partial t=i\omega_k a_k+i\lambda_k\sum_{n=1}^{N} b_n. \tag{41}$$

Note that we use a slightly different Heisenberg picture for our purpose to obtain the final-state vector at time t. We let an operator evolve as $A(t)=\exp(-iHt)A(0)\exp(iHt)$ and it satisfies $i\partial A(t)/\partial t=[H,A(t)]$.

Now we introduce the collective exciton operator $b_c=(1/\sqrt{N})\Sigma_{n=1}^{N} b_n$ which can be considered as a boson since $[b_c,b_c^\dagger]=1$. In terms of the collective boson operator, we rewrite the above equation as

$$\partial b_0/\partial t=i\omega b_0+ig_N b_c,$$

$$\partial b_c/\partial t=i\omega b_c+ig_N b_0+i\sum_k \lambda_{N,k}a_k,$$

$$\partial a_k/\partial t=i\omega_k a_k+i\lambda_{N,k}b_c, \tag{42}$$

where $g_N=g\sqrt{N}$ and $\lambda_{N,k}=\lambda_k\sqrt{N}$. If we let

$$b_0=B_0e^{i\omega t},\quad b_c=B_ce^{i\omega t},\quad a_k=A_ke^{i\omega t}, \tag{43}$$

Eq. (42) reduces to

$$\partial B_0/\partial t=ig_N B_c,$$

$$\partial B_c/\partial t=ig_N B_0+i\sum_k \lambda_{N,k}A_k,$$

$$\partial A_k/\partial t=i(\omega_k-\omega)A_k+i\lambda_{N,k}B_c. \tag{44}$$

To solve the above equation, we make the Laplace transform

$$\overline{f}(s)=\int_0^\infty e^{-st}f(t)dt. \tag{45}$$

We obtain after minor algebra

$$s\overline{B_0}=B_0(0)+ig_N\overline{B_c}, \tag{46}$$

$$\overline{B_c}=\frac{B_c(0)+ig_N\overline{B_0}+i\sum_k \dfrac{\lambda_{N,k}A_k(0)}{s+i(\omega_k-\omega)}}{s+N\sum_k \dfrac{\lambda_k^2}{s+i(\omega_k-\omega)}}. \tag{47}$$

The above equation cannot be solved exactly. So we resort to the Wigner-Weisskopff approximation [37]. After the approximation, from the above two equations, we obtain

$$\overline{B_0}=\frac{\left(s+\dfrac{N\Gamma}{2}\right)B_0(0)+ig\sqrt{N}B_c(0)-\sum_k \dfrac{g_N\lambda_{N,k}A_k(0)}{s+i(\omega_k-\omega)}}{s^2+\dfrac{N\Gamma}{2}s+Ng^2}, \tag{48}$$

where $\Gamma=2\pi\epsilon(\omega)\lambda^2(\omega)$ and $\epsilon(\omega)$ is a distribution function of the multimode electromagnetic field. We assumed that $\Delta\omega=-\int d\omega_k\epsilon(\omega_k)\lambda^2(\omega_k)/(\omega_k-\omega)=0$ in the derivation of the above equation.

From Eq. (48), we obtain the operator $b_0(t)$ in the Heisenberg representation

$$b_0(t)=u(t)b_0(0)+v(t)\sum_{n=1}^{N} b_n(0)+\sum_k w_k(t)a_k(0),$$

$$u(t)=e^{-N\Gamma t/4}\left[\cos(\Delta_N t)+\frac{N\Gamma}{4\Delta_N}\sin(\Delta_N t)\right]e^{i\omega t},$$

$$v(t)=\frac{ig}{\Delta_N}e^{-N\Gamma t/4}\sin(\Delta_N t)e^{i\omega t}, \tag{49}$$

where $\Delta_N=\sqrt{Ng^2-N^2\Gamma^2/16}$. Let the cavity field be an odd coherent state and other systems in the vacuum states. Then, from Eq. (49) we obtain the state vector at time t as

$$|\Psi(t)\rangle=[2-2\exp(-2|\alpha|^2)]\Big(|\alpha_u\rangle\otimes|\alpha_v\rangle^{\otimes N}\otimes\prod_k|\alpha_{w,k}\rangle - |-\alpha_u\rangle\otimes|-\alpha_v\rangle^{\otimes N}\otimes\prod_k|-\alpha_{w,k}\rangle\Big), \tag{50}$$

