

Uncertainty characteristics of generalized quantum measurements

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The effects of any quantum measurement can be described by a collection of measurement operators $\{\hat{M}_m\}$ acting on the quantum state of the measured system. However, the Hilbert space formalism tends to obscure the relationship between the measurement results and the physical properties of the measured system. In this paper, a characterization of measurement operators in terms of measurement resolution and disturbance is developed. It is then possible to formulate uncertainty relations for the measurement process that are valid for arbitrary input states. The motivation of these concepts is explained from a quantum communication viewpoint. It is shown that the intuitive interpretation of uncertainty as a relation between measurement resolution and disturbance provides a valid description of measurement back action. Possible applications to quantum cryptography, quantum cloning, and teleportation are discussed.

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I. INTRODUCTION

One of the most intriguing problems of quantum mechanics is the interpretation of the measurement process [1]. The reason for this central role of the measurement process is the absence of fundamental “elements of reality” that would simultaneously characterize both the dynamics and the measurement results [2–4]. It is therefore not possible to trace the measurement interaction back to microscopic trajectories. Instead, only a summary of the total statistical effects of a measurement is available. Originally, this property of quantum mechanics was explained by Heisenberg in terms of an uncontrollable disturbance in one variable caused by the measurement of another variable [5]. However, this explanation was still based on a classical model of the measurement interaction. Consequently, the general validity of Heisenberg’s original argument has been questioned by a number of researchers [6,7]. In particular, there appear to be some unresolved issues concerning the derivation of uncertainties using correlations between the system and the measurement device [8–10].

On the other hand, the investigation of various methods to prepare and control quantum states, especially in the field of quantum optics, has motivated the development of a generalized measurement theory based on the Hilbert space representation of quantum states. This formalism allows an expression of all relevant statistical properties of a quantum measurement in an extremely compact form [11]. Unfortunately, this compact form tends to obscure the relationship between physical properties of the system and the measurement process. In particular, the relationship of this generalized formulation of measurement with Heisenberg’s original discussion of the uncertainty principle as a relation between measurement resolution and the disturbance of a conjugate variable may not be entirely clear [6].

In this paper, the measurement effects of a generalized

measurement described by a set of operators $\{\hat{M}_m\}$ are characterized in terms of the physical properties of the measured system. This characterization is based on the reliability of quantitative estimates for various physical properties of the system before the measurement. Using these definitions, the uncertainty relations for measurement resolution and disturbance can be derived, thereby establishing the validity of the uncertainty principle for generalized quantum measurements. It is then possible to translate Heisenberg’s original argument into a form closer to present problems in quantum information theory. In particular, it is shown that the concept of disturbance can be understood in terms of a loss of information about the input state caused by the measurement back action. This interpretation can then be applied to problems such as quantum cryptography, quantum cloning, and quantum teleportation.

II. QUANTITATIVE ESTIMATES AND MEASUREMENT RESOLUTION

While classical physics allows a direct identification of measurement results with objective properties of the system, the existence of which is thought to be independent of the measurement process, quantum mechanics is formulated in an abstract probabilistic space from which the measurement statistics must be derived indirectly. Therefore, a special theory is necessary to identify and define the connection between a measurement outcome m and the quantum state of the system. In general, this can be achieved by using a set of measurement operators $\{\hat{M}_m\}$. These operators describe both the measurement probabilities $p(m)$ for each outcome m and the change of the quantum state caused by the measurement back action. For an input density operator $\hat{\rho}_{\text{in}}$,

$$p(m) = \text{tr}\{\hat{\rho}_{\text{in}}\hat{M}_m^\dagger\hat{M}_m\},$$

$$\hat{\rho}_{\text{out}} = \frac{1}{p(m)}\hat{M}_m\hat{\rho}_{\text{in}}\hat{M}_m^\dagger. \quad (1)$$

