

Entwined paths, difference equations, and the Dirac equationG. N. Ord^{1,*} and R. B. Mann^{2,†}¹*MPCS, Ryerson University, Toronto, Ontario, Canada*²*Department of Physics, University of Waterloo, Waterloo, Ontario, Canada*

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Entwined space-time paths are bound pairs of trajectories which are traversed in opposite directions with respect to macroscopic time. In this paper, we show that ensembles of entwined paths on a discrete space-time lattice are simply described by coupled difference equations which are discrete versions of the Dirac equation. There is no analytic continuation, explicit or forced, involved in this description. The entwined paths are “self-quantizing.” We also show that simple classical stochastic processes that generate the difference equations as ensemble averages are stable numerically and converge at a rate governed by the details of the stochastic process. This result establishes the Dirac equation in one dimension as a phenomenological equation describing an underlying classical stochastic process, in the same sense that the diffusion and telegraph equations are phenomenological descriptions of stochastic processes.

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I. INTRODUCTION

This paper is the first of a series of papers exploring the consequences of a recent discovery. The new result is that the Feynman chessboard model of the Dirac propagator can be transplanted into classical statistical mechanics, bypassing formal analytic continuation (FAC) completely [1]. The new feature allowing this is the use of entwined space-time paths which are essentially self-quantizing. These will be discussed shortly, but first we clarify what is meant by the phrase “transplanted into classical statistical mechanics.”

One usually arrives at quantum mechanics through one of two routes. The most common approach is to analytically continue from \mathbb{R} to \mathbb{C} explicitly by imposing operator relations [e.g., $p \rightarrow -i\hbar(\partial/\partial x)$, $E \rightarrow i\hbar(\partial/\partial t)$]. Another alternative is to force a FAC by imposing physical conditions that cannot be met within the original space (e.g., requiring diffusion to be reversible) [2–6]. Both of these procedures extend classical physics to a suitably enlarged regime (i.e., Hilbert space). However, the resultant wave functions are formal objects with no direct interpretation.

We avoid these routes entirely. In our approach, the components of wave functions are classical ensemble averages obtainable by simple counting processes. It is the spatio-temporal geometry of entwined paths that gives rise to the properties described by the standard quantum-mechanical complex wave equations. The relevant algebraic property $i^2 = -1$ appears explicitly through the geometry of entwined paths. It is because the equivalent of this algebraic structure (expressed, as we shall later see, by an anti-Hermitian operator) is built into the geometry of the space-time paths themselves that we never need to introduce it artificially through a FAC. This is the main feature of our results and we will return to the relation between geometry and algebra in the discussion at the end of the paper.

By avoiding the FAC generally used to quantize a system,

entwined paths provide the Dirac equation with a new context that is conceptually very different from its context in quantum mechanics. In quantum theory, the Dirac equation is a “fundamental” equation. Wave-function solutions are thought to contain “all the information about the state of a system,” and the Dirac equation describes the evolution of the wave function.

In the context described in this paper, the Dirac equation is not a fundamental equation at all. Rather it is a phenomenological equation that describes ensemble averages of a classical charge density arising from entwined paths. Just as the diffusion equation is a phenomenological equation describing a density of random paths (Brownian motion), so the Dirac equation describes net densities of entwined paths, where the time-reversed portions of paths add the new qualitative features of interference and reversibility. Because there is a specific underlying stochastic model involved, wave-function solutions do not contain all the information about the state of a system, they are simply ensemble averages of the background stochastic process. In the new context, the stochastic process itself is the fundamental object. This means that there is an identifiable underlying stochastic process involved in the *formation* of wave functions.

In contrast, quantum mechanics has nothing to say about the process of wave-function formation, since there, the wave function is just part of an algorithm. Probability only enters quantum mechanics through the measurement postulates, not through unitary evolution (Fig. 1). Thus, in quantum mechanics we postulate that the modulus squared of the wave function represents a probability density. That the postulate is correct is well verified experimentally, but remains a feature which does not follow from unitary evolution itself. On the other hand, entwined paths support unitary evolution of an ensemble average which itself has an underlying stochastic process that, as we shall see, is amenable to direct simulation. In the future we shall be able to ask “does the underlying stochastic process also mimic the measurement postulates through the stochastic formation of the wave function?” If the answer is yes, then the stochastic process we are proposing for the Dirac equation may have a deeper connec-

