Influence of two parallel plates on atomic levels

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(Received 20 August 2002; published 7 February 2003)

This paper is devoted to the study of the influence of two parallel plates on the atomic levels of a hydrogen atom placed in the region between the plates. We treat two situations, namely, the case where both plates are infinitely permeable and the case, where one of them is a perfectly conducting plate and the other, an infinitely permeable one. We compare our result with those found in literature for two parallel conducting plates. The limiting cases where the atom is near a conducting plate and near a permeable one are also taken.

DOI: 10.1103/PhysRevA.67.022103 PACS number(s): 12.20.Ds

I. INTRODUCTION

It has been known for a long time that the consideration of boundary conditions in the radiation field imposed, for instance, by the presence of material plates, may alter not only the vacuum energy, as it occurs in the Casimir effect [1], but also the properties of atomic systems in interaction with this field. The most common examples are the influence of cavities on the spontaneous emission rate, or on atomic energy levels (Lamb shift modification), or even on the anomalous magnetic moment of the electron ($g-2$ factor). In other words, we can say that the presence of material walls in the vicinity of atomic systems renormalizes their transition frequencies as well as the widths of their spectral lines. The branch of physics that is concerned with the influence of the environment of atomic systems on their radiative properties is usually called cavity QED and the above examples represent only a few of them (for a review see, for instance, Refs. [2–5]).

In this paper we shall investigate how the energy levels of a hydrogenlike atom are altered when it is placed in a region between two parallel plates, where at least one of them is an infinitely permeable plate. These energy modifications are originated from the interaction between the atom and the electromagnetic vacuum fluctuations distorted by the presence of the plates, as pointed out by Power [6] in 1966. For the case of perfectly conducting plates, this problem was first discussed by Barton a long time ago [7] and later on by Lütken and Ravndal [8]. For the particular case where only one plate is present, the interested reader can consult Refs. [9,10]. More recently some generalizations were made by Barton [11,12], and Jhe [13,14]. Cavity QED between parallel dielectric surfaces has also been discussed in the literature [15].

However, although the influence of permeable plates in the spontaneous emission rate has already been considered in the literature [16], its influence in the atomic energy levels has not, at least as far as the authors' knowledge. Our purpose here is to fill this gap in the literature. For simplicity, we shall consider the following situations: (i) a perfectly conducting plate and an infinitely permeable one, which we will refer to as the CP configuration, and (ii) two infinitely permeable parallel plates, which we will refer to as the PP configuration. The former setup, used for the first time by Boyer in order to compute the Casimir effect in the context of stochastic electrodynamics [17], is particularly interesting since it leads to a repulsive Casimir pressure [17] (see also Refs. [18,19] and Tenorio *et al.* [20] for the thermal corrections to this problem). More recently, the influence of this unusual pair of plates was also considered in the context of the Scharnhost effect [21,22]. Regarding the latter setup, although it leads to the same Casimir effect as the usual case (two conducting plates), its influence on the radiative properties of atomic systems are different.

In order to calculate the desired energy shifts, we shall use second-order perturbation theory, regularizing the relevant field correlations with the aid of Schwinger's imaginary time splitting, as in Ref. [8]. The results are compared with the usual case where both plates are perfect conductors, a setup that, from now on, we shall refer to as the CC configuration.

From these results we shall obtain the energy shifts for an atom placed near one single conductor plate, and near an infinitely permeable one.

II. PLATES WITH DIFFERENT NATURE (CP CONFIGURATION)

Let us start by considering the case where the atom is placed between a perfectly conducting plate, located at $z=0$, and an infinitely permeable one, located at $z=L$. For this case, the corresponding boundary conditions are

$$\hat{\mathbf{z}}\times\mathbf{E}(x,y,0,t)=0,\quad \hat{\mathbf{z}}\times\mathbf{B}(x,y,L,t)=0,$$
$$\hat{\mathbf{z}}\cdot\mathbf{B}(x,y,0,t)=0,\quad \hat{\mathbf{z}}\cdot\mathbf{E}(x,y,L,t)=0. \tag{1}$$

In the Coulomb gauge ($\boldsymbol{\nabla}\cdot A=0$) with $A^0=0$, we have

$$\mathbf{E}=-\dot{\mathbf{A}},\quad \mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}. \tag{2}$$

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1050-2947/2003/67(2)/022103(7)/$20.00