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Note that the properties of \hat{M}_m are only restricted by the fact that the sum of all probabilities must be one for any input state, that is,

$$\sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{1}. \quad (2)$$

In this formulation, the information obtained about the measured system is represented by the dependence of the measurement probability $p(m)$ on the input state $\hat{\rho}_{\text{in}}$. In order to characterize the measurement information obtained about an observable \hat{A} , it is necessary to examine how the probability $p(m)$ varies for different eigenstates of \hat{A} . Suppose that the input state is an unknown eigenstate of the observable \hat{A} . It is then possible to estimate the eigenvalue of \hat{A} based on the measurement result m . Assuming that each eigenstate of \hat{A} is equally likely to be the input, the probability $p(A|m)$ that m was obtained as a result of A is given by

$$p(A|m) = \frac{\langle A | \hat{M}_m^\dagger \hat{M}_m | A \rangle}{\sum_{A'} \langle A' | \hat{M}_m^\dagger \hat{M}_m | A' \rangle} = \frac{\langle A | \hat{M}_m^\dagger \hat{M}_m | A \rangle}{\text{tr}\{\hat{M}_m^\dagger \hat{M}_m\}}. \quad (3)$$

In order to provide a single quantitative estimate of the input eigenvalue A , it is necessary to assign a measurement value A_m to each possible outcome m . The reliability of this estimate can be characterized by the average quadratic error obtained from the probabilities $p(A|m)$,

$$\delta A_m^2 = \sum_A (A_m - A)^2 p(A|m) = \frac{\text{tr}\{(A_m - \hat{A})^2 \hat{M}_m^\dagger \hat{M}_m\}}{\text{tr}\{\hat{M}_m^\dagger \hat{M}_m\}}. \quad (4)$$

The best possible estimate is then obtained by minimizing this quadratic error. The result of this optimization is the average value of A in the input state distribution $p(A|m)$,

$$A_m = \sum_A A p(A|m) = \frac{\text{tr}\{\hat{A} \hat{M}_m^\dagger \hat{M}_m\}}{\text{tr}\{\hat{M}_m^\dagger \hat{M}_m\}}. \quad (5)$$

The measurement outcome m can then be identified with a quantitative measurement of the observable \hat{A} , where the measurement result is given by A_m and the measurement resolution is given by δA_m^2 .

This procedure can be applied to any observable. The measurement result m , therefore provides information about all physical properties of the measured system. The information obtained from the measurement result m can be summarized by the normalized statistical measurement operator \hat{R}_m given by

$$\hat{R}_m = \frac{\hat{M}_m^\dagger \hat{M}_m}{\text{tr}\{\hat{M}_m^\dagger \hat{M}_m\}}. \quad (6)$$

This operator is essentially a time-reversed version of the density matrix. Instead of predicting future measurement re-

sults, it is used to “retrodict” properties of the input [12,13]. In particular, the quantitative estimates A_m and the measurement errors δA_m^2 are now defined as expectation values of the operator and fluctuations of the statistical operator \hat{R}_m ,

$$A_m = \text{tr}\{\hat{A} \hat{R}_m\},$$

$$\delta A_m^2 = \text{tr}\{\hat{A}^2 \hat{R}_m\} - A_m^2. \quad (7)$$

The analogy between the statistical operator \hat{R}_m and the density operator indicates that the same uncertainty relations that apply to quantum state preparation also apply to the simultaneous measurement of noncommuting properties (see the Appendix for a general derivation of uncertainty relations). Specifically, the uncertainty limit of the measurement errors δA_m^2 and δB_m^2 of two noncommuting observables \hat{A} and \hat{B} is given by

$$\delta A_m^2 \delta B_m^2 \geq \frac{1}{4} |\text{tr}\{\hat{R}_m [\hat{A}, \hat{B}]\}|^2. \quad (8)$$

This uncertainty relation applies to cases where the same measurement procedure is used to estimate both input eigenvalues of \hat{A} and input eigenvalues of \hat{B} when no additional information on the input state is available. An example of such a situation can be given in terms of a quantum cryptography protocol, where a message can be either encoded in the eigenvalues of \hat{A} or in the eigenvalues of \hat{B} . Uncertainty relation (8) then defines a quantitative limitation on eavesdropping attempts. An important feature of this measurement uncertainty is that it does not depend on the input state at all. Instead, the uncertainty limit is defined by an expectation value of the statistical matrix \hat{R}_m that characterizes the information obtained in the measurement. In general, this expectation value can itself be interpreted as an estimate of a physical property of the measured system. The characterization of measurement uncertainty is thus achieved entirely in terms of information obtained in the measurement, avoiding any ambiguities of assumptions about the physical reality represented by the input state. Nevertheless, uncertainty relation (8) also has implications for the interpretation of the measurement back action, as will be explained in the following section.