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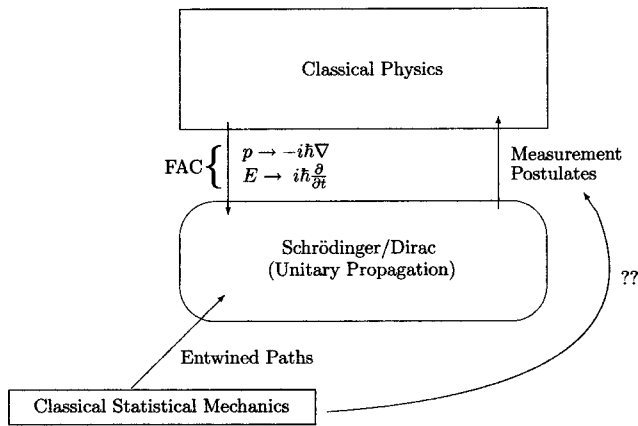


FIG. 1. In conventional interpretations of quantum mechanics, we pass from classical physics (upper box) to the Schrödinger or Dirac equation (rounded box) through a formal analytic continuation (FAC). This brings wave features into the classical particle paradigm as wave functions propagate unitarily. Measurement postulates are then used to interpret wave-function solutions in terms of macroscopic measurements. This paper discusses an alternative route to the Dirac equation from classical statistical mechanics using “entwined paths” (lower left). This route does not require a FAC since the geometry of the paths automatically builds in the relevant wave behavior. Furthermore, in the new context, the Dirac equation appears as a phenomenology for the underlying stochastic process. Further examination of the stochastic process is required to see if the measurement postulates can also be supported in the new context.

tion to quantum mechanics. If not, the two contexts for the Dirac equation will remain distinct.

II. ENTWINED PATHS

We shall be working in a two-dimensional discrete space (z, t) with lattice spacings δ and ϵ , respectively. Although we shall eventually think of t as time, it is convenient at this point to think of t as a spatial coordinate. Entwined paths can be generated in two different ways, both ways being important in understanding the resulting phenomenology. We shall employ two walkers, Eve and Max, who will generate and color random walks, using the two different methods. Eve’s method is explicitly ergodic in the sense that she will generate the full ensemble of entwined paths in a serial fashion after a sufficiently long time. Max will not use an explicitly ergodic process. He will, however, generate an ensemble of paths using a Markovian process which generates paths in parallel, many at a time. At each step, Max’s behavior will depend only on his current state. Both walkers generate and color their paths through (z, t) based on calls to a random process R which generates a string of letters, with the distribution

$$R = \begin{cases} U & \text{with probability } 1 - a\Delta t, \\ M & \text{with probability } a\Delta t. \end{cases} \quad (1)$$

We can think of M as standing for marked and U standing for unmarked, with the whole string of symbols a sequential list or tape T .

Eve’s instructions for coloring her path through the lattice are as follows. Eve starts at the origin and her first step is to the lattice site (δ, ϵ) . At the next step she consults the random process. If a U is the first letter on the tape, she maintains her direction and steps to $(2\delta, 2\epsilon)$. If she gets an M , she changes her direction in z and steps to $(0, 2\epsilon)$. Again she consults the tape. If she gets a U , she maintains her direction and takes another step. If she gets her second M , she drops a “marker” at the site but steps forward maintaining her direction. This process is repeated, alternately changing direction and dropping markers whenever the random process indicates a mark. Eve colors her path blue as she goes. With each step she advances one lattice spacing in t until the first time she is called to drop a marker after a set return “time” t_R . This marker will be the last symbol on the tape that describes this particular path. At this marker, Eve maintains her direction in z but steps back one unit in t and switches to the color red [Fig. 2(a)]. Eve now makes her way back to the origin down the “light-cone” paths [paths with slopes of unit magnitude in the (z, t) plane] which intersect all of the markers, coloring the path red as she goes [Fig. 2(b)]. On her return path, Eve does not need to consult a tape since her trail of markers uniquely defines her return path.

There are several things of interest about the entwined path generated by Eve’s walk. The first thing to notice is that the distribution of distances between the corners in the path and neighboring markers/crossing points is identical to the distribution of steps between the marks on the tape. Furthermore, because Eve returns to the origin at the end of her return journey, we see that her walk is ergodic. She can cover all such paths simply by repeating the process enough times, covering all possible distributions of marks on the tapes with the appropriate frequency from the random process. Thus the time-average contribution from Eve, in the limit of large time, is the same as the ensemble average, for when Eve has returned to the origin after N circuits to t_R and back, we cannot distinguish the pattern of the space-time path from that of N Eves each traversing the lattice once.