It is convenient to write separate expressions for the vector potential for the transverse electric (TE) and magnetic (TM) modes in the following way:

$$\mathbf{A}_{\mathbf{k}}^{TE}(\mathbf{x})=\boldsymbol{\nabla}\times\mathbf{U}_{\mathbf{k}}^{TE}(\mathbf{x}),$$
$$\mathbf{A}_{\mathbf{k}}^{TM}(\mathbf{x})=\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times\mathbf{U}_{\mathbf{k}}^{TM}(\mathbf{x})], \tag{3}$$

where we defined

$$\mathbf{U}_{\mathbf{k}}^{TE}(\mathbf{x})=N\mathbf{e}_z\sin(kz)e^{i\mathbf{k}_T\cdot\mathbf{x}},$$
$$\mathbf{U}_{\mathbf{k}}^{TM}(\mathbf{x})=\frac{N}{i\omega_{\mathbf{k}}}\mathbf{e}_z\cos(kz)e^{i\mathbf{k}_T\cdot\mathbf{x}} \tag{4}$$

and used the following notation: $\mathbf{x}=\mathbf{x}_T+z\,\hat{\mathbf{z}}$. The wave vector is given by

$$\mathbf{k}=(\mathbf{k}_T,k_z)=(k_x,k_y,k_z),\quad k_z=\frac{(n+1/2)\pi}{L}=:k,$$
$$k_x,k_y\in\mathbb{R},\quad n=0,1,2,\ldots \tag{5}$$

and hence, the corresponding frequencies read

$$\omega_{\mathbf{k}}=[\mathbf{k}_T^2+\{(n+1/2)\pi/L\}^2]^{1/2}. \tag{6}$$

The normalization constant $N=\sqrt{2/\mathbf{k}_T^2L}$ is obtained from the condition

$$\int d^3x\mathbf{A}_{\mathbf{k}}^{\lambda*}(\mathbf{x})\cdot\mathbf{A}_{\mathbf{k}'}^{\lambda'}(\mathbf{x})=4\pi^2\delta_{\lambda\lambda'}\delta_{nn'}\delta(\mathbf{k}_T-\mathbf{k}_T'). \tag{7}$$

Therefore, we can write the vector potential between the plates as

$$\mathbf{A}(\mathbf{x})=\sum_{\lambda=E,M}\sum_{n=0}^{\infty}\int\frac{d^2k_T}{(2\pi)^2}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}[a_{\mathbf{k}}^{\lambda}\mathbf{A}_{\mathbf{k}}^{\lambda}(\mathbf{x})e^{-i\omega_{\mathbf{k}}t}+\text{H.c}], \tag{8}$$

where the anihilation and creation operators satisfy the well-known commutation relations

$$[a_{\mathbf{k}'}^{\lambda'},a_{\mathbf{k}}^{\lambda\dagger}]=4\pi^2\delta_{\lambda\lambda'}\delta_{nn'}\delta(\mathbf{k}_T-\mathbf{k}_T'), \tag{9}$$

with all other commutators being zero.

Our purpose here is to study the effect of the vacuum field fluctuations on the energy levels of an atom placed in a region between the plates. With this goal, we shall use perturbation theory and assume that the fields do not vary appreciably in the atomic scales (dipole approximation). The first nonvanishing contributions to the energy shifts are obtained in second order in e from

$$\Delta\varepsilon_n=e^2\sum_{m;\mathbf{k},\lambda}\frac{|\langle n,0|\mathbf{r}\cdot\mathbf{E}(\mathbf{x})|m;\mathbf{k},\lambda\rangle|^2}{\varepsilon_n-\varepsilon_m-\omega_{\mathbf{k}}}, \tag{10}$$

with $|n;\mathbf{k},\lambda\rangle:=|n\rangle\otimes|\mathbf{k},\lambda\rangle$, where $|n\rangle$ designates an atomic state with energy ε_n and $|\mathbf{k},\lambda\rangle$, is a field state with one photon with momentum $\mathbf{k}$ and polarization λ. It can be shown that the perturbation caused by the magnetic field can be neglected [8]. In the above expression, $\mathbf{r}$ is the electron position operator of the atom with the origin taken in its nucleus.