III. BACK ACTION AND DISTURBANCE

The measurement operator \hat{M}_m not only describes the measurement information obtained from the measurement result m , but also the measurement back action effects that change the input state into the output state. In order to characterize this change in terms of physical properties, it is necessary to find a useful definition of the disturbance of an observable \hat{B} caused by the measurement \hat{M}_m . In the following, the definition of disturbance will be based on the measurement error obtained when the input eigenvalue of \hat{B} is estimated by the outcome of a precise projective measurement of \hat{B} performed on the output of the measurement \hat{M}_m .

The disturbance then corresponds to a loss of information about the property \hat{B} suffered as a consequence of the measurement back action.

The measurement sequence defined by an initial measurement result m followed by a final measurement result of B_f is characterized by a statistical matrix \hat{R}_{mf} given by

$$\hat{R}_{mf} = |r_{mf}\rangle\langle r_{mf}| \quad \text{with} \quad |r_{mf}\rangle = \frac{1}{\sqrt{\langle B_f | \hat{M}_m \hat{M}_m^\dagger | B_f \rangle}} \hat{M}_m^\dagger |B_f\rangle. \quad (9)$$

This statistical matrix determines the optimal estimates for the input eigenvalues of both \hat{A} and \hat{B} obtained from the measurement results m and B_f ,

$$A_{mf} = \langle r_{mf} | \hat{A} | r_{mf} \rangle, \quad B_{mf} = \langle r_{mf} | \hat{B} | r_{mf} \rangle. \quad (10)$$

The measurement errors are given by

$$\delta A_{mf}^2 = \langle r_{mf} | \hat{A}^2 | r_{mf} \rangle - A_{mf}^2, \quad \delta B_{mf}^2 = \langle r_{mf} | \hat{B}^2 | r_{mf} \rangle - B_{mf}^2. \quad (11)$$

The proper uncertainty relation for the estimates of \hat{A} and \hat{B} obtained from an initial measurement \hat{M}_m followed by a precise measurement of \hat{B} therefore reads

$$\delta A_{mf}^2 \delta B_{mf}^2 \geq \frac{1}{4} |\langle r_{mf} | [\hat{A}, \hat{B}] | r_{mf} \rangle|^2. \quad (12)$$

Since this uncertainty relation applies to the best estimates of \hat{A} and \hat{B} , it is obvious that it is also fulfilled for estimates of \hat{A} based only on m and estimates of \hat{B} given directly by the final measurement value B_f . In particular, the actual disturbance ΔB_{mf}^2 can be written as

$$\Delta B_{mf}^2 = \langle r_{mf} | (B_f - \hat{B})^2 | r_{mf} \rangle = \delta B_{mf}^2 + (B_f - B_{mf})^2, \quad (13)$$

where the difference between the measurement result B_f and the optimal estimate $B_{mf} = \langle r_{mf} | \hat{B} | r_{mf} \rangle$ corresponds to a systematic error caused by the measurement back action. The disturbance can thus be separated into a random part δB_{mf}^2 limiting the possibility to estimate the initial value of \hat{B} , and a well-defined shift of \hat{B} given by the difference between the output value B_f and the best possible estimate B_{mf} of the unknown input value based on all the available measurement informations.

With the definitions given above, it is possible to formulate Heisenberg's uncertainty principle with respect to the measurement resolution δA_{mf} and the disturbance ΔB_{mf}^2 caused by the measurement back action. In this form, it states that the error δA_{mf}^2 of an estimate of \hat{A} obtained from

the measurement result m is related to the disturbance ΔB_{mf}^2 of a noncommuting property \hat{B} by

$$\delta A_{mf}^2 \Delta B_{mf}^2 \geq \frac{1}{4} |\langle r_{mf} | [\hat{A}, \hat{B}] | r_{mf} \rangle|^2. \quad (14)$$

This relation takes into account the final measurement result B_f and may include correlations between the error of the estimate A_m and the final result B_f . Therefore, a complete characterization of \hat{M}_m requires the determination of measurement resolutions δA_{mf} and disturbances \hat{B}_{mf} for each final result B_f .