We also note that an entwined pair generated by Eve can be viewed as two osculating paths which start at the origin moving in opposite directions and finally merge where Eve changes directions in t . We call these two paths the left and right envelopes of the pair.

The envelopes themselves have simple properties. They always have opposite coloring. They both change color at every second corner of the envelope. The distribution of lengths between corners is the same for both envelopes and is the same as the distribution of waiting times between the marks on the tape.

The simple properties of the envelopes allow Max, our second walker, to traverse and color the paths in a different way. Max will not actually step between each neighboring lattice site as did Eve. We imagine Max to have long arms and he will paint entwined paths as pairs, painting one envelope with each hand. Max uses the same tapes as Eve but he will interpret them in a different way.

Max reads the tape out to the end of the second mark at, say, t_2 . He notes the position of the two marks on the tape and walks to the position where the envelopes touch at the

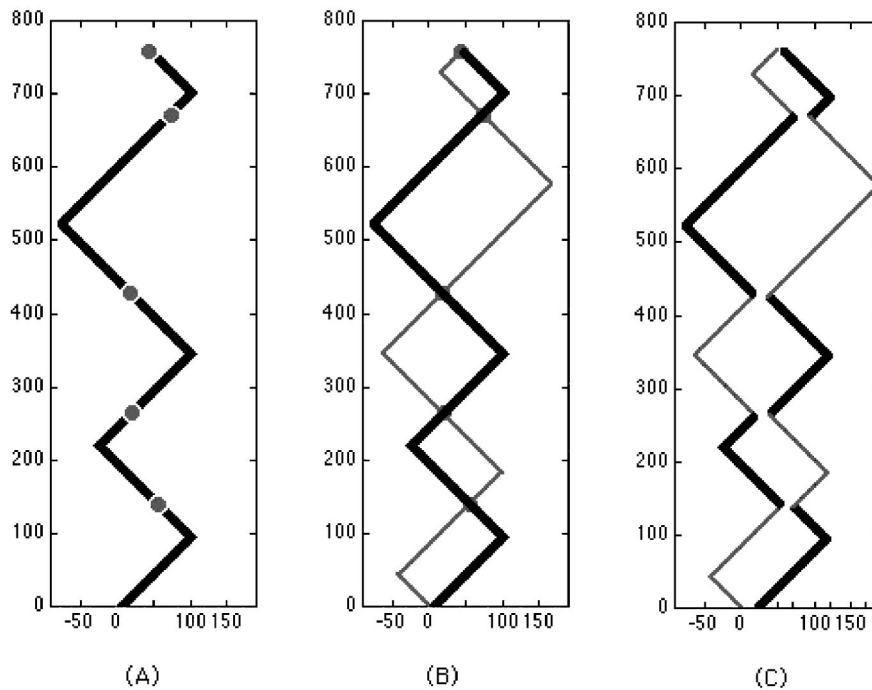


FIG. 2. Forming entwined paths in space-time. z is horizontal, t is vertical. (a) Eve travels at constant speed but occasionally reverses direction in response to a stochastic process. At every other indication from the stochastic process, a marker is dropped instead of a direction change (disks in the figure). Eve paints her path blue (thick line in figure) as she goes. (b) After some specified time t_R ($t_R = 700$), Eve stops at the next marker. She then reverses her direction in time but not in space. She changes from blue to red (thin line in figure) and follows the “light-cone” paths through the markers back to the origin. (c) The entwined path formed in (b) can be regarded as two osculating paths which we call envelopes. These are separated for clarity. Max paints these envelopes with a simple rule specifying a change of color at every second corner, with opposite colors on the envelopes.

second mark. As he does this, he paints the right envelope path blue with his right hand and the left envelope path red with his left hand. Max needed to see where the first two marks on the tape were in order to get both corners on his right and left hand in the appropriate places. At t_2 , Max interchanges the paintbrushes, reads the tape out to the fourth mark, and repeats his double-handed painting. Max continues this process until he brings the paint brushes together at or beyond the return time t_R . At this point, Max picks up his brushes and returns to the origin without painting the lattice. Max can cover the entire ensemble of paths by repeating the process a large number of times, however, since he always returns to the origin without painting, Max’s technique is not continuously ergodic. Note that Eve always painted wherever she moved, including on the return path to the origin. Eve can fully paint any of Max’s paths without having to remove her brushes from the lattice. Note also that a single tape codes for an entwined pair for both Eve and Max. Eve traverses the pair in a forward and reversed path [Fig. 2(a)]; Max “traverses” the pair by painting the two envelopes in parallel.