Separating the contributions for the energy shift (10) due to the degenerate states and nondegenerate states, denoted, respectively, by $|n'\rangle$ and $|\ell\rangle$, we write

$$\Delta\varepsilon_n^{(1)}=-e^2\sum_{n';\mathbf{k},\lambda}\frac{|\langle n,0|\mathbf{r}\cdot\mathbf{E}(\mathbf{x})|n',\mathbf{k},\lambda\rangle|^2}{\omega_{\mathbf{k}}}, \tag{11}$$

$$\Delta\varepsilon_n^{(2)}=e^2\sum_{\ell;\mathbf{k},\lambda}\frac{|\langle n,0|\mathbf{r}\cdot\mathbf{E}(\mathbf{x})|\ell;\mathbf{k},\lambda\rangle|^2}{\varepsilon_n-\varepsilon_\ell-\omega_{\mathbf{k}}}. \tag{12}$$

Using selection rules valid for central potentials, or properties of the vacuum field, the energy contribution coming from the degenerated atomic states $\Delta\varepsilon_n^{(1)}$ can be written as

$$\Delta\varepsilon_n^{(1)}=-e^2\sum_i\sum_{n'}|\langle n|x_i|n'\rangle|^2\sum_{\mathbf{k},\lambda}\frac{|\langle 0|E_i|\mathbf{k},\lambda\rangle|^2}{\omega_{\mathbf{k}}}. \tag{13}$$

Employing Schwinger's method of imaginary time splitting we obtain (see the Appendix)

$$\sum_{\mathbf{k},\lambda}\frac{|\langle 0|E_x|\mathbf{k},\lambda\rangle|^2}{\omega_{\mathbf{k}}}=\sum_{\mathbf{k},\lambda}\frac{|\langle 0|E_y|\mathbf{k},\lambda\rangle|^2}{\omega_{\mathbf{k}}}=\frac{1}{2}\sum_{\mathbf{k},\lambda}\frac{|\langle 0|\mathbf{E}_T|\mathbf{k},\lambda\rangle|^2}{\omega_{\mathbf{k}}}$$
$$=\frac{1}{512L^3\pi}G_+(z)$$
$$\sum_{\mathbf{k},\lambda}\frac{|\langle 0|E_z|\mathbf{k},\lambda\rangle|^2}{\omega_{\mathbf{k}}}=\frac{1}{256L^3\pi}G_-(z), \tag{14}$$

where we defined

$$G_\pm(z)=\zeta_H(3,z/2L)+\zeta_H(3,-z/2L)-\zeta_H(3,1/2+z/2L)$$
$$-\zeta_H(3,1/2-z/2L)+\left(\frac{2L}{z}\right)^3\pm12\zeta_R(3), \tag{15}$$

with ζ_H and ζ_R being the Huruwitz and Reimann zeta functions, respectively. As a consequence, the contribution coming from the degenerated levels to the energy shifts are given by

$$\Delta\varepsilon_n^{(1)}=-\frac{e^2}{512\pi L^3}\sum_{n'}[(|\langle n|x|n'\rangle|^2+|\langle n|y|n'\rangle|^2)G_+(z)$$
$$+2|\langle n|z|n'\rangle|^2G_-(z)]. \tag{16}$$

Let us now address our attention to the contribution coming from the nondegenerate levels. For this case, we shall consider separately two limiting regimes, namely, where the atom is near one plate and when it is far away from the plates.

Near one of the plates, it can be shown that the dominant contribution comes from $\omega_{\mathbf{k}}\gg\varepsilon_n-\varepsilon_\ell$ [8]. Hence, neglecting $\varepsilon_n-\varepsilon_\ell$ and using, as before, arguments based on selection rules for atomic transitions with spherical symmetric potentials, or properties of the vacuum fields, it can be shown that the two contributions $\Delta\varepsilon_n^{(1)}$ and $\Delta\varepsilon_n^{(2)}$ [Eqs. (11) and (12)] take the same form. Consequently, using the completeness of the atomic states, we get

$$\Delta\varepsilon_n=\Delta\varepsilon_n^{(1)}+\Delta\varepsilon_n^{(2)}=-\frac{e^2}{512\pi L^3}[\langle n|(x^2+y^2)|n\rangle G_+(z)+2\langle n|z^2|n\rangle G_-(z)]. \tag{17}$$

Far away from the plates (retarded regime), it can be shown that the dominant contribution comes from $\omega_{\mathbf{k}}\ll\varepsilon_n-\varepsilon_\ell$ [8]. Discarding now $\omega_{\mathbf{k}}$, the contribution $\Delta\varepsilon_n^{(2)}$ becomes

$$\Delta\varepsilon_n^{(2)}=e^2\sum_i\sum_\ell\frac{|\langle n|x_i|\ell\rangle|^2}{\varepsilon_n-\varepsilon_\ell}\sum_{\mathbf{k},\lambda}|\langle 0|\mathbf{E}_i|\mathbf{k},\lambda\rangle|^2. \tag{18}$$