In order to obtain a single expression for the disturbance of \hat{B} caused by the measurement \hat{M}_m , it is necessary to average over all final measurement results B_f . For this purpose, the statistical matrix \hat{R}_m for the measurement of m given by Eq. (6) can be expressed in terms of the eigenstate $|r_{mf}\rangle$ of the statistical matrix \hat{R}_{mf} for the joint measurement given by Eq. (9),

$$\hat{R}_m = \sum_{B_f} w_m(B_f) |r_{mf}\rangle\langle r_{mf}| \quad \text{with} \quad w_m(B_f) = \frac{\langle B_f | \hat{M}_m \hat{M}_m^\dagger | B_f \rangle}{\text{tr}\{\hat{M}_m^\dagger \hat{M}_m\}}. \quad (15)$$

This decomposition shows that the statistical matrix \hat{R}_m can be interpreted as an average over the statistical matrices $|r_{mf}\rangle\langle r_{mf}|$ of each possible final outcome B_f with the appropriate statistical weights $w_m(B_f)$. If no other information on the input state is available, $w_m(B_f)$ is the conditional probability of obtaining the final result B_f following an initial measurement result of m . Since the errors of the estimate A_m obtained from the measurement result m are given by an expectation value of the statistical matrix, it follows from Eqs. (15) and (4) that

$$\begin{aligned} \delta A_m^2 &= \text{tr}\{(A_m - \hat{A})^2 \hat{R}_m\} \\ &= \sum_{B_f} w_m(B_f) \langle r_{mf} | (A_m - \hat{A})^2 | r_{mf} \rangle \\ &= \sum_{B_f} w_m(B_f) \delta A_{mf}^2. \end{aligned} \quad (16)$$

The measurement error δA_m^2 obtained for \hat{M}_m is therefore equal to the statistical average over the measurement errors δA_{mf}^2 obtained for the measurement sequences $\langle B_f | \hat{M}_m$. Likewise, the averaged disturbance of \hat{B} associated with the measurement result m can be obtained by

$$\begin{aligned} \Delta B_m^2 &= \sum_{B_f} w_m(B_f) \Delta B_{mf}^2 \\ &= \sum_{B_i, B_f} w_m(B_f) |\langle r_{mf} | B_i \rangle|^2 (B_f - B_i)^2 \\ &= \sum_{B_i, B_f} \frac{|\langle B_f | \hat{M}_m | B_i \rangle|^2}{\text{tr}\{\hat{M}_m^\dagger \hat{M}_m\}} (B_f - B_i)^2. \end{aligned} \quad (17)$$

This definition of measurement disturbance corresponds to an average of the squared difference between the final value B_f and the initial value B_i over all possible input and output values of \hat{B} . This average can be obtained experimentally and corresponds well with the intuitive idea of disturbance as a random change of \hat{B} .

Since δA_m^2 and ΔB_m^2 can both be expressed as averages over δA_{mf}^2 and ΔB_{mf}^2 , it is now possible to derive an uncertainty relation for δA_m^2 and ΔB_m^2 from relations (14) for each δA_{mf}^2 and ΔB_{mf}^2 . As shown in the Appendix, the uncertainty of a statistical mixture can be derived directly from the individual uncertainties by averaging the corresponding uncertainties as well,

$$\left(\sum_{B_f} w_m(B_f) \delta A_{mf}^2 \right) \left(\sum_{B_f} w_m(B_f) \Delta B_{mf}^2 \right) \geq \frac{1}{4} \left(\sum_{B_f} w_m(B_f) |\langle r_{mf} | [\hat{A}, \hat{B}] | r_{mf} \rangle|^2 \right). \quad (18)$$

The uncertainty limit on the right side of the equation can be simplified by noting that

$$\left(\sum_{B_f} w_m(B_f) |\langle r_{mf} | [\hat{A}, \hat{B}] | r_{mf} \rangle|^2 \right) \geq |\text{tr}\{\hat{R}_m[\hat{A}, \hat{B}]\}|^2. \quad (19)$$