Although Max’s technique is not ergodic, it can be made Markovian. That is, if we allow Max enough arms, and the ability to paint many pairs in parallel, he can paint the entire ensemble of paths in a single pass from the origin to t_R , without ever requiring information beyond the current symbols on an ensemble of tapes, and the current state of each “arm.” Thus, he paints at step n according to his state at the

previous step. He never has to reverse his direction in t , and he never has to read more than one symbol per envelope path at a time. He does not need to read out to the second mark on each tape before he starts painting.

To see how he does this, suppose that Max takes one of Eve’s tapes, T_p say, and regards it as instructions for his right hand only. He interprets it in the following way. Each mark will now correspond to a direction change and every other mark will also be interpreted as a color change, so he only needs to read the tape one symbol at a time. Although this interpretation of the tape by Max will not generate the same right envelope for him as it would for Eve, the difference is a simple, unique permutation of the symbols on the tape. That is, Max’s right envelope for T_p will correspond to Eve’s right envelope for another unique tape $T_{p'}$, and vice versa. If Max is going to paint all distinct tapes in the ensemble simultaneously, he can use his new interpretation of Eve’s tapes to do this, since the mapping between T_p and $T_{p'}$ is invertible. Thus, although Eve’s traversal of entwined paths is not Markovian, the ensemble average generated by her traversing the full ensemble *is* Markovian, and we may write a difference equation for it just by noting how Max interprets and paints trajectories. This is true for both envelopes.

Now we have been thinking of t as a spatial coordinate, but since Max can generate the ensemble of paths via a Markov process which just steps forward in t , we can think of t as a macroscopic time. In this case, the red/blue coloring of Eve’s path, which indicated the direction in t of the traversal,

now indicates particle-antiparticle status. For example, blue indicates particle and red indicates antiparticle. If we associate a plus 1 with blue we have to associate a minus 1 with red. This number associated with color we shall call charge, but it is not to be confused with electromagnetic charge. Our charge is a classical concept which is associated with the (discrete) continuity of the trajectory. Since our entwined pairs all return to the origin, we see them as either pairs of points at fixed t , or as two points superimposed, or no points at all at fixed t . Particle number is not conserved in t but charge is, if we allow particle and antiparticle opposite charge.

How does charge behave in ensembles of entwined pairs? This is easy to calculate if we consider ensembles of paths by their envelopes, generated by Max's Markovian method.

Consider the left envelope path in Fig. 1(c). Note that the rule for its generation is very simple. Starting at the origin, the particle proceeds in the $-z$ direction until a mark on the tape indicates a direction change. At the first (*right*) turn, the particle just changes direction but not color. At the second (*left*) turn, the path also changes color, and the charge changes sign. This process is repeated. Each right corner maintains the color, and each left turn changes the color and the sign of the charge. The reader familiar with the Feynman chessboard model will recognize this rule as a version of Feynman's corner rule [7,8]. In this context, the rule is dictated by the geometry of entwined paths. Left turns in the left envelope are actually crossing points of the particle-antiparticle pair, and the origin of the sign change is physical.

If we now let $\phi_n^1(z)$ be the ensemble charge density from left envelope links parallel to the left light cone at step n and $\phi_n^2(z)$ be the ensemble charge density from left envelope links parallel to the right light cone, we can write

$$\begin{aligned}\phi_n^1(z) &= (1 - a\Delta t)\phi_{n-1}^1(z + c\Delta t) - a\Delta t\phi_{n-1}^2(z - c\Delta t), \\ \phi_n^2(z) &= (1 - a\Delta t)\phi_{n-1}^2(z - c\Delta t) + a\Delta t\phi_{n-1}^1(z + c\Delta t).\end{aligned}\quad (2)$$

That is, regarding Fig. 3, most paths maintain their direction and color as they pass through a lattice site. The proportion which does this is $(1 - a\Delta t)$. However, a proportion $a\Delta t$ changes direction at the site. When they scatter from the right light cone they change charge on scattering, so they *decrease* the net charge in the new direction in proportion to the density in the old direction. However, when they scatter from the left light cone they maintain their charge on scattering, so they *increase* the net charge in the new direction in proportion to the density in the old direction.