Using the definition of the static electric polarizabilities of level n,

$$\alpha_i\equiv 2e^2\sum_\ell\frac{|\langle n|x_i|\ell\rangle|^2}{\varepsilon_\ell-\varepsilon_n}, \tag{19}$$

as well as the matrix elements of the electric-field operator obtained in the Appendix, we have in the diagonal basis of atomic states,

$$\Delta\varepsilon_n^{(2)}=-\frac{\pi^2}{96L^4}\left[(\alpha_x+\alpha_y+\alpha_z)\left(\frac{G(\theta)}{2}+\frac{7}{720}\right)-\alpha_z\frac{7}{360}\right], \tag{20}$$

where we defined

$$G(\theta)=6\frac{\cos\theta}{\sin^4\theta}-\frac{\cos\theta}{\sin^2\theta}. \tag{21}$$

To have the total shift away from the plates we must consider the contributions $\Delta\varepsilon_n^{(1)}$ and $\Delta\varepsilon_n^{(2)}$ of Eqs. (16) and (20).

III. TWO INFINITELY PERMEABLE PLATES (PP CONFIGURATION)

Considering now the PP configuration, that is, two infinitely permeable parallel plates, the boundary conditions on the electromagnetic fields are now given by

$$\hat{\mathbf{z}}\times\mathbf{B}(x,y,0,t)=0,\quad \hat{\mathbf{z}}\times\mathbf{B}(x,y,L,t)=0,$$
$$\hat{\mathbf{z}}\cdot\mathbf{E}(x,y,0,t)=0,\quad \hat{\mathbf{z}}\cdot\mathbf{E}(x,y,L,t)=0. \tag{22}$$

Using the same gauge as before ($\boldsymbol{\nabla}\cdot\mathbf{A}=0$ with $A^0=0$), and writing separately the vector potential for the TE and TM modes, as we did for the CP configuration (see Sec. II), we have

$$\mathbf{U}_{\mathbf{k}}^{TE}(\mathbf{x})=N\mathbf{e}_z\cos(kz)e^{i\mathbf{k}_T\cdot\mathbf{x}},$$
$$\mathbf{U}_{\mathbf{k}}^{TM}(\mathbf{x})=\frac{N}{i\omega_{\mathbf{k}}}\mathbf{e}_z\sin(kz)e^{i\mathbf{k}_T\cdot\mathbf{x}}, \tag{23}$$

$$k=\frac{n\pi}{L},\quad n=0,1,2,\ldots,\quad k_x,k_y\in\mathbb{R}. \tag{24}$$

Following the same procedure as that employed for the CP configuration, we obtain after a lengthy but straightforward calculation that the energy shifts when the atom is near one of the plates are obtained from

$$\Delta\varepsilon_n=\frac{e^2}{64\pi L^3}[\langle n|(x^2+y^2)|n\rangle F_+(z)+2\langle n|z^2|n\rangle F_-(z)], \tag{25}$$

where we defined

$$F_\pm(z)=\zeta_H(3,z/L)+\zeta_H(3,-z/L)\pm 2\zeta_R(3)+\left(\frac{L}{z}\right)^3. \tag{26}$$

Away from the plates, the contributions coming from the degenerated states to the energy shifts are obtained from

$$\Delta\varepsilon_n^{(1)}=\frac{e^2}{64\pi L^3}\sum_{n'}[(|\langle n|x|n'\rangle|^2+|\langle n|y|n'\rangle|^2)F_+(z)+2|\langle n|z|n'\rangle|^2F_-(z)], \tag{27}$$

and the contributions from the nondegenerated states, in a basis that diagonalizes the atomic states, read

$$\Delta\varepsilon_n^{(2)}=\frac{\pi^2}{96L^4}\left[(\alpha_x+\alpha_y+\alpha_z)\left(F(\theta)+\frac{1}{15}\right)-\frac{2}{15}\alpha_z\right], \tag{28}$$

where

$$F(\theta)=\frac{3}{\sin^4(\theta)}-\frac{2}{\sin^2(\theta)}. \tag{29}$$

IV. COMMENTS AND CONCLUSIONS

This section is devoted to comparing the results obtained in this paper, and the results presented in Ref. [8], where we have the CC configuration, that is, two conductor plates. For this boundary condition, the energy shifts of an atom placed near one of the plates come from

$$\Delta\varepsilon_n=-\frac{e^2}{64\pi L^3}[\langle n|(x^2+y^2)|n\rangle F_-(z)+2\langle n|z^2|n\rangle F_+(z)]. \tag{30}$$