With this simplification, it is now possible to formulate the uncertainty given by Eq. (18) without any explicit sums over the final results B_f . For any measurement described by a measurement operator \hat{M}_m , the measurement error δA_m^2 of the best estimate of \hat{A} obtained from m and the disturbance ΔB_m^2 in the property \hat{B} caused by the measurement back action of \hat{M}_m obey the uncertainty relation

$$\delta A_m^2 \Delta B_m^2 \geq \frac{1}{4} |\text{tr}\{\hat{R}_m[\hat{A}, \hat{B}]\}|^2. \quad (20)$$

This limit shows that the uncertainty principle does indeed apply to the relation between measurement resolution and disturbance, contrary to the statement found in the otherwise excellent book by Nielsen and Chuang [6]. Moreover, it suggests that reports on possible violations of measurement uncertainty [7,9,10,14] are based on definitions of measurement resolution and disturbance that are not consistent with those given here. The definition of uncertainties in terms of the information obtained about unknown input states given in Eqs. (4) and (17) may therefore be closer to the original intention of Heisenberg's argument than the alternatives.

A significant feature of uncertainty relation (20) is that it characterizes the actual changes in a physical property caused by the measurement given by the disturbance ΔB_m^2 . This disturbance is given in terms of information that may be available before and after the measurement, but it does not directly refer to the information obtained in the measurement process itself. By relating this disturbance in \hat{B} to the measurement resolution in \hat{A} , uncertainty (20) establishes an in-

separable connection between physics and information that may be one of the most characteristic features of quantum mechanics. Consequently, a complete characterization of quantum measurements must always include both the information aspects given by the measurement resolution and the dynamical aspect given by the disturbance.

Two simple examples of photon number measurements may help to illustrate the different aspects of measurements expressed by resolution and disturbance. Conventional photon detection usually requires the absorption of all photons. The detection of a single photon can therefore be represented by the operator $\hat{M}_{n=1} = |n=0\rangle\langle n=1|$. This operator has a perfect measurement resolution of $\delta n^2=0$, but its disturbance is given by $\Delta n^2=1$. On the other hand, a quantum nondemolition measurement of photon number is represented by a measurement operator $\hat{M}_m = \sum_n M_m(n) |n\rangle\langle n|$. This operator commutes with \hat{n} , and therefore has a disturbance of $\Delta n^2=0$. However, the coefficients $M_m(n)$ are usually given by a slowly varying function of n and the corresponding measurement resolution is very low ($\delta n \gg 1$). These examples show that the measurement resolution and the disturbance of a single property are not usually connected in any way. Interestingly, uncertainty (20) does establish such a connection for noncommuting properties.

IV. APPLICATION TO PROBLEMS IN QUANTUM COMMUNICATION

The application of quantitative concepts to quantum communication may appear to be a bit unusual. Theoretically, it does not make a difference whether the eigenvalue difference of two orthogonal states used in a quantum code is large or small. However, the quantitative aspect may be reintroduced by the specific physical implementation. In multilevel systems, a reasonable choice of operator properties will then represent the fact that weak interactions with the environment are more likely to cause transitions between eigenstates if the eigenvalue difference is small. In the presence of noise, it is then optimal to encode information in such a way that the more likely errors causing small changes in the eigenvalues of \hat{A} or \hat{B} are less serious than the comparatively unlikely errors involving large changes. Such codes will have a quantitative character similar to that of analog signals. In fact, this kind of situation is well known in the case of continuous variable quantum optics, where the concept of uncertainty can be applied directly to implementations of quantum cryptography [15,16].

A quantum cryptography protocol for the general case of noncommuting variables \hat{A} and \hat{B} may be implemented as follows. Alice will randomly choose either an eigenstate of \hat{A} or an eigenstate of \hat{B} to send her information. Likewise, Bob chooses randomly whether to measure \hat{A} or \hat{B} . By later exchanging data on their choices of \hat{A} or \hat{B} , they can then select the valid communication attempts. An eavesdropper can now try to optimize the simultaneous extraction of information about the eigenvalues of \hat{A} and of \hat{B} by choosing various measurement strategies $\{\hat{M}_m\}$ with the appropriate

resolutions δA_m^2 and δB_m^2 . However, this eavesdropping attempt will cause additional noise in the communication between Alice and Bob. This noise is given by the disturbances ΔA_m^2 and ΔB_m^2 and may lead to the detection of the eavesdropper by Alice and Bob. In fact, Alice and Bob can determine the average disturbances by exchanging information about the initial eigenvalues sent by Alice and the final eigenvalues received by Bob. From randomly selected subsets of the valid communication attempts, Alice and Bob can then estimate the maximal resolutions δA_m^2 and δB_m^2 that could have been obtained by the eavesdropper. If these resolutions are sufficient to decode the information encoded in eigenstates of \hat{A} and \hat{B} , the line is not safe. On the other hand, security can be established if the noise levels given by the disturbance are low enough to prevent the required measurement resolution.