The right envelope is similar, except here it is the right turns which change charge. If we let $\phi_n^3(z)$ and $\phi_n^4(z)$ be the right envelope charges parallel to, respectively, left light cones and right light cones, we have

$$\begin{aligned}\phi_n^3(z) &= (1 - a\Delta t)\phi_{n-1}^3(z + c\Delta t) + a\Delta t\phi_{n-1}^4(z - c\Delta t), \\ \phi_n^4(z) &= (1 - a\Delta t)\phi_{n-1}^4(z - c\Delta t) - a\Delta t\phi_{n-1}^3(z + c\Delta t).\end{aligned}\quad (3)$$

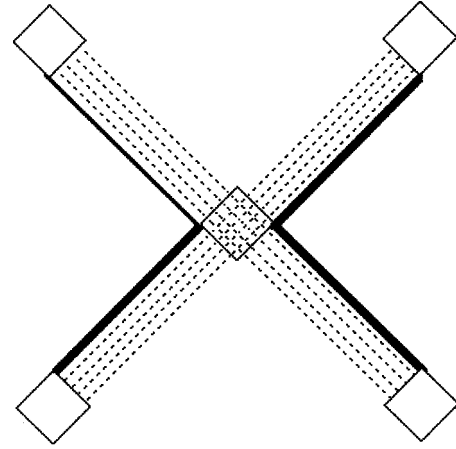


FIG. 3. Left envelope scattering. Most paths do not scatter, but those that do behave differently depending on which direction they are coming from. Paths scattering from right-moving to left-moving change color (charge) when they scatter (leftmost path in figure). Paths scattering from left-moving to right-moving do not change color when they scatter (rightmost path in figure).

Note that the change in signs of the scattering terms in the above equations are a direct result of the geometry of entwined pairs. It is Eve's insistence on entwining forward and reversed paths that allows Max to use his simple coloring rule of a change in color after every second corner. This rule in turn forces the ensemble to alternate the signs of the scattering terms, since it is a detailed feature of every path in the ensemble.

Equations (2) and (3) constitute a set of coupled difference equations in the four densities $\phi^1 - \phi^4$. Although their derivation is straightforward, their consequences as a description of a classical ergodic stochastic process are potentially far-reaching, since these are representations of the discrete Dirac equation. This will be demonstrated in Sec. IV. Furthermore, the above arguments discuss only ensemble averages. It is possible that the above equations are correct for the ensemble average, but that the underlying stochastic process gives rise to such large fluctuations that the ensemble average survives only in the event of a uniformly covered ensemble. In this case, normal stochastic fluctuations would swamp the signal and the system would not exhibit the above equations except under the rare circumstance of an almost perfectly uniform coverage of the ensemble. In other words, if we watch Eve sequentially follow a large number of randomly chosen tapes, adding and subtracting her contributions to the sample average, would this sample average $\bar{\phi}$ converge to the ensemble average predicted in Eqs. (2) and (3)? In the following section, we test this question by stochastic simulation of Eve's generation of entwined paths.

III. NUMERICAL CHECKS

In the preceding section, we discussed entwined paths from the perspective of two walkers, Eve and Max. Eve draws a single entwined path by traversing a pair through a forward passage followed by the entwined reverse passage. Since she returns to the origin after each pair traversal, she

can sequentially cover all members of the ensemble of paths. Each time she traverses the lattice, she records her passage by adding and subtracting 1's as appropriate to the direction she is moving in t to her record of path visits. She thus creates a space-time field which we shall denote by $\tilde{\phi}$. For example, before she starts, all the $\tilde{\phi}$ are 0 for all z with $t > 0$. At her first step she arrives at (δ, ϵ) and adds a 1 to $\tilde{\phi}^4(\delta, \epsilon)$. If her tape gives her a U at this point, her second step is to $(2\delta, 2\epsilon)$ where she will add a 1 to $\tilde{\phi}^4(2\delta, 2\epsilon)$. If she receives an M at the second step she would instead move to $(0, 2\epsilon)$ where she would add a 1 to $\tilde{\phi}^3(0, 2\epsilon)$. She repeats this process until the return time where she traverses the return path, adding -1 's to the appropriate $\tilde{\phi}$ on the way. If Eve were to perform this task so as to cover all distinct paths exactly once, the arguments of the previous section imply that the $\tilde{\phi}$ will satisfy the difference equations (2) and (3) exactly, since these difference equations represent the ensemble averages that Max would generate if he painted all