The contributions to the shifts due to the degenerated states come from

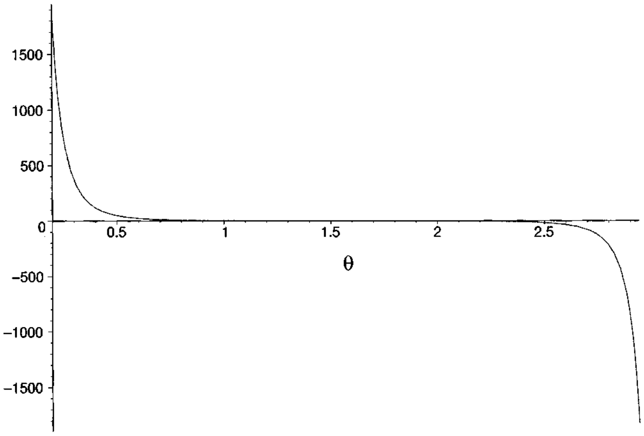

FIG. 1. Graphic for $G(\theta)$.

$$\Delta\varepsilon_n^{(1)} = -\frac{e^2}{64\pi L^3}\sum_{n'}\left[(|\langle n|x|n'\rangle|^2 + |\langle n|y|n'\rangle|^2)F_-(z) + 2|\langle n|z|n'\rangle|^2 F_+(z)\right] \tag{31}$$

for any distance from the plates.

Far away from the plates, the enegy shift contributions coming from the nondegenerated states, in a basis that diagonalizes the atomic states, read

$$\Delta\varepsilon_n^{(2)} = -\frac{\pi^2}{96L^4}\left[(\alpha_x+\alpha_y+\alpha_z)\left(F(\theta)-\frac{1}{15}\right)+\frac{2}{15}\alpha_z\right] \tag{32}$$

with the functions $F(\theta)$ and $F_\pm(z)$ defined in Eqs. (29) and (26), respectively.

It is interesting to note that the function $F(\theta)$ is strictly positive along its domain (see Ref. [8]), but the function $G(\theta)$ can be positive or negative, as shown in Fig. 1, which gives completely different behaviors for the energy shift contributions (28) and (32) compared to Eq. (20).

In order to do a numerical analysis of our results, let us restrict ourselves, from now on, to atoms not too highly excited. In this case it can be shown [8] that the contributions coming from the nondegenarated atomic states, Eqs. (31), (27), and (16), are relevant to the energy shifts. Their signs are determined from the signs of the functions $F_\pm(z)$ and $G_\pm(z)$; the former are strictly positive and the latter can change their signs, giving negative shifts for the CC configuration, while positive shifts for the PP configuration. Note from Eqs. (31) and (27) that these contributions to the energy shifts have opposite signs. Further, the roles of the longitudinal and transverse field fluctuations are also interchanged. For the CP configuration, the shifts can be positive or negative.

Here we can point out some differences between these energy shifts and the Casimir effect, another important manifestation of the vacuum fluctuations. For the Casimir effect, the PP and CC plates give the same attractive Casimir force, while for CP plates we have a repulsive Casimir force (but with the same L dependence as for the other two boundary conditions). In contrast, for the energy shifts we expect different behaviors even for those cases where the Casimir energies are the same, since the atom probes locally the quantum vacuum fluctuations, while the Casimir energy is a global quantity.

Now we present a table with numerical results, showing the energy shifts computed for the lowest hydrogen levels when the atom interacts with the radiation field in vacuum state submitted to the three boundary conditions mentioned above. For simplicity, we assume that the atom is placed at the midle point between the plates ($z=L/2$). The results are in units of $\zeta_R(3)(\alpha m)^{-2}\alpha/32L^3$ (the results for CC plates can be found in Ref. [8]).

Energy shifts	Casimir	Double permeable	Boyer
$\Delta\epsilon_{200}^{(1)}$	-1008	1008	0
$\Delta\epsilon_{210}^{(1)}$	-576	432	-54
$\Delta\epsilon_{211}^{(1)}$	-216	288	27
$\Delta\epsilon_{+}^{(1)}$	$-162(25+\sqrt{241})$	$1296(3+\sqrt{3})$	$-60{,}75(1+\sqrt{33})$
$\Delta\epsilon_{-}^{(1)}$	$-162(25-\sqrt{241})$	$1296(3-\sqrt{3})$	$-60{,}75(1-\sqrt{33})$
$\Delta\epsilon_{310}^{(1)}$	-6156	5184	$-364{,}5$
$\Delta\epsilon_{311}^{(1)}$	-3726	4212	182,2
$\Delta\epsilon_{321}^{(1)}$	-1782	1620	$-60{,}75$
$\Delta\epsilon_{322}^{(1)}$	-972	1296	121,5

(33)

We can see from the above table that the energy shifts for the CC and PP plates are of the same order, while for CP plates they are one order of magnitude smaller. Note that for the Casimir effect, the CP configuration also leads to a smaller force, in modulus, than the CC original configuration, but they are of the same order of magnitude.