Another application of measurement uncertainties is the quantum cloning problem. If it is known that the state to be cloned is either an eigenstate of \hat{A} or an eigenstate of \hat{B} , it is possible to define a quantitative cloning error equal to the average quadratic deviation of the clone's property \hat{A} or \hat{B} from the eigenvalue of the original. This cloning error can then be used to evaluate cloning strategies based on a quantum measurement $\{\hat{M}_m\}$ on the original and a quantum state preparation $|\psi_m\rangle$ for the clones. In this case, the disturbance caused by the measurement $\{\hat{M}_m\}$ characterizes the unavoidable damage done to the original in the cloning process, resulting in an irreversible loss of information about the original properties of the cloned state. If the cloning process extracts the maximal amount of information from the original system by effectively projecting the system onto a pure state, it is also possible to define a set of cloning operators $\hat{C} = |\psi_m\rangle\langle\psi_m|$ for the optimal cloning procedure. This set of operators represents a projective measurement of the input system followed by a preparation of N copies of the corresponding quantum state. Note that it is possible to produce any number of clones in this manner, since $|\psi_m\rangle$ is precisely defined by the classical measurement information m . The total output statistics of the cloning process is then given by a mixture of the product states of $|\psi_m\rangle$ with the respective statistical weight given by the measurement probabilities $p(m)$ for the original input state. However, the cloning errors for each individual clone can be estimated directly from the disturbances caused by the cloning operator \hat{C} , since it represents both the sensitivity of the cloning process to the input and the resulting output statistics of all the clones.

Finally, it is also possible to apply this quantitative characterization to errors and information extraction in quantum teleportation. In this case, the measurement made on the joint system of the input state and one part of the entangled pair may be sensitive to properties of the unknown input state, e.g., because the entanglement is nonmaximal. This effect can be described by a set of transfer operators \hat{T}_m with properties equivalent to the measurement operators \hat{M}_m [16,17,18]. The measurement resolution δA_m^2 then characterizes the information extracted about the input eigenvalue of

\hat{A} , while the disturbance ΔB_m^2 quantifies the teleportation error in \hat{B} . A particularly simple example is given by the classical limit of continuous-variable teleportation, where a pair of uncorrelated vacuum fields is used instead of the entangled pair [19]. The teleportation procedure then corresponds to a measurement projection on a coherent state $|\alpha\rangle$, followed by the preparation of a corresponding state in the output. This method can also be used for quantum cloning or as an eavesdropping strategy. In all cases, the procedure can be represented by the α -dependent measurement operators

$$\hat{M}(\alpha) = \frac{1}{\sqrt{\pi}} |\alpha\rangle\langle\alpha|. \quad (21)$$

These operators can now be characterized using the quadrature components of the light field, \hat{x} and \hat{y} , with $[\hat{x}, \hat{y}] = i/2$, and the definitions of optimized estimates and uncertainties given by Eqs. (7) and (17). The results for the operators $\hat{M}(\alpha)$ then read

$$\begin{aligned} x_\alpha + iy_\alpha &= \alpha, \\ \delta x_\alpha^2 &= \delta y_\alpha^2 = 1/4, \\ \Delta x_\alpha^2 &= \Delta y_\alpha^2 = 1/2. \end{aligned} \quad (22)$$

These uncertainties now define the noise levels in the measurements and in the transmitted signal. Note that the disturbances are twice as high as the measurement resolutions. This is a typical feature of the classical teleportation limit [19]. In an eavesdropping scenario, this strategy therefore extracts maximal information, but makes it easy for Alice and Bob to detect the eavesdropping attempt.