the paths simultaneously with his Markovian algorithm. As Eve sequentially paints path after path, a question that needs testing is whether the difference equations (2) and (3) emerge as approximate descriptions of the sample averages $\tilde{\phi}$ generated by Eve. With no extra controls over the stochastic process, the probabilities $\alpha = a\Delta t$ and $\beta = 1 - \alpha$ as actual frequencies will necessarily fluctuate over the sample. It is important to check that the fluctuations do not destroy the signal, and that Eve's sequentially generated sample does approximately satisfy the difference equations. To check this we note that on the lattice the $\tilde{\phi}^i$ at any time step are vectors indexed by the value of z . At time step m each vector $\tilde{\phi}$ has roughly $m + 1$ nonzero elements. If $\|\cdot\|$ is the Euclidean norm, we can then test the relative measure of error in each separate difference equation in Eqs. (2) and (3) for a sample average generated by Eve. For example, to test Eq. (2) for a sample average $\tilde{\phi}^1$ we consider the error E_n^1 after Eve has traversed the lattice n times,

$$E_n^1 = \frac{\|\tilde{\phi}_n^1(z) - [(1 - a\Delta t)\tilde{\phi}_{n-1}^1(z - c\Delta t) - a\Delta t\tilde{\phi}_{n-1}^2(z - c\Delta t)]\|}{\|\tilde{\phi}_n^1(z)\|} \tag{4}$$

Analogous expressions apply for the other sample averages. Here a is the ensemble parameter, not the sample approximation. If we were to replace the $\tilde{\phi}$ in Eq. (4) by the ensemble average ϕ , we would get zero. However, $E_1^1 > 0$ since, after only one traversal, the ensemble averages cannot be met, there being only a single entwined path through the lattice. If, however, $\lim_{n \rightarrow \infty} E_n^1 = 0$, then the stochastic process is stable and the ensemble average description is a valid approximation of the process for large n .

Figure 4 shows E_n^4 for various values of the sample size n plotted on a log-log scale. The three groups of points represent three values of t . As t increases, the range of z increases and the relative coverage by the sample paths decreases, resulting in a larger error. If Eve randomly chooses at each lattice site whether or not to scatter, we would expect the convergence to have the usual $1/\sqrt{n}$ dependence characteristic of random sampling without replacement. This would yield a very slow rate of convergence. This does indeed happen, so we have modified the underlying stochastic process to a form of sampling without replacement. In our approach, when Eve has to make a decision at a lattice site, she checks her previous decisions and chooses to scatter or not scatter based on her local scattering probability, making the choice that causes this probability to come closer to the ensemble average if this is possible, choosing randomly if not. This improves the convergence so that it is asymptotically $1/n$. The other three errors E_n^1, E_n^2, E_n^3 show the same convergence characteristics. Note that the sampling technique only speeds up the convergence. It does not affect the limit itself since it does not alter the ensemble average for the probabili-

ties α and β . Random sampling with replacement in which the "local" probability is not consulted and each call to the random process is independent gives rise to figures similar to

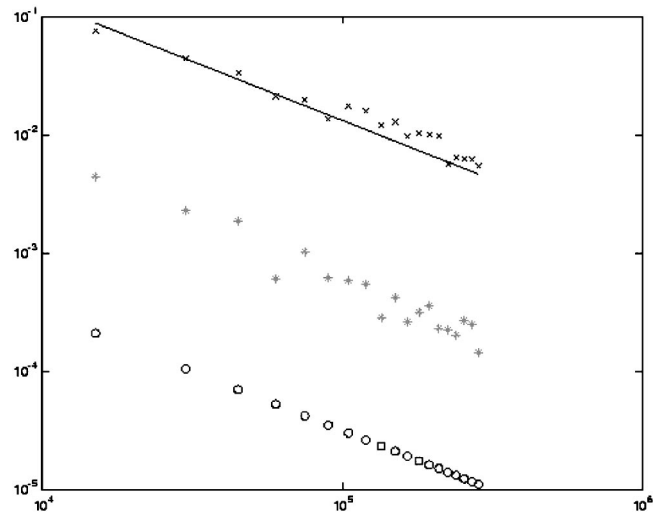


FIG. 4. A numerical test of Eq. (3) using Eq. (4). The numerical error is plotted vs the number of entwined paths in the run. The upper sequence is at t_R , which is 16 time steps for this run. The solid line is a reference line which decreases as $1/n$, suggesting that for this stochastic process the error goes down as $1/n$. The middle sequence is at $t=8$ and the lowest sequence is at $t=2$. As t increases, the error increases since the configuration space is larger and the $\tilde{\phi}_i$ less well covered by the sample. The tendency to converge as $1/n$ appears to hold for all t .