As a last comment, let us analyze the limiting cases where the atom is near a unique perfectly conducting plate as well as a unique perfectly permeable one. For the former case, we take in Eq. (17) the limit of the atom located near the conducting plate, namely, we just make $z/L\to 0$:

$$\Delta\varepsilon_{(n)} = -\frac{e^2}{64\pi z^3}\left[\langle n|(x^2+y^2)|n\rangle + 2\langle n|z^2|n\rangle\right]. \tag{34}$$

This same result can be obtained from Lütken and Ravndal's paper [8].

For the second case, it would be better for calculations to have a formula that gives the energy shifts for an infinitely permeable plate at $z=0$, and a perfectly conducting one at $z=L$. This expression can be obtained by making the substitution $z\to-(z-L)$ in Eq. (17):

$$\Delta\varepsilon_n = -\frac{e^2}{512\pi L^3}\{\langle n|(x^2+y^2)|n\rangle G_+(-[z-L]) + 2\langle n|z^2|n\rangle G_-(-[z-L])\}. \tag{35}$$

We can then take the limit $z/L\to 0$, giving for an atom near one infinitely permeable plate the energy shifts

$$\Delta\varepsilon_{(n)}=\frac{e^2}{64\pi z^3}[\langle n|(x^2+y^2)|n\rangle+2\langle n|z^2|n\rangle]. \quad (36)$$

We could also have obtained this expression taking the limit $z/L\rightarrow 0$ in Eq. (25) as well. Note that the energy shifts for one conducting plate (34) and one perfectly permeable one (36) have opposite signs and same magnitude.

As a last comment, we would like to emphasize the increasing importance of considering the influence of permeable plates in different physical situations [23,24] (see also references therein). This is clear, for instance, if we note that Casimir forces may become dominant at the nanometer scale, and the appropriate consideration of permeable plates can produce repulsive forces. Recall that only attractive forces could lead to restrictive limits on the construction of nanodevices.

APPENDIX

Using the plane-wave expansion of the vector potential (8), we can easily show that

$$\sum_{\mathbf{k},\lambda}|\langle 0|E_i|\mathbf{k},\lambda\rangle|^2=\langle 0|E_iE_i|0\rangle. \quad (A1)$$

These correlators are plagued with infinities and must be regularized. Choosing Schwinger's imaginary time splitting, we write the regularized transverse and longitudinal vacuum fluctuations of the electric-field operator, respectively, as

$$\begin{aligned}\langle 0|E_T^2(z)|0\rangle&=\lim_{t'\rightarrow t}\langle 0|\mathbf{E}_T(z,t)\mathbf{E}_T(z,t')|0\rangle\\&=\frac{1}{L}\lim_{t'\rightarrow t}\sum_{n=0}^{\infty}\int\frac{d^2\mathbf{k}_T}{(2\pi)^2}\left(\omega_{\mathbf{k}}+\frac{k^2}{\omega_{\mathbf{k}}}\right)\\&\quad\times\sin^2(kz)e^{i\omega(t'-t)},\end{aligned} \quad (A2)$$

$$\begin{aligned}\langle 0|E_z^2(z)|0\rangle&=\langle 0|E_z(z,t)E_z(z,t')|0\rangle\\&=\frac{1}{L}\lim_{t'\rightarrow t}\sum_{n=0}^{\infty}\int\frac{d^2\mathbf{k}_T}{(2\pi)^2}\left(\omega_{\mathbf{k}}-\frac{k^2}{\omega_{\mathbf{k}}}\right)\\&\quad\times\cos^2(kz)e^{i\omega(t'-t)},\end{aligned}$$

where $t'-t$ will be substitute by $i\tau$ (this is equivalent to introducing an exponential cutoff). Further, defining

$$\lambda=\frac{\pi}{L},\quad \epsilon=\lambda\tau,\quad \theta=\lambda z, \quad (A3)$$

and

$$\begin{aligned}\Xi_{\pm}(\epsilon,\theta)&=\sum_{n=0}^{\infty}[\pm\cos(2kR)]\\&=\sum_{n=0}^{\infty}\{1\pm\cos[2\pi(n+1/2)z/L]\}\\&=\frac{1}{2\sinh(\epsilon/2)}\pm\frac{1}{4}\left[\frac{1}{\sinh(\epsilon/2-i\theta)}+\text{c.c.}\right],\end{aligned} \quad (A4)$$