V. CONCLUSIONS

The effects of quantum measurements described by sets of measurement operators $\{\hat{M}_m\}$ can be characterized in terms of the physical properties of the measured system by evaluating the effects of the measurement on eigenstates of the corresponding Hermitian operators. It is then possible to define quantitative expressions for the concepts of measurement resolution and disturbance corresponding to the notions expressed in the earliest discussions of quantum measurement [5]. These definitions allow a derivation of Heisenberg's uncertainty principle, demonstrating the general validity of uncertainty for all possible measurement strategies. In particular, it can be shown that the back action of a generalized measurement is indeed uncertainty limited. A complete characterization of generalized quantum measurements in terms of measurement resolutions and disturbances for each relevant physical property may therefore provide practical insights into the nature of quantum measurements.

Since the definitions of measurement uncertainties have been based on quantitative estimates of an unknown eigenstate input, they can also be applied to evaluate errors in various quantum communication scenarios. For example, eavesdropping strategies for quantum cryptography may re-

quire an optimization of both the measurement resolutions δA_m^2 and δB_m^2 for simultaneous estimates of \hat{A} and \hat{B} , and the corresponding disturbances ΔA_m^2 and ΔB_m^2 . Similar considerations may also be useful in the discussion of quantum cloning and quantum teleportation.

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APPENDIX: DERIVATION OF UNCERTAINTY RELATIONS FOR STATISTICAL MIXTURES

Although the basic derivation of uncertainty relations for quantum states and density matrices is well known [6], it may be useful to review it in the general context of statistical mixtures in order to provide a more precise justification of the measurement uncertainties discussed in this paper.

The basic derivation of uncertainty relations for pure states is obtained from the Cauchy-Schwarz inequalities for the two Hilbert space vectors given by

$$(A_m - \hat{A})|\psi\rangle \quad \text{and} \quad (B_m - \hat{B})|\psi\rangle. \quad (\text{A1})$$

Since the product of the squared length of these vectors must be larger or equal to the squared inner product of the vectors, it follows that

$$\begin{aligned} & \langle \psi | (A_m - \hat{A})^2 | \psi \rangle \langle \psi | (B_m - \hat{B})^2 | \psi \rangle \\ & \geq |\langle \psi | (A_m - \hat{A})(B_m - \hat{B}) | \psi \rangle|^2. \end{aligned} \quad (\text{A2})$$

The uncertainty relations are then obtained by taking only the imaginary part of the inner product into account. Since \hat{A} and \hat{B} are the Hermitian operators, this imaginary part is given by one half of the commutation relation, and the result

is the well-known formulation of uncertainty for pure states,

$$\underbrace{\langle \psi | (A_m - \hat{A})^2 | \psi \rangle}_{\delta A^2} \underbrace{\langle \psi | (B_m - \hat{B})^2 | \psi \rangle}_{\delta B^2} \geq \frac{1}{4} |\langle \psi | [\hat{A}; \hat{B}] | \psi \rangle|^2. \quad (\text{A3})$$

In order to generalize this result to density matrices or to any other form of statistical mixtures, it is sufficient to examine the case of a set of uncertainties given by

$$\delta A_i^2 \delta B_i^2 \geq U_i^2, \quad (\text{A4})$$

where the averaged uncertainties are given by

$$\begin{aligned} \delta A^2 &= \sum_i p(i) \delta A_i^2, \\ \delta B^2 &= \sum_i p(i) \delta B_i^2. \end{aligned} \quad (\text{A5})$$

It then follows that:

$$\begin{aligned} \delta A^2 \delta B^2 &= \sum_{i,j} p(i)p(j) \frac{1}{2} (\delta A_i^2 \delta B_j^2 + \delta A_j^2 \delta B_i^2) \\ &\geq \sum_{i,j} p(i)p(j) \delta A_i \delta B_i \delta A_j \delta B_j \\ &\geq \left(\sum_i p(i) U_i \right)^2. \end{aligned} \quad (\text{A6})$$

It is therefore possible to derive an uncertainty relation for the statistical mixture defined by $p(i)$ by averaging over the uncertainties U_i . This derivation can be applied to derive uncertainties for statistical operators such as Eq. (8) by representing the statistical operator as a mixture of pure states. However, it can also be useful in a more general context, as seen in the derivation of the back action uncertainty (20).

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