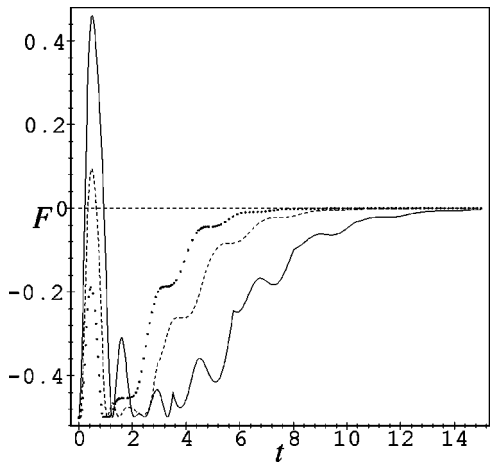

FIG. 4. The quantity $F=2\mathcal{F}-1$ for different numbers of excitons; $N=2$ (solid line), $N=3$ (dashed line), and $N=4$ (dotted line). The parameters $|\alpha|^2=3$, $g=1$, and $\Gamma=0.5$.

where $\alpha_u=\alpha u^*(t)$, $\alpha_v=\alpha v^*(t)$, and $\alpha_{w,k}=\alpha w_k^*(t)$.

We use the state preparation fidelity $\mathcal{F}$ to examine multipartite entanglement in the above state and choose the following GHZ state for consideration:

$$|\psi_{\mathrm{GHZ}}\rangle=\frac{1}{\sqrt{2}}(|\alpha_u\rangle\otimes|\alpha_v\rangle^{\otimes N}-|\alpha_u^{\perp}\rangle\otimes|\alpha_v^{\perp}\rangle^{\otimes N}), \quad (51)$$

where

$$|\alpha_x^{\perp}\rangle=(|-\alpha_x\rangle-p_x|\alpha_x\rangle)/\mathcal{M}_x,$$

$$p_x=\exp(-2|\alpha_x|^2),$$

$$\mathcal{M}_x=\sqrt{1-p_x^2}, \quad x=u,v. \quad (52)$$

From Eqs. (50) and (51), we obtain the state preparation fidelity as

$$\mathcal{F}=\langle\Psi(t)|\psi_{\mathrm{GHZ}}\rangle\langle\psi_{\mathrm{GHZ}}|\Psi(t)\rangle=\frac{1}{4-4e^{-2|\alpha|^2}}[1+2e^{-2|\alpha|^2}$$

$$\times(\mathcal{M}_u\mathcal{M}_v^N p_u^{-1}p_v^{-N}-1)+(\mathcal{M}_u\mathcal{M}_v^N-p_u p_v^N)^2]. \quad (53)$$

In the derivation of the above equation, we have used the relation

$$|\alpha|^2=|\alpha_u|^2+N|\alpha_v|^2+\sum_k |\alpha_{w,k}|^2, \quad (54)$$

which results from the normalization of the state $|\Psi(t)\rangle$ and implies the energy conservation.

Figure 4 shows the fidelity against time t for different numbers of excitons. From the figure, we see that the fidelity is not a periodic function of t due to the dissipation of energy to the environment. When there is no dissipation ($\Gamma=0$), the fidelity attains the ideal case of being a periodic function of t. We observe that multipartite entanglement occurs only in the beginning of the evolution. When the number of excitons becomes larger, the generation of the multipartite entanglement becomes more difficult. In the limit of $t\to\infty$ the fidelity becomes 0.5 as we expected. In this case, there is no multipartite entanglement and all the energy of the cavity-excitons system dissipate to the environment. The environment diminishes the generation of multipartite entanglement when the number of excitons increases. From Eq. (49), we can see that the decay rate is proportional to the number of excitons when the intensity of the cavity field is fixed. In addition, we find a similar result as that of Ref. [13], if we fix Γ and vary $|\alpha|$, i.e., the multipartite entanglement decays rapidly with increasing the cavity field density.

VI. CONCLUSIONS

In conclusion, we have investigated the dynamical evolution of multipartite entanglement in a system of quantum dots embedded in a microcavity. The entanglement is studied via two sufficient conditions for multipartite entanglement and the square of the multiqubit concurrence. We observed the global multipartite entanglement and at certain times the entanglement becomes maximal. We can also produce the multipartite entangled coherent exciton states and multiqubit W state by preparing different initial states. Finally, we study the effects of environment on the generation of multipartite entangled states, and find that the decay rate is proportional to the number of excitons. We also find that the entanglement decays rapidly with increasing the cavity field density.

Although multipartite entanglement studied here has not yet been observed experimentally, the potential application of excitons in quantum computing as well as rapid development of CQED technique suggest that our analysis will find applications in this field.

ACKNOWLEDGMENTS

We appreciate the helpful discussions with Paolo Zanardi, Irene D'Amico, and Ehoud Pazy. This work has been supported by the European Commission through the Research Project SQUID within the FET Program and IST, Q-ACTA, and by an Australian Research Council Large Grant.

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