we have, omitting the limit $\tau\rightarrow 0$,

$$\begin{aligned}\langle 0|E_T^2(z)|0\rangle&=\frac{\lambda^3}{4\pi L}\left(\partial_\epsilon\frac{1}{\epsilon}+\frac{1}{\epsilon}\partial_\epsilon\right)\Xi_-(\epsilon,\theta),\\\langle 0|E_z^2(z)|0\rangle&=\frac{\lambda^3}{4\pi L}\left(\partial_\epsilon\frac{1}{\epsilon}-\frac{1}{\epsilon}\partial_\epsilon\right)\Xi_+(\epsilon,\theta).\end{aligned} \quad (A5)$$

Expanding Eq. (A4) in powers of ϵ, that is,

$$\Xi_{\pm}=\frac{1}{\epsilon}-\frac{\epsilon}{4}T_{\pm}(\theta)+\frac{\epsilon^3}{96}\left(\frac{7}{60}\mp G(\theta)\right)+O(\epsilon^4), \quad (A6)$$

where

$$\begin{aligned}T_{\pm}(\theta)&=\frac{1}{6}\mp\frac{\cos\theta}{\sin^2\theta},\\G(\theta)&=6\frac{\cos\theta}{\sin^4\theta}-\frac{\cos\theta}{\sin^2\theta},\end{aligned} \quad (A7)$$

we obtain

$$\begin{aligned}\langle 0|E_T^2(z)|0\rangle&=\frac{\pi^2}{4L^4}\left[\frac{8}{\epsilon^4}+\frac{1}{12}\left(G(\theta)+\frac{7}{360}\right)\right],\\\langle 0|E_z^2(z)|0\rangle&=\frac{\pi^2}{4L^4}\left[\frac{4}{\epsilon^4}+\frac{1}{24}\left(G(\theta)-\frac{7}{360}\right)\right].\end{aligned} \quad (A8)$$

It is not a difficult task to show that

$$\begin{aligned}\langle 0|E_x^2(z)|0\rangle&=\langle 0|E_y^2(z)|0\rangle\\&=\frac{1}{2}\langle 0|E_T^2(z)|0\rangle\\&=\frac{\pi^2}{4L^4}\left[\frac{4}{\epsilon^4}+\frac{1}{24}\left(G(\theta)+\frac{7}{360}\right)\right].\end{aligned} \quad (A9)$$

The terms proportional to $1/\varepsilon^4$ in Eqs. (A8) and (A9) diverge in the limit $\tau\rightarrow 0$; however, they are L independent, so that they are spurious terms with no physical significance.

Using the same procedure adopted to compute Eqs. (A1) and (A2), we can write

$$\sum_{\mathbf{k},\lambda} \frac{|\langle 0|\mathbf{E}_T|\mathbf{k},\lambda\rangle|^2}{\omega_{\mathbf{k}}} = \lim_{t'\to t}\sum_{\mathbf{k},\lambda} \frac{\langle 0|\mathbf{E}_T(z,t)|\mathbf{k},\lambda\rangle\langle \mathbf{k},\lambda|\mathbf{E}_T(z,t')|0\rangle}{\omega_{\mathbf{k}}} = \frac{1}{L}\lim_{t'\to t}\sum_{n=0}^{\infty}\int \frac{d^2\mathbf{k}_T}{(2\pi)^2}\frac{1}{\omega_{\mathbf{k}}}\left[\omega_{\mathbf{k}}+\frac{k^2}{\omega_{\mathbf{k}}}\right] \times \sin^2(kz)e^{i\omega_{\mathbf{k}}(t'-t)}, \tag{A10}$$

$$\sum_{\mathbf{k},\lambda} \frac{|\langle 0|E_z|\mathbf{k},\lambda\rangle|^2}{\omega_{\mathbf{k}}} = \lim_{t'\to t}\sum_{\mathbf{k},\lambda} \frac{\langle 0|E_z(z,t)|\mathbf{k},\lambda\rangle\langle \mathbf{k},\lambda|E_z(z,t')|0\rangle}{\omega_{\mathbf{k}}} = \frac{1}{L}\lim_{t'\to t}\sum_{n=0}^{\infty}\int \frac{d^2\mathbf{k}_T}{(2\pi)^2}\frac{1}{\omega_{\mathbf{k}}}\left[\omega_{\mathbf{k}}-\frac{k^2}{\omega_{\mathbf{k}}}\right] \times \cos^2(kz)e^{i\omega_{\mathbf{k}}(t'-t)}.$$