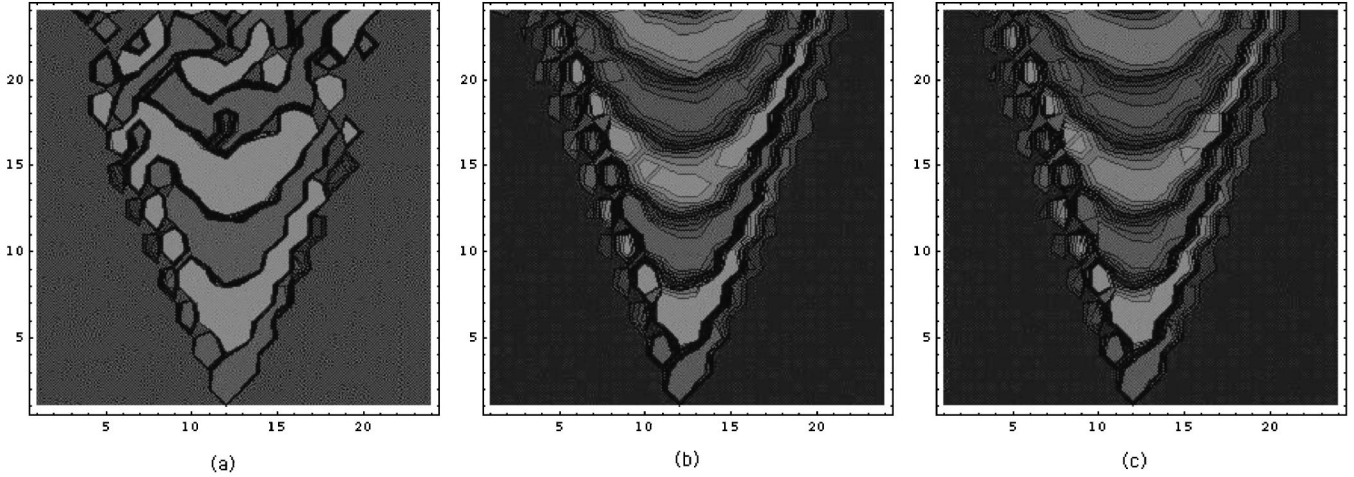


FIG. 5. Entwined paths draw the Dirac propagator. Pictured are contour plots of $\bar{\phi}_3$ for, respectively, 10^3 , 10^5 , and 10^7 entwined paths. In the simulation, the return time is 24 lattice steps and the probability of scattering is $\alpha\Delta t = 1/2$. (c) is visually indistinguishable from the ensemble average solution obtained by solving Eq. (3) exactly. The sawtooth appearance of the light cone is an artifact of the lattice used to store the $\bar{\phi}$.

Fig. 4 except the slope of the trend lines is less, corresponding to $1/\sqrt{n}$ convergence.

Figure 5 shows a contour plot of $\bar{\phi}_3$ as Eve continues to traverse entwined paths. At the resolution of the figure, part (c) is indistinguishable from the exact solution of Eqs. (3) with the source at the origin. That is, Eve's formation of the component of the propagator pictured in Fig. 5(c) is indistinguishable from Max's formation of the ensemble average using his Markovian technique. The other components are similar. Interestingly, whereas Max "draws" Fig. 5(c) sequentially from $t=0$ to $t=t_R$, one step at a time, coloring all paths in parallel and evolving the "picture" in a form of unitary evolution, Eve assembles the picture as a projection from a larger space. If we think of Eve as having her movement in the (z,t) plane parametrized by some variable s , then the sequence of pictures in Fig. 5 represents a projection of Eve's history onto the (z,t) plane with s increasing from (a) to (c). Projection has been used before to bypass FAC [9].

These numerical results confirm the fact that the difference equations (2) and (3) have an underlying stochastic process and are phenomenological equations for entwined pairs. In the next section, we show that the difference equations describing the ensemble averages are discrete versions of the Dirac equation.

IV. COMPARISON WITH DIRAC

By expanding Eqs. (2) and (3) to first order, we can approximate the difference equations by a set of coupled PDE's. The resulting equations are

$$\begin{aligned}\frac{\partial\phi^1}{\partial t} &= c\frac{\partial\phi^1}{\partial z} - a\phi^1 - a\phi^2, \\ \frac{\partial\phi^2}{\partial t} &= -c\frac{\partial\phi^2}{\partial z} - a\phi^2 + a\phi^1,\end{aligned}$$

$$\begin{aligned}\frac{\partial\phi^3}{\partial t} &= c\frac{\partial\phi^3}{\partial z} - a\phi^3 + a\phi^4, \\ \frac{\partial\phi^4}{\partial t} &= -c\frac{\partial\phi^4}{\partial z} - a\phi^4 - a\phi^3.\end{aligned}\quad (5)$$

Removing the exponential decay by writing $u^\mu(z,t) = e^{at}\phi^\mu(z,t)$, the above becomes

$$\begin{aligned}\frac{\partial u^1}{\partial t} &= c\frac{\partial u^1}{\partial z} - au^2, \\ \frac{\partial u^2}{\partial t} &= -c\frac{\partial u^2}{\partial z} + au^1, \\ \frac{\partial u^3}{\partial t} &= c\frac{\partial u^3}{\partial z} + au^4, \\ \frac{\partial u^4}{\partial t} &= -c\frac{\partial u^4}{\partial z} - au^3.\end{aligned}\quad (6)$$

The above is a representation of the Dirac equation in which all the densities u are real. This may be seen by writing the equation in matrix form. If we write $p_z = -i(\partial/\partial z)$, setting $c=1$ and $a=m$ with $\alpha_z = -\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$, $\beta = \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}$, we have

$$i\frac{\partial u}{\partial t} = (\alpha_z p_z + \beta m)u, \quad (7)$$

where the σ are the Pauli matrices. Note that α and β anti-commute as required, and the relativistic energy-momentum relations are obeyed if we associate the usual meanings with E and p . It is important to note that the rewrite of Eqs. (6) in (7) is only cosmetic. The use of p and i is for convention

only; the u are still real densities, the limit of ensemble averages. They are not the formal objects of conventional quantum mechanics.

V. DISCUSSION

The Dirac equation is usually produced along the lines of Dirac's original argument. The start of the argument is canonical quantization $p \rightarrow -i\hbar(\partial/\partial x), E \rightarrow i\hbar(\partial/\partial t)$. From there, Dirac leads us through the construction of his algebra to satisfy the requirements of the relativistic energy-momentum relations. From the perspective of the above work, the formal step in Dirac's argument is the first one, the FAC.

Entwined paths are essentially self-quantizing, as can be seen by the form of Eq. (7). If m is set equal to zero in Eq. (7), the resulting equation is just two two-component forms of the wave equation which are appropriate for classical particles that stay on their initial light cones. It is the $m\beta$ term that both describes scattering and brings in the interference effects characteristic of the Dirac equation. An inspection of the β matrix shows that its Hermitian character arises because of the factor of i absorbed into it in the rewrite from Eq. (6). The predecessor of β in Eq. (6) is *anti-Hermitian* and it is this feature that ultimately results in interference effects. On the other hand, the anti-Hermitian form arises *directly* from the entwined geometry of the paths. It is for this reason that entwined paths make the canonical quantization step

from classical physics unnecessary. The space-time geometry of entwined paths automatically builds in the relevant physics. To see this, note that it is the regular crossing of the entwined paths that produces the alternating signs of the scattering terms in Eqs. (2) and (3), and ultimately the β matrix in Eq. (7).

The existence of an underlying stochastic model for the Dirac equation, as demonstrated in this paper, allows us the opportunity to consider the Dirac equation as a phenomenological equation describing the evolution of a particle moving on an entwined path in space-time. We now have a stochastic basis for the " U process" of quantum mechanics (that process by which a wave function unitarily evolves), to use Penrose's terminology [10,11], at least in the special case of a free particle in one dimension. We hope exploration of the stochastic model will help to clarify further the relationships between classical and quantum physics, and possibly shed some light on the process by which wave functions "collapse" as a consequence of a measurement (the " R " process).

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