With definitions (A3) and (A4), and omitting as before the limit $\tau\to 0$, we obtain

$$\sum_{\mathbf{k},\lambda} \frac{|\langle 0|\mathbf{E}_T|\mathbf{k},\lambda\rangle|^2}{\omega_{\mathbf{k}}} = \frac{\pi}{2L^3}\left[\int_{\epsilon}^{\infty} dx g_-(x) - \frac{1}{\epsilon}\frac{\partial}{\partial\epsilon}\Xi_-(\epsilon,\theta)\right], \tag{A11}$$

$$\sum_{\mathbf{k},\lambda} \frac{|\langle 0|E_z|\mathbf{k},\lambda\rangle|^2}{\omega_{\mathbf{k}}} = \frac{\pi}{2L^3}\left[\frac{1}{\epsilon^2}\Xi_+(\epsilon,\theta) - \int_{\epsilon}^{\infty} dx\, g_+(x)\right],$$

where $g_{\pm}(x)=(1/x^3)\Xi_{\pm}(x,\theta)$. In order to compute the above integrals, we consider the analytically continued function

$$g_{\pm}(z,p)=\frac{1}{z^p}\Xi_{\pm}(z,\theta). \tag{A12}$$

The integral along C_ρ [see Fig. 2] vanishes, which yields, using the residue theorem,

$$\int_{\epsilon}^{\infty} dx g_{\pm}(x,p) = (1-e^{-2\pi p i})^{-1}\left[-\int_{C_\epsilon} dz g_{\pm}(z,p) + 2\pi i \sum_{z\neq 0} \mathrm{Res} g_{\pm}(x,p)\right]. \tag{A13}$$

With the definitions

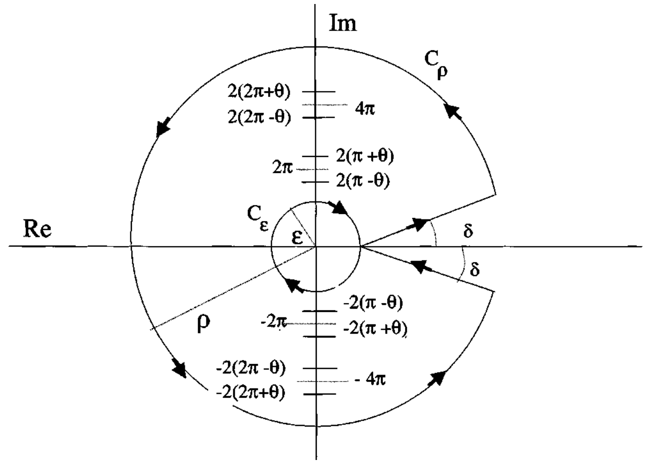

FIG. 2. Integration contour.

$$\zeta_R(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}, \quad \zeta_H(s,a)=\sum_{n=1}^{\infty}\frac{1}{(n+a)^s}, \tag{A14}$$

and using the expansion (A6), we have,

$$\lim_{p\to 3}\left[(1-e^{-2\pi p i})^{-1}(-1)\int_{C_\epsilon} dz g_{\pm}(z,p)\right] = \left[\frac{1}{3\epsilon^3} - \frac{T_{\pm}(\theta)}{4\epsilon}\right] \tag{A15}$$

and

$$\lim_{p\to 3}\left[(1-e^{-2\pi p i})^{-1} 2\pi i \sum_{z\neq 0} \mathrm{Res} g_{\pm}(x,p)\right] = -\frac{1}{2(4\pi)^2}\left[-6\zeta_R(3) \pm \frac{1}{2}\left\{\zeta(3,\theta/2\pi)+\zeta(3,-\theta/2\pi) - \zeta(3,1/2+\theta/2\pi) - \zeta(3,1/2-\theta/2\pi) + \left(\frac{2\pi}{\theta}\right)^3\right\}\right] \equiv \mp \frac{1}{128\pi^2} G_{\mp}(\theta), \tag{A16}$$

which gives the desired integral

$$\int_{\varepsilon}^{\infty} dx g_{\pm}(x) = \int_{\varepsilon}^{\infty} dx \frac{1}{x^3}\Xi_{\pm}(x,\theta) = \frac{1}{3\varepsilon^3} - \frac{T_{\pm}(\theta)}{4\varepsilon} \mp \frac{1}{128\pi^2} G_{\mp}(\theta). \tag{A17}$$

Subsituting this result in Eq. (A11) and using the definition of Ξ, we obtain the result (14) and a term that diverges in the limit $\tau\to 0$. As before it is L independent, so that it is a spurious term with no physical significance.

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